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THE METHOD OF STATIONARY PHASE FOR ONCE INTEGRATED GROUP

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ABSTRACT. We obtain a formula of decomposition for

$$\Phi(A) = A \int\limits_{R^n} S(f(x))\varphi(x)\,dx + \int\limits_{R^n} \varphi(x)\,dx$$

using the method of stationary phase. Here $(S(t))_{t\in R}$ is once integrated, exponentially bounded group of operators in a Banach space X, with generator A, which satisfies the condition:

For every $x \in X$ there exists $\delta = \delta(x) > 0$ such that $\frac{S(t)x}{t^{1/2+\delta}} \to 0$ as $t \to 0$.

The function $\varphi(x)$ is infinitely differentiable, defined on R^n, with values in X, with a compact support $\operatorname{supp}\varphi$, the function $f(x)$ is infinitely differentiable, defined on R^n, with values in R, and $f(x)$ on $\operatorname{supp}\varphi$ has exactly one nondegenerate stationary point x_0.

1. Introduction

In [8] Fedoryuk gives a formula for calculation of the integral $\int_{R^n} \varphi(x)e^{[i\lambda S(x)]}dx$. Here λ is a sufficient large real parameter, $\varphi(x)$ and $S(x)$ are infinitely differentiable functions defined on R^n, with values in R. The function $\varphi(x)$ has a compact support supp φ, x_0 is a critical (stationary) nondegenerate point of the function $S(x)$. By definition, x_0 is a critical (stationary) point of the function $S(x)$ if $\partial S(x_0)/\partial x = 0$; a critical point x_0 is nondegenerate if $\det(\partial^2 S(x_0)/\partial x^2) \neq 0$.

Let $C^\infty(R^n)$ and $C_0^\infty(R^n)$ denote, successively, a space of infinitely differentiable functions defined on R^n with values in R, and a space of infinitely differentiable functions with compact support, defined on R^n with values in R.

THEOREM 1.1. (see [8] or [16]) *Let $\varphi(x) \in C_0^\infty(R^n)$ and $S(x) \in C^\infty(R^n)$, and let S on* $\operatorname{supp}\varphi$ *has exactly one nondegenerate critical point x_0. Then, for every*

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real number λ such that $|\lambda| \geqslant 1$, and for every integer $N \geqslant 1$, it holds

$$(1)\qquad I(\lambda)=\int\limits_{R^n}\varphi(x)e^{[i\lambda S(x)]}dx=\lambda^{-n/2}e^{[i\lambda S(x_0)]}\sum_{j=0}^{N-1}a_j\left(\varphi,S\right)\lambda^{-j}+R_N(\lambda).$$

Here $a_j(\varphi,S)=(P_j\varphi)(x_0)$, where P_j is a linear differential operator of order $2j$, with coefficients in C^∞. For residue $R_N(\lambda)$ it holds

$$(2)\qquad |R_N(\lambda)|\leqslant C_N\lambda^{-\frac{n}{2}-N}\left\|\varphi\right\|_{C^\beta(R^n)},\ \textit{where}\ \beta=\beta(N)<\infty.$$

The formula (1) gives an asymptotic decomposition of the integral $I(\lambda)$ as $\lambda\to\pm\infty$. The method giving decomposition (1) is called the method of stationary phase. Maslov and Fedoryuk in [16] extend this method for calculation of the integral $\Phi(A)=\int\limits_{R^n}e^{[iAS(x)]}\varphi(x)dx$.

Here $\varphi(x)\in C_0^\infty(X)$ (a space of infinitely differentiable functions defined on R^n, with values in a Banach space X, with compact support), $S(x)\in C^\infty(R^n)$ on $\operatorname{supp}\varphi$ has exactly one nondegenerate critical point x_0, a linear and closed operator A in a Banach space X is infinitesimal generator of strongly continuous group $\left(e^{(itA)}\right)_{t\in R}$, of bounded operators in X. It is known that there exist positive real constants M and ω such that $\left\|e^{(itA)}\right\|\leqslant Me^{\omega|t|}$, $t\in R$.

By the conditions given above Maslov and Fedoryuk prove:

Theorem 1.2. [16] *For every complex number λ with $\operatorname{Re}\lambda>\omega$, and every integer $N\geqslant 1$, it holds*

$$(3)\quad \Phi(A)=e^{[iAS(x_0)]}(A+i\lambda I)^{-p/2}(A-i\lambda I)^{-n/2}\sum_{0\leqslant k+l\leqslant N}(A+i\lambda I)^{-k}(A-i\lambda I)^{-l}a_{kl}+g_N.$$

Here, coefficients $a_{kl}\in X$, residue $g_N\in D(A^{N+\left[\frac{n}{2}\right]+1})$, p is number of positive, n is number of negative eigenvalues of matrix $S''_{xx}(x_0)$.

In this paper we extend the method of stationary phase to once integrated group of operators in a Banach space.

2. Preliminaries from the Theory of Integrated Semigroups and Groups

Integrated groups in Banach spaces have been introduced to study abstract Cauchy problems, for example the Schrödinger problem in $L^p(R^n)$ with $p\geqslant 1$, see [12]. n times integrated semigroups were introduced by Arendt [1] and Hieber and Kellermann [13] with $n\in N$, and later Hieber defined α times integrated semigroups with $\alpha\geqslant 0$ [11]. Differential operators in Euclidean operators are examples of integrated groups, see for example [12].

In [20] it is proved that α times integrated groups define algebra homomorphism and smooth distribution groups of fractional order are equivalent to α times integrated groups.

DEFINITION 2.1. [11] Given $\alpha \geqslant 0$, a family of strongly continuous linear and bounded operators $(S(t))_{t\geqslant 0}$ on a Banach space X is said to be an α times integrated semigroup if it satisfies $S(0) = 0$ and for all $x \in X$ and $t, s \geqslant 0$ the following equality holds:

$$S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\left[\int_s^{t+s} (t+s-r)^{\alpha-1}S(r)x\,dr - \int_0^t (t+s-r)^{\alpha-1}S(r)x\,dr\right]. \tag{4}$$

An α times integrated semigroup $(S(t))_{t\geqslant 0}$ is called nondegenerate if $S(t)x = 0$ for all $t \geqslant 0$ implies that $x = 0$. We consider only nondegenerate semigroup.

The generator $(A, D(A))$ of $(S(t))_{t\geqslant 0}$ is defined as follows: $D(A)$ is the set of all $x \in X$ such that there exists $y \in X$ satisfying

$$S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int_0^t S(s)y\,ds, \quad t \geqslant 0 \tag{5}$$

and $Ax := y$. It is straightforward to check that $(A, D(A))$ is a closed operator. The function $t \mapsto S(t)x$, $[0, \infty) \to X$ is differentiable for $t \geqslant 0$ if and only if $S(t)x \in D(A)$ and in this case

$$\frac{d}{dt}S(t)x = AS(t)x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}x. \tag{6}$$

In general, the growth of $\|S(t)\|$, if $t \to \infty$, is bigger than exponential, see for example [13, Example 1.2]. If $\|S(t)\| \leqslant Me^{\omega t}$ with $M, \omega \geqslant 0$, the condition (4) is equivalent (by Laplace transform) to $R(\lambda) := \lambda^\alpha \int_0^\infty e^{-\lambda t}S(t)\,dt$; $\operatorname{Re}\lambda > \omega$ is a pseudoresolvent operator, i.e., $R(\lambda) - R(\mu) = (\mu-\lambda)R(\lambda)R(\mu)$ for any $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega$, see [11]. In this case λ belongs to the resolvent set $\rho(A)$ and $R(\lambda) = R(\lambda, A) := (\lambda - A)^{-1}$ with $\operatorname{Re}\lambda > \omega$.

DEFINITION 2.2. [20] An α times integrated group $(S(t))_{t\in R}$ is a strongly continuous family of linear and bounded operators on a Banach space X such that $(S_+(t) := S(t))_{t\geqslant 0}$ and $(S_-(t) := -S(-t))_{t\geqslant 0}$ are α times integrated semigroups, and if $t < 0 < r$, then

$$S(t)S(r) = \frac{1}{\Gamma(\alpha)}\left[\int_{t+r}^{r} (s-t-r)^{\alpha-1}S(s)\,ds + \int_t^0 (t+r-s)^{\alpha-1}S(s)\,ds\right] \tag{7}$$

holds when $t + r \geqslant 0$, and

$$S(t)S(r) = \frac{1}{\Gamma(\alpha)}\left[\int_t^{t+r} (t+r-s)^{\alpha-1}S(s)\,ds + \int_0^r (s-t-r)^{\alpha-1}S(s)\,ds\right] \tag{8}$$

holds when $t + r \leqslant 0$. The generator of $(S(t))_{t\in R}$ is the generator of $(S_+(t))_{t\geqslant 0}$.

In the case of $(S(t))_{t\in R}$ is exponentially bounded, this definition is equivalent to saying that $(A, D(A))$ and $(-A, D(A))$ are generators of α times integrated semigroups (see [18]).

3. The Method of Stationary Phase for Once Integrated Group

Let X is a Banach space. If $(S(t))_{t\in R}$ is α times integrated group on X for $\alpha = 1$, with generator $(A, D(A))$, we will say that $(S(t))_{t\in R}$ is once integrated group on X, with generator A.

In this section we will obtain a result for once integrated, exponentially bounded group of operators $(S(t))_{t\in R}$ in a Banach space X that satisfies the condition:

(*) For every $x \in X$ there exists $\delta = \delta(x) > 0$ such that $\dfrac{S(t)x}{t^{1/2+\delta}} \to 0$ as $t \to 0$.

Such a family of operators $(S(t))_{t\in R}$ exists. For example, if operators A and $(-A)$ are generators of once integrated local Lipschitz continuously semigroups S_1 and S_2, then $(S(t))_{t\in R}$, defined by: $S(t) = S_1(t)$ for $t \geqslant 0$, and $S(t) = -S_2(-t)$ for $t \leqslant 0$, is once integrated exponentially bounded group with generator A, which satisfies the condition (*). Also, many other integrated groups satisfy that condition.

For the proof of main result in this section we need and give, without proof, the known Morse's lemma. Also, we prove several new lemmas, that we will use.

Lemma 3.1. (Morse [16]) *Let the function $f(x)$ be defined on R^n, with real values, and infinitely differentiable in a neighborhood of the point x_0, where x_0 is nondegenerate critical point of the function $f(x)$. Then, there exist neighborhoods U and V of the points $x = x_0$ and $y = 0$, and a diffeomorphism $g : V \to U$ in the class C^∞, such that*

$$(f \circ g)(y) = f(x_0) + \frac{1}{2}\sum_{j=1}^{n} \mu_j y_j^2 \tag{9}$$

Here, μ_j are eigenvalues of the matrix $f''_{xx}(x_0)$, and $\det g'(0) = 1$.

Lemma 3.2. *Let $(S(t))_{t\geqslant 0}$ be once integrated, exponentially bounded semigroup on a Banach space X, with generator A, which satisfies the condition (*). Let M and ω are positive real constants for which $\|S(t)\| \leqslant Me^{\omega t}$ for every $t \geqslant 0$. Then, for every complex number ε, such that $\operatorname{Re}\varepsilon > \omega$, the operator*

$$[R(\varepsilon, A)]^{1/2} := \frac{2}{\sqrt{\pi}} \int_0^\infty \left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)' S(t)\, dt$$

is a bounded operator from X into X, and, for every $x \in X$ it holds

$$\left[R(\varepsilon, A)^{1/2}\right]^2 x = R(\varepsilon, A)x. \tag{10}$$

PROOF. Using the condition (*) it is easily to see that the operator $[R(\varepsilon,A)]^{1/2}$ is bounded. Let $\varepsilon>\omega$ is an arbitrary number, and $x\in X$. Then,

$$\begin{aligned}
\big[R(\varepsilon,A)^{1/2}\big]^2x&=\left[\frac{2}{\sqrt{\pi}}\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'S(t)\,dt\right]^2x\\
&=\frac{4}{\sqrt{\pi}}\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'S(t)\,dt\int\limits_0^\infty\left(\frac{e^{-\varepsilon s}}{2\sqrt{s}}\right)'S(s)x\,ds\\
&=\frac{4}{\pi}\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'dt\int\limits_0^\infty\left(\frac{e^{-\varepsilon s}}{2\sqrt{s}}\right)'S(s)S(t)x\,ds.
\end{aligned}$$

Since $S(t)S(s)x=\int\limits_0^s[S(t+u)-S(u)]x\,du$, the integration by parts in interior integral gives

$$\begin{aligned}
\big[R(\varepsilon,A)^{1/2}\big]^2x&=\frac{4}{\pi}\frac{e^{-\varepsilon s}}{2\sqrt{s}}S(s)\Big|_{s=0}^\infty\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'S(t)x\,dt\\
&-\frac{4}{\pi}\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'dt\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}[S(t+s)-S(s)]x\,ds.
\end{aligned}$$

Because of the condition (*), the first expression equals zero.

Since A is a closed operator, and $S(t+s)-S(s)=A\int\limits_s^{t+s}S(u)\,du+tI$, we obtain

$$\big[R(\varepsilon,A)^{1/2}\big]^2x=\frac{-4}{\pi}\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'dt\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}\left[A\int\limits_s^{t+s}S(u)x\,du+tx\right]ds=I_1+I_2,$$

where

$$\begin{aligned}
I_1&=\frac{-4}{\pi}\int\limits_0^\infty t\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'dt\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}\,x\,ds,\\
I_2&=\frac{-4}{\pi}A\int\limits_0^\infty\left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)'dt\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}\,ds\int\limits_s^{t+s}S(u)x\,du.
\end{aligned}$$

Further,

$$I_1=\frac{-4}{\pi}\left(t\,\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)\Big|_{t=0}^\infty\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}\,x\,ds+\frac{4}{\pi}\int\limits_0^\infty\frac{e^{-\varepsilon t}}{2\sqrt{t}}\,dt\int\limits_0^\infty\frac{e^{-\varepsilon s}}{2\sqrt{s}}\,x\,ds.$$

Since the first expression equals zero, we have

$$I_1 = \frac{4}{\pi}\left(\int_0^\infty e^{-\varepsilon u^2}\,du\right)^2 x \overset{u\sqrt{\varepsilon}=l}{=} \frac{4}{\pi}\left(\int_0^\infty e^{-l^2}\,\frac{dl}{\sqrt{\varepsilon}}\right)^2 x$$
$$= \frac{4}{\pi\varepsilon}\left(\frac{\sqrt{\pi}}{2}\right)^2 x = \frac{1}{\varepsilon}x. \tag{11}$$

Consider now the integral I_2. We interchange the order of integration and obtain

$$I_2 = \frac{-4}{\pi}A\int_0^\infty \frac{e^{-\varepsilon s}}{2\sqrt{s}}\,ds\int_0^\infty \left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)' dt\int_s^{t+s} S(u)x\,du.$$

Put $v = \dfrac{e^{-\varepsilon t}}{2\sqrt{t}}, u = \int_s^{t+s} S(u)x\,du$. The integration by parts implies

$$\int_0^\infty \left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)' dt\int_s^{t+s} S(u)x\,du = \left[\frac{e^{-\varepsilon t}}{2\sqrt{t}}\int_s^{t+s} S(u)x\,du\right]_{t=0}^\infty - \int_0^\infty \frac{e^{-\varepsilon t}}{2\sqrt{t}} S(t+s)x\,dt.$$

The first summand at the right-hand side is equal to 0,

$$\left\|\frac{e^{-\varepsilon t}}{2\sqrt{t}}\int_s^{t+s} S(u)\,du\right\| \leqslant M\,\frac{e^{-\varepsilon t}}{2\sqrt{t}}(t+s-s)e^{\omega(t+s)} = \frac{M}{2}e^{\omega s}\sqrt{t}e^{(\omega-\varepsilon)t},$$

and $\sqrt{t}\,e^{(\omega-\varepsilon)t}|_{t=0}^\infty = 0$. Hence,

$$\int_0^\infty \left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)' dt\int_s^{t+s} S(u)x\,du = -\int_0^\infty \frac{e^{-\varepsilon t}}{2\sqrt{t}} S(t+s)x\,dt,$$

so that

$$I_2 = \frac{4}{\pi}A\int_0^\infty \frac{e^{-\varepsilon s}}{2\sqrt{s}}\,ds\int_0^\infty \frac{e^{-\varepsilon t}}{2\sqrt{t}} S(t+s)x\,dt.$$

The substitution $t+s=u$ gives

$$I_2 = \frac{4}{\pi}A\int_0^\infty \frac{e^{-\varepsilon s}}{2\sqrt{s}}\,ds\int_s^\infty \frac{e^{-\varepsilon(u-s)}}{2\sqrt{u-s}} S(u)x\,du.$$

We interchange the order of integration and obtain

$$I_2 = \frac{1}{\pi}A\int_0^\infty e^{-\varepsilon u}S(u)x\,du\int_0^u \frac{ds}{\sqrt{s}\sqrt{u-s}}.$$

The substitution $s=uv$ shows that

$$\int_0^u \frac{ds}{\sqrt{s}\sqrt{u-s}} = \int_0^1 \frac{u\,dv}{\sqrt{uv}\sqrt{u-uv}} = \int_0^1 v^{-1/2}(1-v)^{-1/2}dv = B\Big(\frac{1}{2},\frac{1}{2}\Big) = \pi,$$

where $B(\alpha,\beta)$ denotes beta-function. Therefore,

$$I_2 = A\int_0^\infty e^{-\varepsilon u} S(u)x\,du = A\frac{R(\varepsilon,A)}{\varepsilon}\,x = R(\varepsilon,A)x - \frac{1}{\varepsilon}\,x \tag{12}$$

Finally, (11) and (12) imply

$$\big[R(\varepsilon,A)^{1/2}\big]^2 x = \left[\frac{2}{\sqrt{\pi}}\int_0^\infty \left(\frac{e^{-\varepsilon t}}{2\sqrt{t}}\right)' S(t)\,dt\right]^2 x = I_1 + I_2 = R(\varepsilon,A)x. \qquad \square$$

Lemma 3.3. *Let the family of operators $(S(t))_{t\geqslant 0}$ satisfy the conditions of the previous lemma and let $j \geqslant 0$ be any integer. Then, for every complex number ε such that $\operatorname{Re}\varepsilon > \omega$, we have*

$$\int_0^\infty \Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' S(t)\,dt = \Gamma\Big(\frac{j+1}{2}\Big)\,[R(\varepsilon,A)]^{(j+1)/2}\,. \tag{13}$$

Remark 3.1. The condition (*) is necessary only for $j = 0$. Then we obtain the assertion of Lemma 3.2.

Proof. Let us take any $x \in X$ and put

$$I = \int_0^\infty \Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' S(t)x\,dt.$$

By analogy with the proof of the previous lemma we obtain

$$\begin{aligned} I^2 &= -\int_0^\infty \Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' dt \int_0^\infty e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1}[S(t+s)-S(s)]x\,ds \\ &= -\int_0^\infty \Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' dt \int_0^\infty e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1}\left[A\int_s^{t+s} S(u)x\,du + tx\right] ds = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= -\int_0^\infty t\Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' dt \int_0^\infty e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} x\,ds, \\ I_2 &= -A\int_0^\infty \Big[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big]' dt \int_0^\infty e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} ds \int_s^{t+s} S(u)x\,du. \end{aligned}$$

Further,

$$I_1 = (-t)e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\Big|_{t=0}^{\infty} \int_0^{\infty} e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} x\, ds$$
$$+ \int_0^{\infty} e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1} dt \int_0^{\infty} e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} x\, ds.$$

The first expression equals zero, so that we obtain

$$I_1 = \left[\int_0^{\infty} e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1} dt\right]^2 x \overset{\varepsilon t = u}{=} \left[\int_0^{\infty} e^{-u}\left(\frac{u}{\varepsilon}\right)^{(j-1)/2} \frac{du}{\varepsilon}\right]^2 x \tag{14}$$
$$= \left[\frac{1}{\varepsilon^{(j+1)/2}}\,\Gamma\left(\frac{j+1}{2}\right)\right]^2 x = \Gamma^2\left(\frac{j+1}{2}\right)\frac{1}{\varepsilon^{j+1}} x.$$

Consider now I_2. We interchange the order of integration and obtain

$$I_2 = -A \int_0^{\infty} e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} ds \int_0^{\infty} \left[e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1}\right]' \int_s^{t+s} S(u)x\, du\, dt.$$

The integration by parts implies that the interior integral equals

$$-\int_0^{\infty} e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1} S(t+s)x\, dt,$$

so that

$$I_2 = A \int_0^{\infty} e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} ds \int_0^{\infty} e^{-\varepsilon t}\big(\sqrt{t}\big)^{j-1} S(t+s)x\, dt.$$

The substitution $t + s = u$ gives

$$I_2 = A \int_0^{\infty} e^{-\varepsilon s}\big(\sqrt{s}\big)^{j-1} ds \int_s^{\infty} e^{-\varepsilon(u-s)}\left(\sqrt{u-s}\right)^{j-1} S(u)x\, du.$$

We interchange the order of integration and obtain

$$I_2 = A \int_0^{\infty} e^{-\varepsilon u} S(u)x\, du \int_0^{u} \left(\sqrt{s}\sqrt{u-s}\right)^{j-1} ds.$$

The substitution $s = uv$ gives

$$\int_0^{u} \left(\sqrt{s(u-s)}\right)^{j-1} ds = \int_0^{1} \left(\sqrt{uv(u-uv)}\right)^{j-1} u\, dv = u^j \int_0^{1} v^{(j-1)/2}(1-v)^{(j-1)/2} dv$$
$$= u^j B\Big(\frac{j+1}{2}, \frac{j+1}{2}\Big) = u^j \Gamma^2\Big(\frac{j+1}{2}\Big)\frac{1}{j!}.$$

Here B and Γ denote beta and gamma function. Therefore,

$$I_2 = \frac{1}{j!}\Gamma^2\Big(\frac{j+1}{2}\Big)A\int_0^\infty u^j e^{-\varepsilon u}S(u)x\,du.$$

Since

$$\int_0^\infty (-u)^j e^{-\varepsilon u}S(u)x\,du = \left(\frac{R(\varepsilon,A)}{\varepsilon}\right)^{(j)} x,$$

we have

$$I_2 = \frac{(-1)^j}{j!}\Gamma^2\left(\frac{j+1}{2}\right)A\left(\frac{R(\varepsilon,A)}{\varepsilon}\right)^{(j)} x \tag{15}$$

By induction one can easily prove that for every integer $j \in N_0$,

$$A\left(\frac{R(\varepsilon,A)}{\varepsilon}\right)^{(j)} x = (-1)^j j!\left(R(\varepsilon,A)^{j+1} - \frac{I}{\varepsilon^{j+1}}\right)x \tag{16}$$

The relations (15) and (16) imply

$$I_2 = \Gamma^2\left(\frac{j+1}{2}\right)\left(R(\varepsilon,A)^{j+1} - \frac{I}{\varepsilon^{j+1}}\right)x \tag{17}$$

Finally, using (14) and (17) we obtain

$$I^2 = I_1 + I_2 = \Gamma^2\left(\frac{j+1}{2}\right)R(\varepsilon,A)^{j+1}x. \qquad \square$$

LEMMA 3.4. *Let $(S(t))_{t\geqslant 0}$ be once integrated, exponentially bounded semigroup on a Banach space X, with generator A, which satisfies the condition (*). Let M and ω are positive real constants for which $\|S(t)\| \leqslant Me^{\omega t}$ for every $t \geqslant 0$. Let ε be a complex number with $Re\varepsilon > \omega$, $\delta > 0$, $j \geq -1$ integer, and $x \in X$. Then, for every integer $N \geqslant 1$ the next decomposition holds*

$$\begin{aligned}\int_0^\delta \left(e^{-\varepsilon t^2}t^j\right)' S(t^2)x\,dt = \Gamma\left(\frac{j+2}{2}\right)R(\varepsilon,A)^{(j+2)/2}x + \delta^j e^{-\varepsilon\delta^2}S(\delta^2)x \\ + e^{-\varepsilon\delta^2}\sum_{k=1}^N c_k AR(\varepsilon,A)^k S(\delta^2)x + x_N,\end{aligned} \tag{18}$$

where c_k are constants, and residue $x_N \in D(A^{N+1})$.

PROOF. Denote

$$I = \int_0^\delta \left(e^{-\varepsilon t^2}t^j\right)' S(t^2)\,dt = \int_0^\infty \left(e^{-\varepsilon t^2}t^j\right)' S(t^2)\,dt - \int_\delta^\infty \left(e^{-\varepsilon t^2}t^j\right)' S(t^2)\,dt. \tag{19}$$

From previous lemma (the relation (13)) we see that for every $j \geqslant 0$,

$$\int_0^\infty \left[e^{-\varepsilon t}(\sqrt{t})^{j-1}\right]' S(t)\,dt = \Gamma\left(\frac{j+1}{2}\right)[R(\varepsilon,A)]^{(j+1)/2}.$$

The substitution $\sqrt{t} = u$ gives

$$\begin{aligned}\int_0^\infty \left[e^{-\varepsilon t}(\sqrt{t})^{j-1}\right]' S(t)\,dt &= \int_0^\infty \left[e^{-\varepsilon u^2} u^{j-1}\right]'_u \frac{du}{dt}\, S(u^2)\,d(u^2)\\ &= \int_0^\infty \left[e^{-\varepsilon u^2} u^{j-1}\right]' S(u^2)\,du = \Gamma\left(\frac{j+1}{2}\right)[R(\varepsilon,A)]^{(j+1)/2}.\end{aligned}$$

Hence, for every integer $j \geqslant -1$ and every $x \in X$ it holds

$$\int_0^\infty \left[e^{-\varepsilon t^2} t^j\right]' S(t^2)x\,dt = \Gamma\left(\frac{j+2}{2}\right)[R(\varepsilon,A)]^{(j+2)/2}\,x \tag{20}$$

Consider now $J = \int_\delta^\infty \left[e^{-\varepsilon t^2} t^j\right]' S(t^2)\,dt$. Since $S(t) = tI + A\int_0^t S(u)\,du$, we have $J = J_1 + J_2$, where

$$J_1 = \int_\delta^\infty \left(e^{-\varepsilon t^2} t^j\right)' t^2 I dt \quad \text{and} \quad J_2 = \int_\delta^\infty \left(e^{-\varepsilon t^2} t^j\right)' A \int_0^{t^2} S(u)\,du\,dt.$$

The integration by parts gives

$$J_1 = t^{j+2} e^{-\varepsilon t^2}\Big|_{t=\delta}^\infty - 2\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2} dt = -\delta^{j+2} e^{-\varepsilon \delta^2} - 2\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2} dt.$$

Since the operator A is closed, we have

$$\begin{aligned}J_2 &= e^{-\varepsilon t^2} t^j A \int_0^{t^2} S(u)\,du\Big|_{t=\delta}^\infty - A\int_\delta^\infty e^{-\varepsilon t^2} t^j S(t^2)\,2t\,dt\\ &= -e^{-\varepsilon\delta^2}\delta^j\left[S(\delta^2) - \delta^2 I\right] - 2A\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2} S(t^2)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}J &= \int_\delta^\infty \left[e^{-\varepsilon t^2} t^j\right]' S(t^2)\,dt\\ &= -\delta^j e^{-\varepsilon\delta^2} S(\delta^2) - 2I\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2} dt - 2A\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2} S(t^2)\,dt\end{aligned} \tag{21}$$

The relations (19) (20) and (21) give

$$(22)\quad I=\int_0^{\delta}\left(e^{-\varepsilon t^2}t^j\right)' S(t^2)\,dt=\Gamma\left(\frac{j+2}{2}\right)[R(\varepsilon,A)]^{(j+2)/2}+\delta^j e^{-\varepsilon\delta^2}S(\delta^2)$$
$$+2I\int_\delta^\infty t^{j+1}e^{-\varepsilon t^2}dt+2A\int_\delta^\infty t^{j+1}e^{-\varepsilon t^2}S(t^2)\,dt$$

Consider now $\bar{I}=\int_\delta^\infty 2t^{j+1}e^{-\varepsilon t^2}S(t^2)\,dt$. For the integral $\bar{I}$ we will use the integration by parts by putting $U=t^j$, $dV=2t\,e^{-\varepsilon t^2}S(t^2)\,dt$. Then,

$$V=\int_0^{t^2}e^{-\varepsilon u}S(u)\,du=R(\varepsilon,A)(\varepsilon I-A)\int_0^{t^2}e^{-\varepsilon u}S(u)\,du.$$

Consider now the expression $(\varepsilon I-A)\int_0^{t^2}e^{-\varepsilon u}S(u)\,du$. We have

$$(\varepsilon I-A)\int_0^{t^2}e^{-\varepsilon u}S(u)\,du=\varepsilon\int_0^{t^2}e^{-\varepsilon u}S(u)\,du-A\int_0^{t^2}e^{-\varepsilon u}S(u)\,du$$

putting $S(u)du=d\bar{V}$ and $e^{-\varepsilon u}=\bar{U}$

$$=\varepsilon\int_0^{t^2}e^{-\varepsilon u}S(u)\,du-A\left[e^{-\varepsilon u}\int_0^u S(s)\,ds\Big|_{u=0}^{t^2}+\varepsilon\int_0^{t^2}e^{-\varepsilon u}\int_0^u S(s)\,ds\,du\right]$$
$$=\varepsilon\int_0^{t^2}e^{-\varepsilon u}S(u)\,du-e^{-\varepsilon t^2}A\int_0^{t^2}S(s)\,ds-\varepsilon\int_0^{t^2}e^{-\varepsilon u}A\int_0^u S(s)\,ds\,du$$
$$=\varepsilon\int_0^{t^2}e^{-\varepsilon u}S(u)\,du-e^{-\varepsilon t^2}\left[S(t^2)-t^2I\right]-\varepsilon\int_0^{t^2}e^{-\varepsilon u}\left[S(u)-uI\right]du$$
$$=\varepsilon\int_0^{t^2}u\,e^{-\varepsilon u}du+e^{-\varepsilon t^2}\left[t^2I-S(t^2)\right].$$

Because of

$$\varepsilon\int_0^{t^2}u\,e^{-\varepsilon u}du=\frac{1-e^{-\varepsilon t^2}}{\varepsilon}-t^2e^{-\varepsilon t^2},$$

we have

$$(\varepsilon I - A)\int_0^{t^2} e^{-\varepsilon u} S(u)\,du = \frac{1-e^{-\varepsilon t^2}}{\varepsilon} I - e^{-\varepsilon t^2} S(t^2).$$

Since $R(\varepsilon, A)/\varepsilon$ is independent of t, we can put

$$V = R(\varepsilon, A)\Big(\frac{-e^{-\varepsilon t^2}}{\varepsilon} I - e^{-\varepsilon t^2} S(t^2)\Big).$$

Now, we have

$$\bar{I} = \int_\delta^\infty 2t^{j+1} e^{-\varepsilon t^2} S(t^2)\,dt = t^j R(\varepsilon, A)\Big(\frac{-e^{-\varepsilon t^2}}{\varepsilon} I - e^{-\varepsilon t^2} S(t^2)\Big)\Big|_{t=\delta}^{\infty} + R(\varepsilon, A)\int_\delta^\infty \Big(\frac{e^{-\varepsilon t^2}}{\varepsilon} I + e^{-\varepsilon t^2} S(t^2)\Big)(t^j)' dt.$$

Hence,

$$\bar{I} = \delta^j e^{-\varepsilon\delta^2} R(\varepsilon, A) S(\delta^2) + 2R(\varepsilon, A)\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2}\,dt + jR(\varepsilon, A)\int_\delta^\infty e^{-\varepsilon t^2} t^{j-1} S(t^2)\,dt \tag{23}$$

Now (22) and (23) give

$$I = \int_0^\delta \left(e^{-\varepsilon t^2} t^j\right)' S(t^2)\,dt = \Gamma\left(\frac{j+2}{2}\right)[R(\varepsilon, A)]^{(j+2)/2} + \delta^j e^{-\varepsilon\delta^2} S(\delta^2) + 2I\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2}\,dt + \delta^j e^{-\varepsilon\delta^2} AR(\varepsilon, A) S(\delta^2) + 2AR(\varepsilon, A)\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2}\,dt + jAR(\varepsilon, A)\int_\delta^\infty e^{-\varepsilon t^2} t^{j-1} S(t^2)\,dt,$$

i.e.,

$$I = \int_0^\delta \left(e^{-\varepsilon t^2} t^j\right)' S(t^2)\,dt = \Gamma\left(\frac{j+2}{2}\right)[R(\varepsilon, A)]^{(j+2)/2} + \delta^j e^{-\varepsilon\delta^2} S(\delta^2) + \delta^j e^{-\varepsilon\delta^2} AR(\varepsilon, A) S(\delta^2) + 2\varepsilon R(\varepsilon, A)\int_\delta^\infty t^{j+1} e^{-\varepsilon t^2}\,dt + jAR(\varepsilon, A)\int_\delta^\infty e^{-\varepsilon t^2} t^{j-1} S(t^2)\,dt \tag{24}$$

Now, for the last integral in (24) we use the same procedure as for the integral $\bar{I}$. If we continue to repeat this procedure, we conclude that for every $N \geqslant 1$, every

$j \geq -1$, and every $x \in X$,

$$\int_0^\delta \left(e^{-\varepsilon t^2} t^j\right)' S(t^2)x\,dt = \Gamma\left(\frac{j+2}{2}\right) R(\varepsilon, A)^{(j+2)/2}x + \delta^j e^{-\varepsilon\delta^2} S(\delta^2)x$$
$$+ e^{-\varepsilon\delta^2} \sum_{k=1}^{N} c_k A R(\varepsilon, A)^k S(\delta^2)x + x_N,$$

where c_k are constants, and residue $x_N \in D(A^{N+1})$. □

LEMMA 3.5. *Let $(S(t))_{t\in R}$ be once integrated, exponentially bounded group on a Banach space X, with generator A, which satisfies the condition (*). Let M and ω are positive real constants for which $\|S(t)\| \leqslant Me^{\omega|t|}$ for every $t \in R$. Let $\varphi(x) \in C_0^\infty(X)$. Then, for every complex number ε, with* $\operatorname{Re}\varepsilon > \omega$, *and every integer $L \geqslant 0$ it holds*

$$\int_{-\infty}^{\infty} S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx = R(\varepsilon, A)^{1/2} \sum_{k=0}^{L} b_k R(\varepsilon, A)^k + a_L, \tag{25}$$

for $b_k \in X$, $a_L \in D(A^{L+1})$.

PROOF. Denote $J = \int_0^\infty S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx$. Since $S(t) = tI + A\int_0^t S(u)\,du$, we have

$$J = \frac{\varphi(x)}{2x} S(x^2)\Big|_0^\infty - \int_0^\infty \varphi(x)\,dx - A\int_0^\infty S(x^2)\varphi(x)\,dx.$$

First of all we assume that the function $\varphi(x)$ has a null of order $m \geqslant 1$ at the point $x = 0$. Hence, $\frac{\varphi(x)}{2x} S(x^2)\big|_0^\infty = 0$, so that

$$J = -\int_0^\infty \varphi(x)dx - A\int_0^\infty S(x^2)\varphi(x)dx. \tag{26}$$

Let now $\varphi(x) \in C_0^\infty(X)$ be an arbitrary function. Choose any $\delta > 0$ and introduce the function $\chi(x) \in C_0^\infty(R)$ with $\chi(x) \equiv 1$ for every real x for which $|x| \leqslant \delta$.

For any fixed ε with $\operatorname{Re}\varepsilon > \omega$, define the function $\varphi_\varepsilon(x) \in C_0^\infty(X)$ by $\varphi_\varepsilon(x) := e^{\varepsilon x^2}\varphi(x)$ $(x \in R)$. If the function $\varphi_\varepsilon(x)$ has the Maclaurin's series $\varphi_\varepsilon(x) = \sum_{j=0}^N \varphi_j x^j + \psi_N(x)$ ($\varphi_j \in X$, N a positive integer), then the function $\psi_N(x)$ at the point $x = 0$ has a null of order $N + 1$.

From the fact that $\varphi_\varepsilon(x)$ and the functions x^j are infinitely differentiable, the same holds for the residue $\psi_N(x)$. Hence, $\psi_N \in C^\infty(X)$. Consider the integral

$$\bar{J} = -\int_0^\infty \varphi(x)\chi(x)\,dx - A\int_0^\infty S(x^2)\varphi(x)\chi(x)\,dx.$$

Decompose J as

$$J = \bar{J} + (J - \bar{J}). \tag{27}$$

By (26) we have

$$\begin{aligned} J - \bar{J} &= \int_0^\infty \varphi(x)\,[\chi(x) - 1]\;dx + A\int_0^\infty S(x^2)\varphi(x)\,[\chi(x) - 1]\;dx \\ &= \int_\delta^\infty \varphi(x)\,[\chi(x) - 1]\;dx + A\int_\delta^\infty S(x^2)\varphi(x)\,[\chi(x) - 1]\;dx \end{aligned} \tag{28}$$

Consider $\int_\delta^\infty S(x^2)\varphi(x)\,[\chi(x)-1]\,dx$. If we take $dv = 2xe^{-\varepsilon x^2}S(x^2)dx$, then, from the proof of the previous lemma we see that

$$v = R(\varepsilon, A)\biggl(\frac{-e^{-\varepsilon x^2}}{\varepsilon}I - e^{-\varepsilon x^2}S(x^2)\biggr).$$

Therefore, we obtain

$$\begin{aligned} \int_\delta^\infty S(x^2)\,\varphi(x)\,[\chi(x)-1]\,dx &= \int_\delta^\infty 2xe^{-\varepsilon x^2}S(x^2)\frac{\varphi_\varepsilon(x)\,[\chi(x)-1]}{2x}\,dx \\ &= \frac{\varphi(x)\,[\chi_\varepsilon(x)-1]}{2x}\,R(\varepsilon, A)\biggl(\frac{-e^{-\varepsilon x^2}}{\varepsilon}I - e^{-\varepsilon x^2}S(x^2)\biggr)\bigg|_\delta^\infty \\ &\quad + R(\varepsilon, A)\int_\delta^\infty \biggl(\frac{e^{-\varepsilon x^2}}{\varepsilon}I + e^{-\varepsilon x^2}S(x^2)\biggr)\biggl(\frac{\varphi_\varepsilon(x)\,[\chi(x)-1]}{2x}\biggr)'\,dx. \end{aligned}$$

Hence $\int_\delta^\infty S(x^2)\varphi(x)\,[\chi(x)-1]\,dx \in D(A)$.

Integrating by parts again we conclude that $A\int_\delta^\infty S(x^2)\varphi(x)\,[\chi(x)-1]\,dx \in D(A^{N+1})$ for every integer $N \geqslant 1$. Hence, for every fixed $N \geqslant 1$ we have

$$J - \bar{J} = \int_\delta^\infty \varphi(x)\,[\chi(x)-1]\,dx + g_N \qquad \bigl(g_N \in D(A^{N+1})\bigr). \tag{29}$$

Consider now

$$\begin{aligned} &A\int_0^\infty S(x^2)\varphi(x)\chi(x)\,dx \\ &\qquad = A\int_0^\delta S(x^2)e^{-\varepsilon x^2}\varphi_\varepsilon(x)\,dx + A\int_\delta^\infty S(x^2)e^{-\varepsilon x^2}\varphi_\varepsilon(x)\chi(x)\,dx = I_1 + I_2, \end{aligned} \tag{30}$$

where

$$I_1 = A\int_0^\delta S(x^2)e^{-\varepsilon x^2}\varphi_\varepsilon(x)\,dx, \quad I_2 = A\int_\delta^\infty S(x^2)e^{-\varepsilon x^2}\varphi_\varepsilon(x)\chi(x)\,dx.$$

Then

$$I_1 = \sum_{j=0}^N \varphi_j A\int_0^\delta S(x^2)e^{-\varepsilon x^2}x^j\,dx + A\int_0^\delta S(x^2)e^{-\varepsilon x^2}\psi_N(x)\,dx. \tag{31}$$

For every j we have

$$\begin{aligned}\int_0^\delta \left(e^{-\varepsilon x^2}x^j\right)' S(x^2)\,dx \\ = e^{-\varepsilon x^2}x^j S(x^2)\Big|_0^\delta - 2\int_0^\delta x^{j+1}e^{-\varepsilon x^2}dx - 2A\int_0^\delta x^{j+1}e^{-\varepsilon x^2}S(x^2)\,dx.\end{aligned} \tag{32}$$

From the Lemma 3.4 we see that for integers $j \geqslant -1$ and $M \geqslant 1$ it holds

$$\begin{aligned}\int_0^\delta \left(e^{-\varepsilon x^2}x^j\right)' S(x^2)\,dx = \Gamma\Big(\frac{j+2}{2}\Big)R(\varepsilon, A)^{(j+2)/2} + \delta^j e^{-\varepsilon\delta^2}S(\delta^2) \\ + e^{-\varepsilon\delta^2}\sum_{k=1}^M c_k AR(\varepsilon, A)^k S(\delta^2) + x_M\end{aligned} \tag{33}$$

where c_k are constants, and residue $x_M \in D(A^{M+1})$.

From (32) and (33) for $j \geqslant -1$ we have

$$\begin{aligned}A\int_0^\delta x^{j+1}e^{-\varepsilon x^2}S(x^2)\,dx \\ = \frac{1}{2}\delta^j e^{-\varepsilon\delta^2}S(\delta^2) - \int_0^\delta x^{j+1}e^{-\varepsilon x^2}dx - \frac{1}{2}\Gamma\Big(\frac{j+2}{2}\Big)R(\varepsilon, A)^{(j+2)/2} \\ - \frac{1}{2}\,\delta^j e^{-\varepsilon\delta^2}S(\delta^2) - \frac{1}{2}\,e^{-\varepsilon\delta^2}\sum_{k=1}^M c_k AR(\varepsilon, A)^k S(\delta^2) - \frac{1}{2}\,x_M.\end{aligned}$$

If we put j instead of $j+1$ in the last relation, then we obtain for $j \geqslant 0$,

$$\begin{aligned}A\int_0^\delta x^j e^{-\varepsilon x^2}S(x^2)\,dx = -\int_0^\delta x^j e^{-\varepsilon x^2}dx - \frac{1}{2}\Gamma\left(\frac{j+1}{2}\right)R(\varepsilon, A)^{(j+1)/2} \\ - \frac{1}{2}e^{-\varepsilon\delta^2}\sum_{k=1}^M c_k AR(\varepsilon, A)^k S(\delta^2) - \frac{1}{2}\,x_M.\end{aligned} \tag{34}$$

The relations (31) and (34) give now

$$I_1 = \sum_{j=0}^{N} \Big(-\frac{1}{2}\varphi_j\Big)\Big[\Gamma\Big(\frac{j+1}{2}\Big) R(\varepsilon, A)^{(j+1)/2} + 2\int_0^{\delta} x^j e^{-\varepsilon x^2}\, dx + e^{-\varepsilon\delta^2} \sum_{k=1}^{M} c_k A R(\varepsilon, A)^k S(\delta^2) + x_M\Big] + A\int_0^{\delta} S(x^2) e^{-\varepsilon x^2} \psi_N(x)\, dx \tag{35}$$

$$I_2 = \sum_{j=0}^{N} \varphi_j A \int_{\delta}^{\infty} S(x^2) e^{-\varepsilon x^2} x^j \chi(x)\, dx + A\int_{\delta}^{\infty} S(x^2) e^{-\varepsilon x^2} \psi_N(x)\chi(x)\, dx \tag{36}$$

The sum of the last integrals in (35) and (36) equals

$$h_N = A\int_0^{\infty} S(x^2) e^{-\varepsilon x^2} \psi_N(x)\chi(x)\, dx \quad \text{and} \quad h_N \in D(A^{N+1}) \quad \text{for every} \quad N.$$

Integration by parts gives

$$\begin{aligned}\int_{\delta}^{\infty} S(x^2) e^{-\varepsilon x^2} x^j \chi(x)\, dx &= \int_{\delta}^{\infty} 2x e^{-\varepsilon x^2} S(x^2)\, \frac{1}{2}\, x^{j-1}\chi(x)\, dx \\ &= \frac{1}{2}\, x^{j-1}\chi(x) R(\varepsilon, A)\Big(\frac{-e^{-\varepsilon x^2}}{\varepsilon} I - e^{-\varepsilon x^2} S(x^2)\Big)\Big|_{\delta}^{\infty} \\ &\quad + R(\varepsilon, A)\int_{\delta}^{\infty}\Big(\frac{e^{-\varepsilon x^2}}{\varepsilon} I + e^{-\varepsilon x^2} S(x^2)\Big)\Big(\frac{1}{2} x^{j-1}\chi(x)\Big)'\, dx.\end{aligned}$$

Since $\chi(\delta) = 1$, we have

$$\begin{aligned}\int_{\delta}^{\infty} S(x^2) e^{-\varepsilon x^2} x^j \chi(x)\, dx &= \frac{1}{2}\delta^{j-1} R(\varepsilon, A)\Big(\frac{e^{-\varepsilon\delta^2}}{\varepsilon} I + e^{-\varepsilon\delta^2} S(\delta^2)\Big) \\ &\quad + R(\varepsilon, A)\int_{\delta}^{\infty}\Big(\frac{e^{-\varepsilon x^2}}{\varepsilon} I + e^{-\varepsilon x^2} S(x^2)\Big)\Big(\frac{1}{2} x^{j-1}\chi(x)\Big)'\, dx.\end{aligned}$$

If we repeat the integration by parts further, we conclude that the integrals

$$A\int_{\delta}^{\infty} S(x^2) e^{-\varepsilon x^2} x^j \chi(x)\, dx \quad (\text{by } I_2)$$

annul with the expression

$$\Big(-\frac{1}{2}\Big)\Big[e^{-\varepsilon\delta^2} \sum_{k=1}^{M} c_k A R(\varepsilon, A)^k S(\delta^2) + x_M\Big] \quad (\text{by } I_1 \text{ in } (35)).$$

Hence,

$$I_1 + I_2 = A \int_0^\infty S(x^2)\varphi(x)\chi(x)\,dx$$
$$= \sum_{j=0}^{N} \Big(-\frac{1}{2}\Big)\varphi_j \left[\Gamma\Big(\frac{j+1}{2}\Big) R(\varepsilon, A)^{(j+1)/2} + 2\int_0^\delta x^j e^{-\varepsilon x^2} dx\right] + h_N.$$

From the last relation and

$$\bar{J} = -\int_0^\infty \varphi(x)\chi(x)\,dx - A\int_0^\infty S(x^2)\varphi(x)\chi(x)\,dx$$

we obtain that

$$\bar{J} = -\int_0^\infty \varphi(x)\chi(x)\,dx - h_N + \frac{1}{2}\sum_{j=0}^{N} \varphi_j \left[\Gamma\Big(\frac{j+1}{2}\Big) R(\varepsilon, A)^{(j+1)/2} + 2\int_0^\delta x^j e^{-\varepsilon x^2} dx\right].$$

Using (27) and (29) we have

$$J = \int_0^\infty S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx = \int_\delta^\infty \varphi(x)\,[\chi(x) - 1]\,dx - \int_0^\infty \varphi(x)\chi(x)\,dx + g_N - h_N$$
$$+ \frac{1}{2}\sum_{j=0}^{N} \varphi_j \left[\Gamma\Big(\frac{j+1}{2}\Big) R(\varepsilon, A)^{(j+1)/2} + 2\int_0^\delta x^j e^{-\varepsilon x^2} dx\right],$$

where g_N and h_N belong to $D(A^{N+1})$.

Analogously calculating the integral $J^* = \int_{-\infty}^{0} S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx$ we obtain

$$J^* = \int_{-\infty}^{0} S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx = -\int_\delta^\infty \varphi(x)\,[\chi(x) - 1]\,dx + \int_0^\infty \varphi(x)\chi(x)\,dx + \bar{g}_N - \bar{h}_N$$
$$+ \frac{1}{2}\sum_{j=0}^{N} \varphi_j \left[\Gamma\Big(\frac{j+1}{2}\Big)(-1)^j R(\varepsilon, A)^{(j+1)/2} - 2\int_0^\delta x^j e^{-\varepsilon x^2} dx\right],$$

where $\bar{g}_N$ and $\bar{h}_N$ belong to $D(A^{N+1})$.

Finally, for $L \geqslant 0$ the following decomposition holds

$$\int_{-\infty}^{\infty} S(x^2)\left[\frac{\varphi(x)}{2x}\right]' dx = J + J^* = 2\frac{1}{2}R(\varepsilon, A)^{1/2}\sum_{k=0}^{L} \varphi_{2k+1}\Gamma\left(\frac{2k+2}{2}\right) R(\varepsilon, A)^k + a_L.$$

Hence, it holds (25) for $b_k = k!\varphi_{2k+1} \in X$, $a_L \in D(A^{L+1})$. □

Analogously one can prove

LEMMA 3.6. *Let $(S(t))_{t\in R}$ be once integrated, exponentially bounded group on a Banach space X, with generator A, which satisfies the condition (*). Let M and ω are positive real constants for which $\|S(t)\| \leqslant Me^{\omega|t|}$ for every $t \in R$. Let $\varphi(x) \in C_0^\infty(X)$. Then, for every complex number ε, with $\operatorname{Re}\varepsilon < -\omega$, and every integer $L \geqslant 0$, we have*

$$\int_{-\infty}^{\infty} S(-x^2)\left[\frac{\varphi(x)}{2x}\right]' dx = R(-\varepsilon, A)^{1/2}\sum_{k=0}^{L} c_k R(-\varepsilon, A)^k + d_L, \tag{37}$$

for $c_k \in X$, $d_L \in D(A^{L+1})$.

THEOREM 3.1. *Let $(S(t))_{t\in R}$ be once integrated, exponentially bounded group on a Banach space X, with generator A, which satisfies the condition (*). Let M and ω are positive real constants for which $\|S(t)\| \leqslant Me^{\omega|t|}$ for every $t \in R$. Let $\varphi(x) \in C_0^\infty(X)$, $f(x) \in C^\infty(R^n)$, and $f(x)$ on $\operatorname{supp}\varphi$ has exactly one nondegenerate stationary point x_0. Then, for every complex number ε, with $|\operatorname{Re}\varepsilon| > \omega$, the following decomposition holds*

$$\begin{aligned}\Phi(A) &= A\int_{R^n} S(f(x))\varphi(x)\,dx + \int_{R^n}\varphi(x)\,dx \\ &= [AS(f(x_0)) + I]\prod_{r=1}^{n}\left[R(\pm\varepsilon, A)^{1/2}\sum_{k_r=0}^{L_r} b_{k_r}R(\pm\varepsilon, A)^{k_r} + a_{L_r}\right],\end{aligned} \tag{38}$$

where $b_{k_r} \in X$, $a_{L_r} \in D(A^{L_r}+1)$, $L_r \in N \cup \{0\}$ for $r = 1, 2, \dots, n$.

PROOF. We will use mathematical induction, Morse's lemma, and next two formulas proved in Lemmas 3.5 and 3.6.

For every complex number ε with $\operatorname{Re}\varepsilon > \omega$, and every integer $L \geqslant 0$ one has

$$A\int_{-\infty}^{\infty} S(x^2)\varphi(x)\,dx + \int_{-\infty}^{\infty}\varphi(x)\,dx = R(\varepsilon, A)^{1/2}\sum_{k=0}^{L} b_k R(\varepsilon, A)^k + a_L \tag{39}$$

for $b_k \in X$, $a_L \in D(A^{L+1})$.

For every complex number ε with $\operatorname{Re}\varepsilon < -\omega$, and every integer $L \geqslant 0$,

$$A\int_{-\infty}^{\infty} S(-x^2)\varphi(x)\,dx + \int_{-\infty}^{\infty}\varphi(x)\,dx = R(-\varepsilon, A)^{1/2}\sum_{k=0}^{L} c_k R(-\varepsilon, A)^k + d_L \tag{40}$$

for $c_k \in X$, $d_L \in D(A^{L+1})$.

By Morse's lemma, in some neighborhood of x_0, there exists a smooth mapping $x = g(y)$ such that $f(x) = f(g(y)) = f(x_0) + \frac{1}{2}\sum_{i=1}^{n}\mu_i y_i^2$, where μ_i are the eigenvalues of the matrix $f''_{xx}(x_0)$ and $\det g'(0) = 1$. Denote $a_i := \frac{1}{2}\mu_i$ $(i = 1, 2, \dots, n)$.

(i) Check the assertion of the theorem first of all for $n = 1$. By Morse's lemma, in some neighborhood of the point x_0, there exists a smooth mapping $x = g(y_1)$

such that $f(x) = f(x_0) + a_1 y_1^2$. Then $\varphi(x)\,dx = (\varphi \circ g)(y_1)g'(y_1)\,dy_1 = \bar{\varphi}(y_1)\,dy_1$, where $\bar{\varphi}(y_1) = (\varphi \circ g)(y_1)g'(y_1)$. Therefore,

$$\Phi(A) = A\int_{R^1} S(f(x))\varphi(x)\,dx + \int_{R^1} \varphi(x)\,dx \tag{41}$$
$$= A\int_{R^1} S(f(x_0) + a_1 y_1^2)\bar{\varphi}(y_1)\,dy_1 + \int_{R^1} \bar{\varphi}(y_1)\,dy.$$

Definition 2.2 implies that for once integrated group $(S(t))_{t\in R}$ it holds

$$S(t)S(s) = \int_0^s [S(t+u) - S(u)]\,du \quad (t, s \in R).$$

If we work from the left with the operator A, then we obtain

$$AS(t)S(s) = S(t+s) - S(t) - S(s).$$

Hence,

$$S(t+s) = AS(t)S(s) + S(t) + S(s) \quad (t, s \in R) \tag{42}$$

From (41) and (42) we get

$$\Phi(A) = A\int_{R^1} \left[AS(f(x_0))S(a_1 y_1^2) + S(f(x_0)) + S(a_1 y_1^2)\right]\bar{\varphi}(y_1)\,dy_1 + \int_{R^1} \bar{\varphi}(y_1)\,dy_1$$
$$= [AS(f(x_0)) + I]\left[A\int_{R^1} S(a_1 y_1^2)\bar{\varphi}(y_1)\,dy_1 + \int_{R^1} \bar{\varphi}(y_1)\,dy_1\right].$$

We use now the relation (39) or (40), depending on the sign of a_1, and we obtain

$$\Phi(A) = [AS(f(x_0)) + I]\left[R(\pm\varepsilon, A)^{1/2}\sum_{k=0}^{L} b_k R(\pm\varepsilon, A)^k + a_L\right].$$

Hence, the relation (38) holds for $n = 1$.

(ii) Assume that (38) holds for $\Phi(A)$ on R^{n-1}.

(iii) Prove that the decomposition (38) holds on R^n, too. Let

$$y = (y_1, y_2, \ldots, y_n) = (\bar{y}, y_n), \text{ where } \bar{y} = (y_1, y_2, \ldots, y_{n-1}).$$

Using Morse's lemma, we get

$$\Phi(A) = A\int_{R^n} S(f(x))\varphi(x)\,dx + \int_{R^n} \varphi(x)\,dx$$
$$= A\int_{R^n} S(u(\bar{y}) + a_n y_n^2)\bar{\varphi}(\bar{y}, y_n)\,dy + \int_{R^n} \bar{\varphi}(\bar{y}, y_n)\,dy,$$

where $u(\bar{y}) = f(x_0) + a_1 y_1^2 + \cdots + a_{n-1} y_{n-1}^2$. Now using the formula (42) we obtain

$$\Phi(A) = A \int_{R^n} \left[AS(u(\bar{y}))S(a_n y_n^2) + S(u(\bar{y})) + S(a_n y_n^2) \right] \bar{\varphi}(\bar{y}, y_n)\, d\bar{y}\, dy_n + \int_{R^n} \bar{\varphi}(\bar{y}, y_n)\, d\bar{y}\, dy_n.$$

Using the assumption (ii) we conclude that (38) holds on R^n. □

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