PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(91) (2005), 93–98

ON SOME RESULTS FOR λ-SPIRALLIKE AND λ-ROBERTSON FUNCTIONS OF COMPLEX ORDER

M. K. Aouf, F. M. Al-Oboudi, and M. M. Haidan

Communicated by Miroljub Jevtić

Abstract. We give some results of various kinds concerning λ-spirallike functions of complex order and λ-Robertson functions of complex order in the unit disc $U = \{z : |z| < 1\}$. They represent extensions and generalizations of many previous results. We mainly used the subordination method.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

For a function $f(z)$ belonging to the class A we say that $f(z)$ is λ-spirallike of complex order in U if and only if

$$\operatorname{Re}\left\{\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda\right]\right\} > 0, \tag{1.2}$$

for some real λ, $|\lambda| < \pi/2$, $b \neq 0$, complex. We denote this class by $S^\lambda(b)$. It was introduced and studied by Al-Oboudi and Haidan [1].

Also for a function $f(z)$ belonging to the class A we say that $f(z)$ is λ-Robertson function of complex order in U if and only if

$$\operatorname{Re}\left\{\frac{1}{b\cos\lambda}\left[e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right) - (1-b)\cos\lambda - i\sin\lambda\right]\right\} > 0, \tag{1.3}$$

for some real λ, $|\lambda| < \pi/2$, $b \neq 0$, complex. We denote this class by $C^\lambda(b)$.

If follows from (1.2) and (1.3) that

$$f(z) \in C^\lambda(b) \text{ if and only if } zf'(z) \in S^\lambda(b). \tag{1.4}$$

2000 *Mathematics Subject Classification*: Primary 30C45.
Key words and phrases: λ-Spirallike, λ-Robertson functions, subordination.

We note that:

(1) $S^\lambda(1) = S^\lambda$, is the class of λ-spirallike univalent functions defined by Spacek [16], $S^0(b) = S(b)$, is the class of starlike functions of complex order studied by Nasr and Aouf [7], $S^\lambda(1-\alpha) = S^\lambda(\alpha)$, $0 \leqslant \alpha < 1$, is the class of λ-spirallike functions of order α studied by Libera [4] and $S^0(1-\alpha) = S^*(\alpha)$, $0 \leqslant \alpha < 1$, is the class of starlike functions of order α, studied by Robertson [12].

(2) $C^\lambda(1) = C^\lambda$, is the class of λ-Robertson functions studied by Robertson [13], $C^\lambda(1-\alpha) = C^\lambda(\alpha)$, $0 \leqslant \alpha < 1$, is the class of λ-Robertson functions of order α studied by Chichra [3] and $C^0(b) = C(b)$, is the class of convex functions of complex order studied by Waitrowski [18], Nasr and Aouf [8] and Aouf [2] and $C^0(1-\alpha) = C(\alpha)$, $0 \leqslant \alpha < 1$, is the class of convex functions of order α studied by Robertson [12].

The object of this paper is to obtain some results for the classes $S^\lambda(b)$ and $C^\lambda(b)$ using mainly the method of subordination. In that sense, we give some definitions, notations and lemmas we need in the next part.

Let f and F be analytic in the unit disc U. The function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

The general theory of differential subordinations was introduced by Miller and Mocanu [5]. Some classes of the first-order differential subordinations were considered by the same authors in [6]. Namely let $\psi : C^2 \to C$ be analytic in a domain D, let h be univalent in U, and let $p(z)$ be analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then $p(z)$ is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z). \tag{1.5}$$

The univalent function q is said to be a dominant of the differential subordination (1.5) if $p \prec q$ for all p satisfying (1.5). If $\tilde{q}$ is a dominant of (1.5) and $\tilde{q} \prec q$ for all dominants q of (1.5), then $\tilde{q}$ is said to be the best dominant of (1.5).

First we cite the following lemma on differential subordinations due to Miller and Mocanu [6].

Lemma 1. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \qquad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

(i) *Q is starlike (univalent) in U with $Q(0) = 0$ and $Q'(0) \neq 0$,*

(ii) $\operatorname{Re}\left\{z\dfrac{h'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\dfrac{\theta'(q(z))}{\phi(q(z))} + z\dfrac{Q'(z)}{Q(z)}\right\} > 0$, $z \in U$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.6}$$

then $p \prec q$, and q is the best dominant of (1.6).

For the proof of Theorem 2, we need the following result of Robertson [14].

LEMMA 2. *Let $f(z) \in A$ be univalent in U. For $0 \leqslant t \leqslant 1$, let $F(z,t)$ be analytic in U, let $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let r be positive real number for which*

$$F(z) = \lim_{t \to 0^+} \frac{F(z,t) - F(z,0)}{zt^r}$$

exists. Let $F(z,t)$ be subordinate to $f(z)$ in U for $0 \leqslant t \leqslant 1$. Then $\operatorname{Re}\left\{\frac{F(z)}{f'(z)}\right\} \leqslant 0$, *$z \in U$. If, in addition, $F(z)$ is also analytic in U and $\{F(0)\} \neq 0$, then*

$$\operatorname{Re}\left\{\frac{f'(z)}{F(z)}\right\} < 0, \quad z \in U. \tag{1.7}$$

2. Results and consequences

First we use the differential subordinations to obtain:

THEOREM 1. *Let $f \in S^\lambda(b)$ ($|\lambda| < \pi/2$, $b \neq 0$, complex), then*

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}}, \tag{2.1}$$

where $a \neq 0$ is complex and either $|2abe^{-i\lambda}\cos\lambda + 1| \leqslant 1$ or $|2abe^{-i\lambda}\cos\lambda - 1| \leqslant 1$, and this is the best dominant.

PROOF. If we put $q(z) = (1-z)^{-2abe^{-i\lambda}\cos\lambda}$, $\phi(w) = (abe^{-i\lambda}\cos\lambda)^{-1}w^{-1}$ and $\theta(w) = 1$ in Lemma 1, then it is easy to check that the conditions (i) and (ii) in that lemma are satisfied. Namely, $q(z)$ is univalent in U [15], while

$$h(z) = \theta(q(z)) + zq'(z)\phi(q(z)) = \frac{1+z}{1-z}\cdot$$

Consequently, for $p(z) = 1 + p_1 z + \cdots$ analytic in U with $p(z) \neq 0$ for $0 < |z| < 1$, from (1.6) we get

$$1 + \frac{e^{i\lambda}}{ab\cos\lambda}\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} \Rightarrow p(z) \prec q(z)\cdot \tag{2.2}$$

Now, if in (2.2) we choose $p(z) = \left(\frac{f(z)}{z}\right)^a$, then we have

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}},$$

which evidently completes the proof of Theorem 1. □

REMARK 1. (1) Putting (i) $\lambda = 0$, (ii) $b = 1$, and (iii) $b = 1-\alpha$, $0 \leqslant \alpha < 1$, in Theorem 1, we get the results obtained by Obradović, Aouf and Owa [10] for the classes $S(b)$, S^λ and $S^\lambda(\alpha)$, respectively.

(2) Putting (i) $\lambda = 0$ and $b = 1$, (ii) $\lambda = 0$ and $b = 1-\alpha$, $0 \leqslant \alpha < 1$, in Theorem 1, we get the corresponding results for the classes S^* and $S^*(\alpha)$, especially, the well-known results for the classes S^* and $S^*(\alpha)$ when $a = 1$.

From Theorem 1 and using (1.4), we directly get:

COROLLARY 1. *Let $f(z) \in C^\lambda(b)$, $(|\lambda| < \pi/2$, $b \neq 0$, complex), and let $a \neq 0$ be a complex number and either $|2abe^{-i\lambda}\cos\lambda + 1| \leqslant 1$ or $|2abe^{-i\lambda}\cos\lambda - 1| \leqslant 1$. Then*

$$(f'(z))^a \prec (1-z)^{-2abe^{-i\lambda}\cos\lambda}$$

and this is the best dominant.

Putting $b = 1-\alpha$, $0 \leqslant \alpha < 1$, in Corollary 1, we get the following result for the class $C^\lambda(\alpha)$:

COROLLARY 2. *Let $f(z) \in C^\lambda(\alpha)$ $(|\lambda| < \pi/2$, $0 \leqslant \alpha < 1)$, and let a be a complex number such that*

$$|2a(1-\alpha)\cos\lambda e^{-i\lambda} - 1| \leqslant 1 \text{ or } |2a(1-\alpha)\cos\lambda e^{-i\lambda} + 1| \leqslant 1.$$

Then

$$(f'(z))^a \prec (1-z)^{-2a(1-\alpha)e^{-i\lambda}\cos\lambda} \tag{2.3}$$

and this is the best dominant.

Putting $\lambda = 0$ in Corollary 2 we get the result obtained by Obradović, Aouf and Owa [10].

If we put $a = -\dfrac{e^{i\lambda}}{2b\cos\lambda}$ in Theorem 1, we get:

COROLLARY 3. *Let $f(z) \in S^\lambda(b)$ $(|\lambda| < \pi/2$, $b \neq 0)$, then*

$$\left(\frac{z}{f(z)}\right)^{\frac{e^{i\lambda}}{2b\cos\lambda}} \prec (1-z), \tag{2.4}$$

and this is the best dominant.

From (2.4), we have the following inequality for $f(z) \in S^\lambda(b)$

$$\left|\left(\frac{z}{f(z)}\right)^{\frac{e^{i\lambda}}{2b\cos\lambda}} - 1\right| \leqslant |z|, \quad z \in U. \tag{2.5}$$

REMARK 2. (i) Putting $\lambda = 0$ in (2.5), we get the result obtained by Obradović, Aouf and Owa [10], (ii) Putting $b = 1-\alpha$, $0 \leqslant \alpha < 1$ and $\lambda = 0$ in (2.5), we get the result obtained by Obradović, Aouf and Owa [10] and Todorov [17].

By using Lemma 2 we give a criterion for a function $f(z) \in A$ to be in the class $S^\lambda(b)$.

THEOREM 2. *Let $f(z) \in A$ with $f(z)/z \neq 0$ in U, and let the function*

$$g(z) = \frac{e^{i\lambda}}{b\cos\lambda}\left[f(z) - (1 - be^{-i\lambda}\cos\lambda)\int_0^z \frac{f(s)}{s}ds\right] = z + \cdots, \tag{2.6}$$

be univalent in U, where $|\lambda| < \pi/2$ and $b \neq 0$, is a complex number. If the function

$$G(z,t) = \frac{e^{i\lambda}}{b\cos\lambda}\left[(1 - tbe^{-i\lambda}\cos\lambda)f(z) - (1 - be^{-i\lambda}\cos\lambda)(1-t^2)\int_0^z \frac{f(s)}{s}ds\right] \tag{2.7}$$

is subordinate to $g(z)$ for a fixed $b, |\lambda| < \pi/2$, and for each $0 \leqslant t \leqslant 1$, then $f(z) \in S^\lambda(b)$.

PROOF. It is evident that $G(z,0) \equiv g(z)$ and $G(0,t) \equiv 0$. In Lemma 2, we choose $r = 1$ and $F(z,t)$ to be the function $G(z,t)$ defined by (2.7). Then we have

$$G(z) = \lim_{t\to 0^+} \frac{G(z,t) - G(z,0)}{zt} = \frac{1}{z} \lim_{t\to 0} \frac{\partial G(z,t)}{\partial t} = -\frac{f(z)}{z}$$

and $G(z)$ is analytic in U with $\operatorname{Re}\{G(0)\} = -1 \neq 0$. Since from (2.6)

$$g'(z) = \frac{e^{i\lambda}}{b\cos\lambda}\Big[f'(z) - (1 - be^{-i\lambda}\cos\lambda)\frac{f(z)}{z}\Big],$$

then from (1.7) we have $\operatorname{Re}\Big\{\dfrac{g'(z)}{G(z)}\Big\} < 0$, $z \in U$, which is equivalent to

$$\operatorname{Re}\Big\{1 + \frac{e^{i\lambda}}{b\cos\lambda}\Big(\frac{zf'(z)}{f(z)} - 1\Big)\Big\} > 0 \quad z \in U,$$

i.e., $f(z) \in S^\lambda(b)$. □

REMARK 3. (1) Putting (i) $\lambda = 0$ (ii) $\lambda = 0$ and $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, in Theorem 2, we get the results for the classes $S(b)$ and $S^*(\alpha)$ obtained by Obradović, Aouf and Owa [10] and Obradović [9], respectively.

(2) Putting $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, Theorem 2 we get the following result for the class $S^\lambda(\alpha)$ ($|\lambda| < \pi/2$, $0 \leqslant \alpha < 1$).

COROLLARY 4. *Let $f(z) \in A$, and let the function $g(z)$ defined by*

$$g(z) = \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\Big[f(z) - \big(1 - (1-\alpha)e^{-i\lambda}\cos\lambda\big)\int_0^z \frac{f(s)}{s}ds\Big] = z + \cdots$$

be univalent in U, where $|\lambda| < \pi/2$ and $0 \leqslant \alpha < 1$. If the function

$$G(z,t) = \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\Big[\big(1 - t(1-\alpha)e^{-i\lambda}\cos\lambda\big)f(z) \\ - \big(1 - (1-\alpha)e^{-i\lambda}\cos\lambda\big)(1 - t^2)\int_0^z \frac{f(s)}{s}ds\Big],$$

is subordinate to $g(z)$ in the unit disc U for fixed λ ($|\lambda| < \pi/2$) and α ($0 \leqslant \alpha < 1$), and for each t ($0 \leqslant t \leqslant 1$), then $f(z)$ is in the class $S^\lambda(\alpha)$.

REMARK 4. The result obtain in Corollary 4 corrected the result obtained by Obradović and Owa [11, Theorem 3] for the class S^λ.

References

[1] F. M. Al-Oboudi and M. M. Haidan, *Spirallike functions of complex order*, J. Natural Geom. 19 (2000), 53–72.

[2] M. K. Aouf, *p-valent classes related to convex functions of complex order*, Rocky Mountain J. Math. 16 (1985), 775–790.

[3] P. N. Chichra, *Regular functions $f(z)$ for which $zf'(z)$ is α-spiral-like*, Proc. Amer. Math. Soc. 49 (1975), 151–160.

[4] R. L. Libera, *Univalent α-spiral functions*, Canad. J. Math. 19 (1967), 449–456.

[5] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. 65 (1978), 289–305.

[6] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J. 32 (1985), 185–195.
[7] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, J. Natural Sci. Math. 25 (1985), 1–12.
[8] M. A. Nasr and M. K. Aouf, *On convex functions of complex order*, Mansoura Sci. Bull. Egypt 9 (1982), 565–582.
[9] M. Obradović, *Two applications of the one Robertson's result*, Mat. Vesnik 35 (1983), 283–287.
[10] M. Obradović, M. K. Aouf and S. Owa, *On some results for starlike functions of complex order*, Publ. Inst. Math., Nouv. Sér. 46(60) (1989), 79–85.
[11] M. Obradović and S. Owa, *On some results for λ-spiral functions of order α*, Internat. J. Math. Math. Sci. 9 (1986), 430–446.
[12] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. 37 (1936), no. 2, 374–408.
[13] M. S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. 16 (1969), 97–101.
[14] M. S. Robertson, *Applications of the subordination principle to univalent functions*, Pacific J. Math. 11 (1961), 315–324.
[15] W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. 12 (1965), 385–387.
[16] L. Spacek, *Contribution a la theori des function univalentes*, Casopis Pest. Math. (1932), 12–19.
[17] P. Todorov, *The domains of the values of certain functionals defined over the classes of starlike and convex functions of order alpha*, Dokl. Bolg. AN 39 (1986), no. 9, 19–22.
[18] P. Waitrowski, *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Univ. Lodzk. Nauk. Math. Przyrod. Ser. II, Zeszyt (39) Math. (1971), 75–85.

(Received 21 10 2003)

Mathematics Department
Girls College Of Education
Science Sections
Jeddah, Saudi Arabia

Mathematics Department
Girls College of Education
Science Sections
Riyadh, Saudi Arabia
fma34@yahoo.com

Mathematics Department
Girls College of Education
Science Sections
Abha, Saudi Arabia
majbh2001@yahoo.com