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COMPLETENESS THEOREM FOR A LOGIC WITH IMPRECISE AND CONDITIONAL PROBABILITIES

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Abstract. We present a propositional probability logic which allows making formulas that speak about imprecise and conditional probabilities. A class of Kripke-like probabilistic models is defined to give semantics to probabilistic formulas. Every possible world of such a model is equipped with a probability space. The corresponding probabilities may have nonstandard values. The proposition "the probability is close to r" means that there is an infinitesimal ϵ, such that the probability is equal to $r-\epsilon$ (or $r+\epsilon$). We provide an infinitary axiomatization and prove the corresponding extended completeness theorem.

1. Introduction

Although the problem of reasoning with uncertain knowledge is a very old problem dating, at least, from Leibnitz and Boole, formal systems for reasoning in the presence of uncertainty have been recognized as a useful tool in many fields of Computer Science and Artificial Intelligence only since the Nils Nilsson's paper [**16**]. Using the semantical approach, Nilsson introduced some procedures for calculating the bounds on the absolute probability of a consequence given the probabilities of the premisses. It motivated many semantical and some proof-theoretical approaches to logics with absolute probabilities (see [**2, 4, 5, 6, 9, 10, 15, 18, 19, 20**], and the references given there). In the latter discussion on the subject [**17**] Nilsson argued that a more natural generalization of the classical modus ponens could be handled by conditional probabilities. However, there are not too many papers discussing conditional probabilities from the logical point of view [**6, 7, 14, 22**]. In [**6**] a logic which allows formal reasoning about conditional probabilities using the machinery of real closed fields is introduced. In [**14**], a fuzzy modal logic $FCP(L\Pi)$

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is introduced along the ideas about coherent probabilities proposed by de Finetti. For each pair of classical propositional formulas α and β, the probability of the conditional event "α given β" is taken as the truth-value of the (fuzzy) modal proposition $P(\alpha \mid \beta)$. An axiomatic system is introduced and shown to be sound and complete with respect to the class of probabilistic Kripke structures induced by coherent conditional probabilities. A close approach can be found in [**7**], where a treatment of nonstandard conditional probability by means of fuzzy logic is given. One of the areas of application of probabilistic logics is the area of default reasoning, which for a while suffered from a lack of proper syntactic characterization. In the seminal paper [**12**] a set of properties which form a core of default reasoning and the corresponding formal system denoted P are proposed. Many semantics for default entailment have been introduced and proven to be characterized by P [**8**]. One such approach uses conditional probability to semantically characterize defaults [**12**], but in that context probability function with non-standard values has to be used. Another similar line of research concerns the study of conditionals (or counter-factuals) [**3**].

In this paper we present a propositional probability logic $LICP^S$ (L for logic, I for imprecise, CP for conditional probabilities, while S denotes a special countable set which will be discussed later on) and explore the issue of completeness. The logic allows making formulas that speak about imprecise and conditional probabilities. The corresponding language is obtained by adding probability operators to the classical propositional language. There are two kinds of new operators. The operators of the first type ($\{P_{\geqslant s}\}_{s\in S}$, for the probability of a single formula, and $\{CP_{\geqslant s}\}_{s\in S}$, $\{CP_{=s}\}_{s\in S}$, for the conditional probability) concern standard ("crisp") probabilities with the intended meaning "the (conditional) probability is at least s" and "is s", respectively. The operators of the second type ($CP_{\approx r}$, $P_{\approx r}$, where r belongs to the unit interval of rational numbers) concern imprecise probabilities with the intended meaning that "the (conditional) probability is close to r". The set S mentioned above is the unit interval of a recursive nonarchimedean field containing all rational numbers. An example of such field is the Hardy field $Q[\epsilon]$, where ϵ is an infinitesimal. In this paper S is used to syntactically define the range of the probability functions that will appear in the interpretation. Thus, the proposition "the (conditional) probability is close to r" means that there is an infinitesimal ϵ_0, such that the (conditional) probability is equal to $r - \epsilon_0$ (or $r + \epsilon_0$).

Probabilistic models based on Kripke models are used to give semantics to the formulas so that interpreted formulas are either true or false. Every world from a probability model is equipped with a probability space. The corresponding probability measures are defined on sets of subsets of possible worlds, while the range of the probabilities is the set S. More precisely, we consider the so called class of *measurable models*. A model is measurable if only sets of possible worlds definable by formulas are measurable. We give an *infinitary* axiomatic system for $LICP^S$. Here the terms *finitary* and *infinitary* concern the proof system only: our object language is countable, formulas are finite, while only proofs are allowed to be infinite. For that axiomatic system and the mentioned class of measurable probability models we prove the extended completeness theorem ('every consistent set of formulas is satisfiable'). The reason for introducing infinitary rules is that for our logic the compactness theorem does not hold, i.e., there exists a countably infinite set of formulas that is unsatisfiable although every finite subset is satisfiable. For instance, consider $\{\neg P_{=0}\alpha\} \cup \{P_{<\epsilon^n}\alpha : n \text{ is a positive integer}\}$. It follows that it is not possible to give a usual finitary axiomatic system for $LICP^S$ since the compactness theorem follows easily from the extended completeness theorem if the corresponding axiomatization is finitary. There are four infinitary rules in our system. One of them (Rule 3) enables us to syntactically define the range of the probability functions which will appear in the interpretation. A similar rule was given in [**1**] but restricted to rationals only.

We showed in [**21, 22**] how, thanks to our nonstandard probabilistic semantics, a restriction (denoted LPP^S) of $LICP^S$ can be used to syntactically describe the behavior of the defaults in a probabilistic framework. LPP^S allows only one imprecise conditional probability operator $CP_{\approx 1}$, and does not allow iteration of probabilistic operators. The axioms and rules of the system P can be translated into LPP^S-valid formulas, so that the formula $CP_{\approx 1}(\beta, \alpha)$ syntactically describes the behavior of the default 'if α, then generally β'. Thus, the corresponding LPP^S-axiom system can be used to characterize the default consequence relation. In [**13**] a combination of probabilistic knowledge and default reasoning is considered. Since the axioms and rules of the system P are applied to probabilistic knowledge, and since LPP^S does not allow iteration of probabilistic operators, it is not possible to extend our approach from [**21, 22**] to the framework discussed there. Thus, we consider our logic $LICP^S$ as a natural generalization of LPP^S which might be useful in characterization of the system proposed in [**13**].

2. Syntax

Let $Q[0,1]$ denote the set of all rational numbers from the unit interval. Let S be the unit interval of a recursive nonarchimedean field containing $Q[0,1]$. An example of such field is the Hardy field $Q[\epsilon]$. $Q[\epsilon]$ contains all rational functions of a fixed infinitesimal ϵ which belongs to a nonstandard elementary extension R^* of the standard real numbers (see [**11, 23**]). A typical positive element of $Q[\epsilon]$ is of the form

$$\epsilon^k \frac{\sum_{i=0}^{n} a_i \epsilon^i}{\sum_{i=0}^{m} b_i \epsilon^i},$$

where $a_0 \cdot b_0 \neq 0$.

The language of the logic consists of:

- a denumerable set $\mathrm{Var} = \{p, q, r, \ldots\}$ of propositional letters,
- classical connectives $\neg$, and $\wedge$,
- a denumerable list of unary probabilistic operators $(P_{\geqslant s})_{s \in S}$,
- a denumerable list of unary probabilistic operators $(P_{\approx r})_{r \in Q[0,1]}$,
- a denumerable list of binary probabilistic operators $(CP_{\geqslant s})_{s \in S}$,
- a denumerable list of binary probabilistic operators $(CP_{=s})_{s \in S}$ and
- a denumerable list of binary probabilistic operators $(CP_{\approx r})_{r \in Q[0,1]}$.

The set of formulas is defined inductively as the smallest set containing propositional letters and closed under formation rules: if α and β are formulas, $s \in S$ and $r \in Q[0,1]$, then $\neg\alpha$, $\alpha \wedge \beta$, $P_{\geqslant s}\alpha$, $P_{\approx r}\alpha$, $CP_{\geqslant s}(\alpha,\beta)$, $CP_{=s}(\alpha,\beta)$ and $CP_{\approx r}(\alpha,\beta)$ are formulas. Let For_S denote this set of formulas.

For example, $(\neg p \wedge P_{\approx 0.5}q) \wedge CP_{\geqslant 1-2\epsilon_1}(r \wedge q, CP_{=0.8}(r,p))$ is a formula.

The other classical connectives ($\vee$, $\to$, $\leftrightarrow$) can be defined as usual, while we denote $\neg P_{\geqslant s}\alpha$ by $P_{<s}\alpha$, $P_{\geqslant 1-s}\neg\alpha$ by $P_{\leqslant s}\alpha$, $\neg P_{\leqslant s}\alpha$ by $P_{>s}\alpha$, $P_{\geqslant s}\alpha \wedge \neg P_{>s}\alpha$ by $P_{=s}\alpha$, $\neg P_{=s}\alpha$ by $P_{\neq s}\alpha$, $\neg CP_{\geqslant s}(\alpha,\beta)$ by $CP_{<s}(\alpha,\beta)$, $CP_{<s}(\alpha,\beta) \vee CP_{=s}(\alpha,\beta)$ by $CP_{\leqslant s}(\alpha,\beta)$, and $CP_{\geqslant s}(\alpha,\beta) \wedge \neg CP_{=s}(\alpha,\beta)$ by $CP_{>s}(\alpha,\beta)$. Finally, we use $\perp$ to denote $\neg p \wedge p$.

3. Semantics

The semantics for our logic will be based on the Kripke-style models.

Definition 3.1. An $LICP^S$-model is a structure $M = \langle W, \mathrm{Prob}, v\rangle$ where:

- W is a nonempty set of elements called worlds,
- Prob is a probability assignment which assigns to every $w \in W$ a probability space $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w)$ is a non empty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \to S$ is a finitely additive probability measure, and
- $v : W \times \mathrm{Var} \to \{\mathrm{true}, \mathrm{false}\}$ is a valuation which associates with every world $w \in W$ a truth assignment $v(w)$ on the propositional letters.

Let $M = \langle W, \mathrm{Prob}, v\rangle$ be an $LICP^S$-model, $w, u \in W$, $p \in \mathrm{Var}$, $\alpha, \beta \in \mathrm{For}_S$, $s \in S$ and $r \in Q[0,1]$. The satisfiability relation $\models$ is inductively defined as follows:

(1) $(w,M) \models p$ if $v(w)(p) = \mathrm{true}$,
(2) $(w,M) \models \neg\alpha$ if it is not $(w,M) \models \alpha$,
(3) $(w,M) \models \alpha \wedge \beta$ if $(w,M) \models \alpha$ and $(w,M) \models \beta$,
(4) $(w,M) \models P_{\geqslant s}\alpha$ if $\mu(w)(\{u \in W(w) : (u,M) \models \alpha\}) \geqslant s$,
(5) $(w,M) \models P_{\approx r}\alpha$ if for every integer $n > 0$, $\mu(w)(\{u \in W(w) : (u,M) \models \alpha\}) \in [\max\{0, r-1/n\}, \min\{1, r+1/n\}]$,
(6) $(w,M) \models CP_{\geqslant s}(\alpha,\beta)$ if either $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) = 0$ or $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) > 0$ and $\frac{\mu(w)(\{u \in W(w):(u,M)\models \alpha\wedge\beta\})}{\mu(w)(\{u \in W(w):(u,M)\models\beta\})} \geqslant s$,
(7) $(w,M) \models CP_{=s}(\alpha,\beta)$ if either $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) = 0$ and $s = 1$ or $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) > 0$ and $\frac{\mu(w)(\{u \in W(w):(u,M)\models \alpha\wedge\beta\})}{\mu(w)(\{u \in W(w):(u,M)\models\beta\})} = s$,
(8) $(w,M) \models CP_{\approx r}(\alpha,\beta)$ if either $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) = 0$ and $r = 1$ or $\mu(w)(\{u \in W(w) : (u,M) \models \beta\}) > 0$ and for every positive integer n, $\frac{\mu(w)(\{u \in W(w):(u,M)\models \alpha\wedge\beta\})}{\mu(w)(\{u \in W(w):(u,M)\models\beta\})} \in [\max\{0, r-1/n\}, \min\{1, r+1/n\}]$.

In the sequel, we will omit M from $(w,M) \models \alpha$ and write $w \models \alpha$ if M is clear from the context.

Note that the condition 5 is equivalent to saying that the probability of α equals $r-\epsilon_0$ (or $r+\epsilon_0$) for some infinitesimal $\epsilon_0 \in S$. The similar holds for the condition 8.

In an $LICP^S$-model $M = \langle W, \mathrm{Prob}, v\rangle$ the set $\{u \in W(w) : u \models \alpha\}$, is denoted by $[\alpha]_w$. In the sequel we consider only so-called measurable models. An $LICP^S$-model $M = \langle W, \mathrm{Prob}, v\rangle$ is said to be measurable if for every $w \in W$, $H(w) = \{[\alpha]_w : \alpha \in \mathrm{For}_S\}$, i.e., if every set of the form $[\alpha]_w$ is measurable and every $H(w)$ contains only sets definable by formulas. It is not hard to see that $H(w)$'s are algebras. Let us denote this class of models by $LICP^S_{\mathrm{Meas}}$.

A set of formulas T is $LICP^S_{\mathrm{Meas}}$-satisfiable if there is a world w in an $LICP^S_{\mathrm{Meas}}$-model M such that for every formula $\alpha \in T$, $w \models \alpha$. A formula α is $LICP^S_{\mathrm{Meas}}$-satisfiable if the set $\{\alpha\}$ is $LICP^S_{\mathrm{Meas}}$-satisfiable. A formula α is $LICP^S_{\mathrm{Meas}}$-valid (denoted by $\models \alpha$) if it is satisfied in each world in each model.

4. Axiomatization

The set of all valid formulas can be characterized by the following set of axiom schemata:

(1) all the axioms of the classical propositional logic
(2) $P_{\geqslant 0}\alpha$
(3) $P_{\leqslant s}\alpha \to P_{<t}\alpha$, $t > s$
(4) $P_{<s}\alpha \to P_{\leqslant s}\alpha$
(5) $P_{\geqslant 1}(\alpha \leftrightarrow \beta) \to (P_{=s}\alpha \to P_{=s}\beta)$
(6) $(P_{=s}\alpha \wedge P_{=t}\beta \wedge P_{\geqslant 1}\neg(\alpha \wedge \beta)) \to P_{=\min(1,s+t)}(\alpha \vee \beta)$
(7) $P_{\approx r}\alpha \to P_{\geqslant r_1}\alpha$, for every rational $r_1 \in [0, r)$
(8) $P_{\approx r}\alpha \to P_{\leqslant r_1}\alpha$, for every rational $r_1 \in (r, 1]$
(9) $CP_{=s}(\alpha,\beta) \to \neg CP_{=t}(\alpha,\beta)$, $s \neq t$
(10) $P_{=0}\beta \to CP_{=1}(\alpha,\beta)$
(11) $(P_{=t}\beta \wedge P_{=s}(\alpha \wedge \beta)) \to CP_{=s/t}(\alpha,\beta)$, $t \neq 0$, $s \leqslant t$
(12) $CP_{=s}(\alpha,\beta) \to \neg CP_{\geqslant t}(\alpha,\beta)$, $s < t$
(13) $CP_{=s}(\alpha,\beta) \to CP_{\geqslant t}(\alpha,\beta)$, $s \geqslant t$
(14) $CP_{=s}(\alpha,\beta) \to (P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=t}\beta)$, $t \neq 0$
(15) $CP_{\approx r}(\alpha,\beta) \to CP_{\geqslant r_1}(\alpha,\beta)$, for every rational $r_1 \in [0, r)$
(16) $CP_{\approx r}(\alpha,\beta) \to CP_{\leqslant r_1}(\alpha,\beta)$, for every rational $r_1 \in (r, 1]$
(17) $CP_{=r}(\alpha,\beta) \to CP_{\approx r}(\alpha,\beta)$

and inference rules:

(1) From α and $\alpha \to \beta$ infer β.
(2) From α infer $P_{\geqslant 1}\alpha$.
(3) From $\beta \to P_{\neq s}\alpha$, for every $s \in S$, infer $\beta \to \bot$.
(4) From $\gamma \to (P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta)$, for every $s \in S \setminus \{0\}$, infer $\gamma \to CP_{=t}(\alpha,\beta)$.
(5) For every $r \in Q[0,1]$, from $\gamma \to P_{\geqslant r-1/n}\alpha$, for every integer $n \geqslant 1/r$, and $\gamma \to P_{\leqslant r+1/n}\alpha$, for every integer $n \geqslant 1/(1-r)$, infer $\gamma \to P_{\approx r}\alpha$.
(6) For every $r \in Q[0,1]$, from $\gamma \to CP_{\geqslant r-1/n}(\alpha,\beta)$, for every integer $n \geqslant 1/r$, and $\gamma \to CP_{\leqslant r+1/n}(\alpha,\beta)$, for every integer $n \geqslant 1/(1-r)$, infer $\gamma \to CP_{\approx r}(\alpha,\beta)$.

We denote this axiomatic system by Ax_{LICP^S}.

The axioms 2– 6 and Rule 2 concern the "crisp" probabilities. They are given in [**18, 19**]. For example, Axiom 2 says that every formula is satisfied in a set of worlds of the probability at least 0. By substituting $\neg\alpha$ for α in Axiom 2, the formula $P_{\leqslant 1}\alpha$ $(= P_{\geqslant 0}\neg\alpha)$ is obtained. This formula means that every formula is satisfied in a set of worlds of the probability at most 1. Let us denote it by 2'. Axiom 5 means that the equivalent formulas must have the same probability. Axiom 6 corresponds to the property of the finite additivity of probability. Rule 2 may be considered as the analogue of the rule of necessitation in modal logic. From Axiom 2' and Rule 2 we obtain another inference rule 2': from α infer $P_{=1}\alpha$. Rule 3 guarantees that the probability of a formula belongs to the set S, enabling us thus to syntactically specify the range of the probability functions that will appear in the semantics.

The axioms 9–14 and Rule 4 concern properties of the "crisp" conditional probabilities and the relationship between the probability and conditional probability.

The axioms 7–8 and 15–17, and the rules 5 and 6 syntactically determine imprecise probabilities. For example, Rule 5 says that if the probability of α belongs to the interval $[r-1/n, r+1/k]$ for all integers integers n and k such that $r-1/n \geqslant 0$, $r + 1/k \leqslant 1$, then the probability of α is infinitesimally close to r.

Note that the rules 3–6 are infinitary.

A formula α is deducible from a set T of formulas (denoted $T \vdash_{Ax_{LICP^S}} \alpha$) if there is an at most denumerable sequence (called proof) of formulas $\alpha_0, \alpha_1, \ldots, \alpha$, such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Rule 2 can be applied to the theorems only. A formula α is a theorem ($\vdash \alpha$) if it is deducible from the empty set. $T \nvdash_{Ax_{LICP^S}} \alpha$ means that $T \vdash_{Ax_{LICP^S}} \alpha$ does not hold. A set T of formulas is consistent if there is at least one formula α such that $T \nvdash_{Ax_{LICP^S}} \alpha$. A consistent set T of formulas is said to be maximal consistent if for every formula α either $\alpha \in T$ or $\neg\alpha \in T$. A set T is deductively closed if for every formula α, if $T \vdash \alpha$, then $\alpha \in T$.

The limitation of application of Rule 2 in deductions allows us to obtain Deduction theorem (Theorem 5.1) which is one of the main steps in our approach to proving completness of Ax_{LICP^S}. Also, note that, in the presence of Deduction theorem, unrestricted applications of Rule 2 would produce undesirable consequences. For example, consider the following deduction:

$\alpha \vdash \alpha$
$\alpha \vdash P_{\geqslant 1}\alpha$, by Rule 2
$\vdash \alpha \to P_{\geqslant 1}\alpha$ by Deduction theorem,

but the last formula $\alpha \to P_{\geqslant 1}\alpha$ certainly is not a valid formula of our logic.

5. Soundness and completeness

Soundness of the system follows from the soundness of the classical propositional logic, as well as from the properties of probabilistic measures, so the proof is straightforward and we omit it here. For the detailed arguments see the proof of Theorem 13 in [**15**].

In order to prove the completeness theorem for our logic, we show that every consistent set of sentences is satisfiable. We begin with the deduction theorem and some auxiliary statements. Then, we describe how a consistent set T of formulas can be extended to a maximal consistent set, and how a canonical model can be constructed out of maximal consistent sets. Finally, we prove that for every world w from the canonical model and every formula α, $w \models \alpha$ if and only if $\alpha \in w$, and as a consequence we obtain that the set T is satisfiable.

THEOREM 5.1 (Deduction theorem). *If T is a set of formulas and $T \cup \{\alpha\} \vdash \beta$, then $T \vdash \alpha \to \beta$.*

PROOF. We use the transfinite induction on the length of the inference. Let us first consider the case where $\beta = P_{\geqslant 1}\gamma$ is obtained from $T \cup \{\alpha\}$ by an application of Rule 2. In that case γ and β must be theorems. Then, from $\vdash \beta \to (\alpha \to \beta)$, we obtain $T \vdash \alpha \to \beta$. Next, suppose that $\beta = \gamma \to CP_{=t}(\varphi, \psi)$ is obtained from $T \cup \{\alpha\}$ by an application of Rule 4. Then:

$T, \alpha \vdash \gamma \to (P_{=ts}(\varphi \wedge \psi) \leftrightarrow P_{=s}\psi)$, for every $s \in S \smallsetminus \{0\}$
$T \vdash \alpha \to (\gamma \to (P_{=ts}(\varphi \wedge \psi) \leftrightarrow P_{=s}\psi))$, for every $s \in S \smallsetminus \{0\}$, by the induction hypothesis
$T \vdash (\alpha \wedge \gamma) \to (P_{=ts}(\varphi \wedge \psi) \leftrightarrow P_{=s}\psi))$, for every $s \in S \smallsetminus \{0\}$
$T \vdash (\alpha \wedge \gamma) \to CP_{=t}(\varphi, \psi)$, by Rule 4
$T \vdash \alpha \to \beta$.

The other cases follow similarly. □

THEOREM 5.2. *For every formula α, the following hold:*

(1) $\vdash P_{\geqslant t}\alpha \to P_{\geqslant s}\alpha$, $t > s$
(2) $\vdash P_{\leqslant t}\alpha \to P_{\leqslant s}\alpha$, $t < s$
(3) $\vdash P_{=t}\alpha \to \neg P_{=s}\alpha$, $t \neq s$
(4) $\vdash P_{=t}\alpha \to \neg P_{\geqslant s}\alpha$, $t < s$
(5) $\vdash P_{=t}\alpha \to \neg P_{\leqslant s}\alpha$, $t > s$
(6) $\vdash P_{=r}\alpha \to P_{\approx r}\alpha$, $r \in Q[0,1]$.

PROOF. (1–2) Let us call the property expressed by these two formulas the monotonicity of the probability. The formulas follow from the axioms 3 and 4.

(3) Note that $P_{=t}\alpha$ denotes $P_{\geqslant t}\alpha \wedge P_{\leqslant t}\alpha$, while $\neg P_{\geqslant s}\alpha = P_{<s}\alpha$. From Axiom 3, we have $\vdash P_{\leqslant t}\alpha \to \neg P_{\geqslant s}\alpha$ for every $s > t$. Similarly, by the contraposition, from Axiom 3, we obtain $\vdash P_{\geqslant t}\alpha \to \neg P_{\leqslant s}\alpha$, for every $s < t$. It follows that $\vdash (P_{\leqslant t}\alpha \wedge P_{\geqslant t}\alpha) \to (\neg P_{\leqslant s}\alpha \vee \neg P_{\geqslant s}\alpha)$, and that $\vdash P_{=t}\alpha \to \neg P_{=s}\alpha$ for every $s \neq t$.

(4–5) Similarly as the above item (3) of this Theorem.

(6) This statement give us an example of an application of an infinitary inference rule. Namely, by the above items (4–5) of this Theorem, we have $\vdash P_{=r}\alpha \to P_{\geqslant r-1/n}\alpha$ for every $n \geqslant 1/r$, and $\vdash P_{=r}\alpha \to P_{\leqslant r+1/n}\alpha$ for every $n \geqslant 1/(1-r)$. Then, by Rule 5, it follows $\vdash P_{=r}\alpha \to P_{\approx r}\alpha$. □

THEOREM 5.3. *Every consistent set can be extended to a maximal consistent set.*

PROOF. Let T be a consistent set, and α_0, $\alpha_1, \dots$ an enumeration of all formulas. We define a sequence of sets T_i, $i = 0, 1, 2, \dots$ such that:

(1) $T_0 = T$
(2) for every $i \geqslant 0$,
   (a) if $T_{2i} \cup \{\alpha_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{\alpha_i\}$;
   (b) otherwise, if α_i is of the form $\gamma \to CP_{=t}(\alpha, \beta)$, then $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i, \gamma \to \neg(P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta)\}$, for some $s > 0$ so that T_{2i+1} is consistent;
   (c) otherwise, if α_i is of the form $\gamma \to P_{\approx r}\alpha$, then $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i, \gamma \to \neg P_{\geqslant r-1/n}\alpha\}$, or $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i, \gamma \to \neg P_{\leqslant r+1/n}\alpha\}$, for some integer n so that T_{2i+1} is consistent;
   (d) otherwise, if α_i is of the form $\gamma \to CP_{\approx r}(\alpha, \beta)$, then
   $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i, \gamma \to \neg CP_{\geqslant r-1/n}(\alpha, \beta)\}$, or
   $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i, \gamma \to \neg CP_{\leqslant r+1/n}(\alpha, \beta)\}$, for some integer n so that T_{2i+1} is consistent;
   (e) otherwise, $T_{2i+1} = T_{2i} \cup \{\neg\alpha_i\}$,
(3) for every $i \geqslant 0$, $T_{2i+2} = T_{2i+1} \cup \{P_{=s}\alpha_i\}$, for some $s \in S$, so that T_{2i+2} is consistent,

The sets obtained by the steps 1 and 2a are obviously consistent. The step 2e produces consistent sets, too. For if $T_{2i}, \alpha_i \vdash \bot$, by Deduction theorem we have $T_{2i} \vdash \neg\alpha_i$, and since T_{2i} is consistent, so it is $T_{2i} \cup \{\neg\alpha_i\}$. The same holds for the steps 2b– 2d. Let us first consider the step 2b. Suppose that α_i is of the form $\gamma \to CP_{=t}(\alpha, \beta)$, and that neither $T_{2i} \cup \{\gamma \to CP_{=t}(\alpha, \beta)\}$ nor $T_{2i} \cup \{\neg(\gamma \to CP_{=t}(\alpha, \beta)), \gamma \to \neg(P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta)\}$, for every $s > 0$, are consistent. It means that:

(1) $T_{2i}, \neg(\gamma \to CP_{=t}(\alpha, \beta)), \gamma \to \neg(P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta) \vdash \bot$, for every $s > 0$,
(2) $T_{2i}, \neg(\gamma \to CP_{=t}(\alpha, \beta)) \vdash (\gamma \to \neg(P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta)) \to \bot$, for every $s > 0$, by Deduction theorem,
(3) $T_{2i}, \neg(\gamma \to CP_{=t}(\alpha, \beta)) \vdash \neg(\gamma \to \neg(P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta))$, for every $s > 0$,
(4) $T_{2i}, \neg(\gamma \to CP_{=t}(\alpha, \beta)) \vdash \gamma \to (P_{=ts}(\alpha \wedge \beta) \leftrightarrow P_{=s}\beta))$, for every $s > 0$, by the classical tautology $\neg(\alpha \to \beta) \to (\alpha \to \neg\beta)$,
(5) $T_{2i}, \neg(\gamma \to CP_{=t}(\alpha, \beta)) \vdash \gamma \to CP_{=t}(\alpha, \beta)$, by Rule 4,
(6) $T_{2i} \vdash \neg(\gamma \to CP_{=t}(\alpha, \beta)) \to (\gamma \to CP_{=t}(\alpha, \beta))$, by Deduction theorem,
(7) $T_{2i} \vdash \gamma \to CP_{=t}(\alpha, \beta)$, by classical reasoning

which contradicts consistency of T_{2i} since $T_{2i} \cup \{\gamma \to CP_{=t}(\alpha, \beta)\}$ is not consistent. Next, consider the step 2c, suppose that α_i is of the form $\gamma \to P_{\approx r}\alpha$, and that $T_{2i} \cup \{\gamma \to P_{\approx r}\alpha\}$ is not consistent. Also, let all the sets $T_{2i} \cup \{\neg(\gamma \to P_{\approx r}\alpha), \gamma \to \neg P_{\geqslant r-1/n}\alpha\}$, for every integer $n : r - 1/n \geqslant 0$, and $T_{2i} \cup \{\neg(\gamma \to P_{\approx r}\alpha), \gamma \to \neg P_{\leqslant r+1/n}\alpha\}$ for every integer $n : r + 1/n \leqslant 1$ be inconsistent Then, we have:

(1) $T_{2i}, \neg(\gamma \to P_{\approx r}\alpha), \gamma \to \neg P_{\geqslant r-1/n}\alpha \vdash \bot$, for every integer n,
(2) $T_{2i}, \neg(\gamma \to P_{\approx r}\alpha), \gamma \to \neg P_{\leqslant r+1/n}\alpha \vdash \bot$, for every integer n,
(3) $T_{2i}, \neg(\gamma \to P_{\approx r}\alpha) \vdash \gamma \to P_{\geqslant r-1/n}\alpha$, for every integer n,
(4) $T_{2i}, \neg(\gamma \to P_{\approx r}\alpha) \vdash \gamma \to P_{\leqslant r+1/n}\alpha$, for every integer n,
(5) $T_{2i}, \neg(\gamma \to P_{\approx r}\alpha) \vdash \gamma \to P_{\approx r}\alpha$, by Rule 5

which, similarly as above, contradicts consistency of T_{2i}. The step 2d and the case when $\alpha_i = \gamma \to CP_{\approx r}(\alpha, \beta)$ follows in the same way using Rule 6.

Finally, consider the step 3 of the construction, and suppose that for every $s \in S$, $T_{2i+1} \cup \{P_{=s}\alpha_i\}$ is not consistent. Let $T_{2i+1} = T_0 \cup T^+_{2i+1}$, where T^+_{2i+1} denotes the set of all formulas β that are added to T_0 in the previous steps of the construction. Then:

(1) $T_0, T^+_{2i+1}, P_{=s}\alpha_i \vdash \bot$, for every $s \in S$, by the hypothesis
(2) $T_0, T^+_{2i+1} \vdash \neg P_{=s}\alpha_i$, for every $s \in S$, by Deduction theorem
(3) $T_0 \vdash (\bigwedge_{\beta \in T^+_{2i+1}} \beta) \to \neg P_{=s}\alpha_i$, for every $s \in S$, by Deduction theorem
(4) $T_0 \vdash (\bigwedge_{\beta \in T^+_{2i+1}} \beta) \to \bot$, by Rule 3
(5) $T_{2i+1} \vdash \bot$

which contradicts consistency of T_{2i+1}.

Let $T^* = \cup_i T_i$. We have to prove that T^* is a maximal consistent set.

First, we can show that if $P_{=s}\alpha \in T^*$, then for every formula β, $\beta \to P_{=s}\alpha \in T^*$. Suppose that it is not the case. Then, according to the above construction, for some β, and some j, $P_{=s}\alpha$ and $\neg(\beta \to P_{=s}\alpha)$ (i.e., $\beta \wedge \neg P_{=s}\alpha$) belong to T_j. It means that $T_j \vdash P_{=s}\alpha \wedge \neg P_{=s}\alpha$, a contradiction.

The steps 2a–2e guarantee that for every formula α, α or $\neg\alpha$ belongs to T^*, i.e., that T^* is maximal. On the other hand, there is no formula α such that $\alpha, \neg\alpha \in T^*$. To prove that, suppose that $\alpha = \alpha_i$ and $\neg\alpha = \alpha_j$ for some i and j. If $\alpha, \neg\alpha \in T^*$, then also $\alpha, \neg\alpha \in T_{\max(2i,2j)+1}$, a contradiction with the consistency of $T_{\max(2i,2j)+1}$.

We continue by showing that T^* is a deductively closed set, and since it does not contain all formulas, it follows that T^* is consistent.

First, note that if for some i, $T_i \vdash \alpha$, it must be $\alpha \in T^*$. Otherwise, if $\neg\alpha \in T^*$, there must be some k such that $T_k \vdash \alpha$ and $T_k \vdash \neg\alpha$ which contradicts the consistency of T_k.

Now, suppose that the sequence γ_1, γ_2,..., α forms a proof of α from T^*. If the sequence is finite, there must be a set T_i such that $T_i \vdash \alpha$, and $\alpha \in T^*$. Thus, suppose that the sequence is countably infinite. We can show that for every i, if γ_i is obtained by an application of an inference rule, and all the premises belong to T^*, then it must be $\gamma_i \in T^*$. If the rule is a finitary one, then there must be a set T_j which contains all the premises and $T_j \vdash \gamma_i$. Reasoning as above, we conclude $\gamma_i \in T^*$.

So, let us consider the infinitary rules.

Let $\gamma_i = \beta \to \bot$ be obtained from the set of premises $\{\gamma_i^k = \beta \to \neg P_{=s_k}\varphi : s_k \in S\}$ by Rule 3. Suppose that $\gamma_i \notin T^*$. By the induction hypothesis, $\gamma_i^k \in T^*$ for every k. The step 3 of the construction guaranties that there are some l and $s_l \in S$

such that $P_{=s_l}\varphi \in T_l$. Reasoning as above, we conclude that $\beta \to P_{=s_l}\gamma \in T^*$. Thus, there must be some j such that both $\beta \to \neg P_{=s_l}\gamma$, and $\beta \to P_{=s_l}\gamma$ belongs to T_j. It follows that $T_j \vdash \beta \to \neg P_{=s_l}\gamma$, $T_j \vdash \beta \to P_{=s_l}\gamma$, and $T_j \vdash \beta \to \bot$, which means that $\beta \to \bot \in T^*$, i.e., $\gamma_i \in T^*$, a contradiction.

Let $\gamma_i = \beta \to CP_{=t}(\varphi, \psi)$ be obtained from the set of premises $\{\gamma_i^k = \beta \to (P_{=ts}(\varphi \wedge \psi) \leftrightarrow P_{=s}\psi) : s \in S \setminus \{0\}\}$ by rule Rule 4. Suppose that $\gamma_i \notin T^*$. By the step 2b of the construction, there are some $s' > 0$ and l such that $\beta \to \neg(P_{=ts'}(\varphi \wedge \psi) \leftrightarrow P_{=s'}\psi) \in T_l$. It follows that there must be some j such that both $\beta \to \neg(P_{=ts'}(\varphi \wedge \psi) \leftrightarrow P_{=s'}\psi)$ and $\beta \to (P_{=ts'}(\varphi \wedge \psi) \leftrightarrow P_{=s'}\psi)$ belongs to T_j. Then, $T_j \vdash \beta \to \bot$ and $T_j \vdash \beta \to CP_{=t}(\varphi, \psi)$. Thus, $\beta \to CP_{=t}(\varphi, \psi) \in T^*$, a contradiction.

Let $\gamma_i = \beta \to P_{\approx r}\varphi$ be obtained by Rule 5 from the set of premises of the form $\beta \to P_{\geqslant r-1/n}\varphi$, for every integer $n : r - 1/n \geqslant 0$ and $\beta \to P_{\leqslant r+1/n}\varphi$, for every integer $n : r + 1/n \leqslant 1$. Suppose that $\gamma_i \notin T^*$. By the step 2c of the construction, there are some n and j such that $\beta \to \neg P_{\geqslant r-1/n}\varphi$ or $\beta \to \neg P_{\leqslant r+1/n}\varphi$ belongs to T_j. Let us suppose the former case, while the latter one will follow similarly. It means that there is some l such that $\beta \to P_{\geqslant r-1/n}\varphi, \beta \to \neg P_{\geqslant r-1/n}\varphi \in T_l$. Then, $T_l \vdash \beta \to \bot$, and $T_l \vdash \beta \to P_{\approx r}\varphi$. It follows that $\gamma_i \in T^*$, a contradiction.

Finally, the case concerning the formulas of the form $\gamma_i = \beta \to CP_{\approx r}(\varphi, \psi)$ and Rule 6 can be proved in the same way.

Hence, from $T^* \vdash \alpha$, we have $\alpha \in T^*$. It follows that T^* is a maximal consistent set. $\square$

Being a maximal consistent set, T^* has all the expected properties summarized in the next statement.

THEOREM 5.4. *Let T^* be defined as above. Then, the following hold:*

(1) *T^* contains all theorems.*
(2) *If $\alpha \in T^*$, then $\neg\alpha \notin T^*$.*
(3) *$\alpha \wedge \beta \in T^*$ iff $\alpha \in T^*$ and $\beta \in T^*$.*
(4) *If $\alpha, \alpha \to \beta \in T^*$, then $\beta \in T^*$.*
(5) *There is exactly one $s \in S$ such that $P_{=s}\alpha \in T^*$.*
(6) *If $P_{\geqslant s}\alpha \in T^*$, there is some $t \in S$ such that $t \geqslant s$ and $P_{=t}\alpha \in T^*$.*
(7) *If $P_{\leqslant s}\alpha \in T^*$, there is some $t \in S$ such that $t \leqslant s$ and $P_{=t}\alpha \in T^*$.*
(8) *There is exactly one $s \in S$ such that $CP_{=s}(\alpha, \beta) \in T^*$.*
(9) *If $CP_{\geqslant s}(\alpha, \beta) \in T^*$, there is some $t \in S$ such that $t \geqslant s$ and $CP_{=t}(\alpha, \beta) \in T^*$.*
(10) *If $CP_{\approx r_1}(\alpha, \beta) \in T^*$ and $r_2 \in Q[0,1] \setminus \{r_1\}$, then $CP_{\approx r_2}(\alpha, \beta) \notin T^*$*

PROOF. (1– 4) The proof is standard and left to the reader.

(5) First, note that, according to Theorem 5.2. 3, if $P_{=s}\alpha \in T^*$, then for every $t \neq s$, $P_{=t}\alpha \notin T^*$. On the other hand, suppose that for every $s \in S$, $\neg P_{=s}\alpha \in T^*$. It follows that $T^* \vdash \neg P_{=s}\alpha$ for every $s \in S$, and by Rule 3, $T^* \vdash \bot$ which contradicts consistency of T^*. Thus, for every α, there is exactly one $s \in S$ such that $P_{=s}\alpha \in T^*$.

(6) Since $P_{\geqslant s}\alpha \in T^*$, we have that $\neg P_{<s}\alpha \in T^*$. By the above item (5) of this Theorem, for every α there is some $t \in S$ such that $P_{=t}\alpha \in T^*$. It means that

$P_{\geqslant t}\alpha \in T^*$, and $P_{\leqslant t}\alpha \in T^*$. If $t < s$, then by Axiom 3 from $P_{\leqslant t}\alpha \in T^*$ it follows that $P_{<s}\alpha \in T^*$ (i.e., $\neg P_{\geqslant s}\alpha \in T^*$), a contradiction. Thus, it must be $t \geqslant s$.

(7) Similarly as the above item (6) of this Theorem.

(8) According to Axiom 9 there cannot be two different $t, s \in S$ such that $CP_{=t}(\alpha,\beta) \in T^*$ and $CP_{=s}(\alpha,\beta) \in T^*$. From the above item (6) of this Theorem, we have that for exactly one t' and exactly one t'', $P_{=t'}\beta \in T^*$ and $P_{=t''}(\alpha\wedge\beta) \in T^*$. If $t' = 0$, then $CP_{=1}(\alpha,\beta) \in T^*$, by Axiom 10. Let $t' \neq 0$, and $s = \frac{t''}{t'}$. Using Axiom 11 we have that $CP_{=s}(\alpha,\beta) \in T^*$. Thus, for all α and β, there is exactly one $s \in S$ such that $CP_{=s}(\alpha,\beta) \in T^*$.

(9) Let $CP_{\geqslant s}\alpha \in T^*$. From the above item (8) of this Theorem, there is exactly one $t \in S$ such that $CP_{=t}(\alpha,\beta) \in T^*$. It follows from Axiom 12 that t cannot be less than s. Thus, it must be $t \geqslant s$.

(10) Let $CP_{\approx r_1}(\alpha,\beta) \in T^*$ and $r_2 \in Q[0,1] \setminus \{r_1\}$. Suppose that $r_2 < r_1$. If $CP_{\approx r_2}(\alpha,\beta) \in T^*$, from the axioms 15 and 16, for every $r_3 \in (r_2, r_1) \cap Q[0,1]$ it must be $CP_{\geqslant r_3}(\alpha,\beta) \in T^*$ and $CP_{\leqslant r_3}(\alpha,\beta) \in T^*$, which implies $CP_{=r_3}(\alpha,\beta) \in T^*$, a contradiction to the above item (9) of this Theorem. The same conclusion follows from the assumption that $r_2 > r_1$. □

Let the tuple $M = \langle W, \mathrm{Prob}, v\rangle$ be defined as follows:

- W is the set of all maximal consistent sets of formulas,
- $[\alpha] = \{w \in W : \alpha \in w\}$,
- for every world $w \in W$, $\mathrm{Prob}(w)$ is defined as follows:
  - $W(w) = W$,
  - $H(w)$ is a class of all sets of the form $[\alpha] = \{w \in W : \alpha \in w\}$ and
  - $\mu(w)([\alpha]) = s$ iff $P_{=s}\alpha \in w$ and
- for every propositional letter $p \in \mathrm{Var}$, $v(w)(p) = \mathit{true}$ iff $p \in w$.

The next theorem states that M is an $LICP^S_{\mathrm{Meas}}$-model.

THEOREM 5.5. *Let $M = \langle W, \mathrm{Prob}, v\rangle$ be defined as above. Then, the following hold for every $w \in W$:*

(1) *$\mu(w)$ is a well-defined function.*
(2) *$\{[\alpha]\}$ is an algebra of subsets of $W(w)$.*
(3) *$\mu(w) : \{[\alpha]\} \to S$ is a finitely additive probability measure.*

PROOF. (1) It follows from Theorem 5.4. 5 that for every α there is exactly one $s \in S$ such that $P_{=s}\alpha \in w$, i.e., that $\mu(w)([\alpha]) = s$. On the other hand, let $[\alpha] = [\beta]$ for some α and β. It means that for every $w \in W$, $\alpha \leftrightarrow \beta \in w$, that $\neg(\alpha \leftrightarrow \beta)$ is inconsistent, and that $\vdash \alpha \leftrightarrow \beta$ and $\vdash P_{=1}(\alpha \leftrightarrow \beta)$. Axiom 5 guarantees that $P_{=s}\alpha \in w$ iff $P_{=s}\beta \in w$. Thus, $[\alpha] = [\beta]$ implies that $\mu(w)([\alpha]) = \mu(w)([\beta])$.

(2) For an arbitrary β, $W = [\beta \vee \neg\beta]$, and $W \in H(w)$. If $[\beta] \in H(w)$, then the complement of $[\beta]$ is $[\neg\beta]$, and it belongs to $H(w)$ as well. If $[\beta_1], \ldots, [\beta_k] \in H(w)$, then also the union $[\beta_1] \cup \ldots \cup [\beta_k] \in H(w)$ because $[\beta_1] \cup \ldots \cup [\beta_k] = [\beta_1 \vee \ldots \vee \beta_k]$.

(3) It follows from the step 3 of the construction in Theorem 5.3 that $\mu(w) : H(w) \to S$. Since for an arbitrary formula α, $W = [\alpha \vee \neg\alpha]$, $\vdash \alpha \vee \neg\alpha$ and by

the Rule 2' we have $\vdash P_{=1}(\alpha \vee \neg\alpha)$, it follows that $P_{=1}(\alpha \vee \neg\alpha) \in w$. It means that $\mu(w)(W) = 1$. Let $[\alpha] \cap [\beta] = \emptyset$, $\mu(w)([\alpha]) = s$ and $\mu(w)([\beta]) = t$. Since, $[\alpha] \cap [\beta] = \emptyset$, we have that $[\neg(\alpha \wedge \beta)] = W$, and $\mu(w)([\neg(\alpha \wedge \beta)]) = 1$. From the assumptions we have that $P_{=s}\alpha$, $P_{=t}\beta$, $P_{\geqslant 1}\neg(\alpha \wedge \beta) \in w$. Using Axiom 6 it follows that $P_{=s+t}(\alpha \vee \beta) \in w$, i.e., $\mu(w)([\alpha \vee \beta]) = s + t$. □

Now we can prove the extended completeness theorem for the class $LICP^S_{\mathrm{Meas}}$:

Theorem 5.6 (Extended completeness theorem). *A set T of formulas is consistent if and only if T has an $LICP^S_{\mathrm{Meas}}$-model.*

Proof. The ($\Leftarrow$)-direction follows from the soundness of the axiomatic system Ax_{LICP^S}. In order to prove the ($\Rightarrow$)-direction we construct the canonical $LICP^S_{\mathrm{Meas}}$-model $M = \langle W, \mathrm{Prob}, v\rangle$ as above, and show that for all α and $w \in W$, $w \models \alpha$ iff $\alpha \in w$.

Let α be a propositional letter from Var. By the definition of the valuation $v(w)$, $w \models \alpha$ iff $\alpha \in w$. The classical cases $\alpha = \neg\beta$ and $\alpha = \beta \wedge \gamma$ follow by the standard arguments.

Let $\alpha = P_{\geqslant s}\beta$. If $P_{\geqslant s}\beta \in w$, then, by Theorem 5.4. 6, there is some $t \geqslant s$ such that $P_{=t}\beta \in w$, i.e., such that $\mu(w)([\beta]) = t \geqslant s$. Thus, $w \models P_{\geqslant s}\beta$. On the other hand suppose that $w \models P_{\geqslant s}\beta$, i.e., that $\mu(w)([\beta]) = t \geqslant s$, and $P_{=t}\beta \in w$. It means that $P_{\geqslant t}\beta \in w$, and by Theorem 5.2. 1 it follows that $P_{\geqslant s}\beta \in w$.

Let $\alpha = P_{\approx r}\beta$. If $P_{\approx r}\beta \in w$, by the axioms 7 and 8, for every rational $r' < r$, $P_{\geqslant r'}\beta \in w$, and for every rational $r' > r$, $P_{\leqslant r'}\beta \in w$. Suppose that $w \not\models P_{\approx r}\beta$. It means that there is some $t \in S$ such that $t \neq r$ and $\mu(w)([\beta]) = t$, i.e., $P_{=t}\beta \in w$. Also, there must be an interval $[r_1, r_2]$ containing r such that r_1 and r_2 are rational and $t \notin [r_1, r_2]$, otherwise $w \models P_{\approx r}\beta$. Now, if $t < r$, then according to Theorem 5.2. 4 $\neg P_{\geqslant r_1}\beta \in w$, a contradiction. The similar conclusion follows from the assumption that $t > r$. Thus, it must be $w \models P_{\approx r}\beta$. For the other direction, suppose that $w \models P_{\approx r}\beta$. It follows that $w \models P_{\geqslant r'}\beta$ for every rational $r' < r$ and $w \models P_{\leqslant r''}\beta$ for every rational $r'' > r$. However, if $P_{\approx r}\beta \notin w$, by the step 2c of the construction in Theorem 5.3, there must be some rational $r_1 < r$ (or some rational $r_2 > r$) such that $\neg P_{\geqslant r_1}\beta \in w$ (or $\neg P_{\leqslant r_2}\beta \in w$) which contradicts consistency of w. Thus, $P_{\approx r}\beta \in w$.

Let $\alpha = CP_{=s}(\beta, \gamma)$. Suppose that $CP_{=s}(\beta, \gamma) \in w$. If $P_{=0}\gamma \in w$, from Axiom 10 and Theorem 5.4. 8 it must be $CP_{=1}(\beta, \gamma) \in w$, and $s = 1$. Since $\mu(w)([\gamma]) = 0$, we have $w \models CP_{=1}(\beta, \gamma)$. Otherwise, let $P_{=t}\gamma \in w$, $t \neq 0$. It follows from Axiom 14 that $P_{=st}(\beta \wedge \gamma) \in w$. Since $\mu(w)([\gamma]) = t \neq 0$ and $\mu(w)([\beta \wedge \gamma]) = st$, we have $w \models CP_{=s}(\beta, \gamma)$. For the other direction, suppose that $w \models CP_{=s}(\beta, \gamma)$. If $w \models P_{=0}\gamma$, then $s = 1$ and $P_{=0}\gamma \in w$. From Axiom 10 we have that $CP_{=1}(\beta, \gamma) \in w$. Otherwise, let $w \models P_{=t}\gamma$, $t \neq 0$, and $CP_{=s}(\beta, \gamma) \notin w$. Then, by the step 2b of the construction in Theorem 5.3, there is some $u > 0$ such that $\neg(P_{=su}(\beta \wedge \gamma) \leftrightarrow P_{=u}\gamma) \in w$. It means that $(P_{=su}(\beta \wedge \gamma) \wedge \neg P_{=u}\gamma) \in w$ or $(\neg P_{=su}(\beta \wedge \gamma) \wedge P_{=u}\gamma) \in w$, i.e., $P_{=su}(\beta \wedge \gamma) \in w$ and $\neg P_{=u}\gamma \in w$ or $\neg P_{=su}(\beta \wedge \gamma) \in w$ and $P_{=u}\gamma \in w$. In the former case $\mu(w)([\beta \wedge \gamma]) = su$ and $\mu(w)([\gamma]) = t$, $t \notin \{0, u\}$. It follows that $\frac{\mu(w)([\beta \wedge \gamma])}{\mu(w)([\gamma])} \neq s$, a contradiction since $w \models CP_{=s}(\beta, \gamma)$. In the latter case, we have that $u = t$, $\mu(w)([\beta \wedge \gamma]) \neq st$ and $\mu(w)([\beta]) = t$, $t \neq 0$, which implies a contradiction in the same way as above. Thus, $CP_{=s}(\beta, \gamma) \in w$.

Let $\alpha = CP_{\geqslant s}(\beta, \gamma)$. Suppose that $CP_{\geqslant s}(\beta, \gamma) \in w$. From Theorem 5.4. 8 and Theorem 5.4. 9, there is exactly one $t \geqslant s$ such that $CP_{=t}(\beta, \gamma) \in w$, and $w \models CP_{=t}(\beta, \gamma)$. It follows that $w \models CP_{\geqslant s}(\beta, \gamma)$. For the other direction, suppose that $w \models CP_{\geqslant s}(\beta, \gamma)$. If $\mu(w)([\gamma]) = 0$, we have that $M \models CP_{=1}(\beta, \gamma)$, and since $P_{=0}\gamma \in w$, by Axiom 10 it follows that $CP_{=1}(\beta, \gamma) \in w$. Then, by Axiom 13 we have that $CP_{\geqslant s}(\beta, \gamma) \in w$. Otherwise, let $\mu(w)([\gamma]) \neq 0$. From Theorem 5.4. 5 there is exactly one u' such that $P_{=u'}(\beta \wedge \gamma) \in w$, and exactly one $u'' \neq 0$ such that $P_{=u''}\gamma \in w$. It means that $\mu(w)([\beta \wedge \gamma]) = u'$ and $\mu(w)([\gamma]) = u''$, $w \models CP_{=u'/u''}(\beta, \gamma)$, and $CP_{=u'/u''}(\beta, \gamma) \in w$. Since $w \models CP_{\geqslant s}(\beta, \gamma)$, Axiom 12 guarantees that it must be $s \leqslant \frac{u'}{u''}$. It follows from Axiom 13 that $CP_{\geqslant s}(\beta, \gamma) \in w$.

Finally, let $\alpha = CP_{\approx r}(\beta, \gamma)$. Suppose that $CP_{\approx r}(\beta, \gamma) \in w$. If $\mu(w)([\gamma]) = 0$, it follows that $w \models CP_{\approx 1}(\beta, \gamma)$, and by the axioms 10 and 17 that $CP_{\approx 1}(\beta, \gamma) \in w$. By Theorem 5.4. 10 we have that $r = 1$ and $w \models CP_{\approx r}(\beta, \gamma)$. Next, suppose that $\mu(w)([\gamma]) \neq 0$. From the axioms 15 and 16 we have that for every rational $r_1 \in [0, r)$, $CP_{\geqslant r_1}(\beta, \gamma) \in w$, and for every rational $r_2 \in (r, 1]$, $CP_{\leqslant r_2}(\beta, \gamma) \in w$. Suppose that $w \not\models CP_{\approx r}(\beta, \gamma)$. It means that there is some $t \in S$ such that $t \neq r$ and $\mu(w)([\gamma]) > 0$, $\frac{\mu(w)([\beta \wedge \gamma])}{\mu(w)([\gamma])} = t$ and $CP_{=t}(\beta, \gamma) \in w$. Furthermore, there must be an interval $[r_1, r_2]$ containing r such that r_1 and r_2 are rational and $t \notin [r_1, r_2]$. If it is not the case, $w \models CP_{\approx r}(\beta, \gamma)$. Suppose that $t < r$. According to Axiom 12, $\neg CP_{\geqslant r_1}(\beta, \gamma) \in w$, a contradiction. The similar conclusion follows from the assumption that $t > r$. It follows $w \models CP_{\approx r}(\beta, \gamma)$. For the other direction, suppose that $w \models CP_{\approx r}(\beta, \gamma)$. Then, $w \models CP_{\geqslant r'}(\beta, \gamma)$ for every rational $r' < r$ and $w \models CP_{\leqslant r''}(\beta, \gamma)$ for every rational $r'' > r$. However, if $CP_{\approx r}(\beta, \gamma) \notin w$, by the step 2d of the construction in Theorem 5.3, there must be some rational $r_1 < r$ (or $r_2 > r$) such that $\neg CP_{\geqslant r_1}(\beta, \gamma) \in w$ (or $\neg CP_{\leqslant r_2}(\beta, \gamma) \in w$) which contradicts consistency of w. Thus, $CP_{\approx r}(\beta, \gamma) \in w$. $\square$

6. Conclusion

In this paper we consider a language, a class of probabilistic models and a sound and complete axiomatic system for reasoning about conditional probabilities.

We are aware of only one paper [**6**] in which conditional probability is defined syntactically. The approach taken there includes among the axioms the complicated machinery of real closed fields. It is needed to obtain the sound and complete axiomatization. In our approach, since the parts of field theory are moved to the meta theory, the axioms are rather simple. Also, we are able to prove the extended completeness theorem ('every consistent set of formulas has a model') which is impossible for the system in [**6**], although at a price of introducing infinitary deduction rules.

Note that in the canonical $LICP^S_{\mathrm{Meas}}$-model $M = \langle W, \mathrm{Prob}, v\rangle$ constructed above, for all w, $w' \in W$, the following conditions hold:

- $W(w) = W(w')$ and
- $H(w) = H(w')$.

Thus, the same proof can be used to show that Ax_{LICP^S} is complete with respect to the subclass of $LICP^S_{\text{Meas}}$ containing only models that satisfy those conditions.

As we mentioned in the introduction, a fragment of the presented logic (denoted LPP^S) is used in [**21, 22**] to model defaults. Besides the mentioned syntactical restrictions in LPP^S, we use there an additional semantical requirement called "neatness". The requirement means that only the empty set has the zero probability and it is important in modelling default reasoning. In this paper we do not consider the "neatness"-condition, but, following the ideas from [**22**], we can easily handle it. The only change in the completeness proof concerns an additional step in building a maximal consistent extension of a set of formulas, where we have to add the following step: if the current extension of the considered consistent set is enlarged by $P_{=0}\alpha$, then $\neg\alpha$ must be added as well. In allowing the nesting of probabilistic operators our logic resembles conditional logic [**3**], i.e., it treats the conditional probability operator as a standard binary logical operator (except that it has a denumerable list of such operators). However, our logic is much more expressive since it also includes the (absolute) probability operators ($P_{\geqslant s}$) which behave like a sort of modal operators. This enables us to combine defeasible and probabilistic knowledge in the same context.

There are many possible directions for further investigations. For example, the question of decidability of our logic naturally arises. In [**6**] decidability of a similar logic was proven. For the present approach the problem is still open since we do not consider real-valued probabilities, but the range of probability is the unit interval of a recursive nonarchimedean field.

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