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GOOD DECOMPOSITION IN THE CLASS OF CONVEX FUNCTIONS OF HIGHER ORDER

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Abstract. The problems investigated in this article are connected to the fact that the class of slowly varying functions is not closed with respect to the operation of subtraction. We study the class of functions $\mathcal{F}_{k-1}$, which are nonnegative and i-convex for $0 \leqslant i < k$, where k is a positive integer. We present necessary and sufficient condition that guarantee that, no matter how we decompose an additively slowly varying function $L \in \mathcal{F}_{k-1}$ into a sum $L = F + G$, $F, G \in \mathcal{F}_{k-1}$, then necessarily F and G are additively slowly varying.

1. Introduction

A positive measurable function $l : [a, +\infty) \to \mathbb{R}^+$, is *slowly varying* if, for every $s > 0$,

$$\frac{l(st)}{l(t)} \to 1, \quad t \to +\infty.$$

Karamata introduced this class of functions in [4] and proved the basic properties; see [1] and [7] for subsequent development of this theory and its applications. A measurable function $L : [a, +\infty) \to \mathbb{R}^+$, is *additively slowly varying* if, for every $h \in \mathbb{R}$,

$$\frac{L(x+h)}{L(x)} \to 1, \quad x \to +\infty.$$

In order to distinguish between the two introduced classes of slowly varying functions, we shall call functions from the first class *multiplicatively slowly varying* functions. It is not difficult to see that the following relationship exists between the two classes of functions: every additively slowly varying function L can be represented as $L = l \circ \exp$, where l is a multiplicatively slowly varying function, and conversely, if a function L can be represented as $L = l \circ \exp$, where l is a multiplicatively slowly varying function, then it is additively slowly varying.

The classes of multiplicatively and additively slowly varying functions are closed with respect to the arithmetic operations: addition, multiplication and division, but

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not with respect to subtraction. The question whether a linear combination of two slowly varying functions is again slowly varying appears in probability theory and in differential equations with the regularly varying coefficients. Shimura (see [8] and [9]) investigated related problems connected to probability theory, and Maric [5] mentioned this question, related to differential equations. In connection with the fact that the class of slowly varying functions is not closed under subtraction, the following general problem appears:

Let $\mathcal{F}$ be a class of functions mapping $[a, +\infty)$ into $\mathbb{R}^+$, which is closed under the operation of addition. Find a subclass of the class $\mathcal{F}$, consisting of functions with the following property: no matter how we represent this function as a sum of two functions from $\mathcal{F}$, both summands are necessarily slowly varying (either in the multiplicative, or in the additive sense).

For a function that has the property above we say that it has *the property of good decomposition in the class $\mathcal{F}$.*

In the article [2], we studied the following problem: given a nondecreasing to infinity slowly varying function f, find conditions under which, no mater how we decompose f into a sum of two nondecreasing functions $f = f_1 + f_2$, f_1 and f_2 are necessarily slowly varying. In this case, the class $\mathcal{F}_0$, consisting of positive, nondecreasing to infinity functions, was investigated. It was proved in [2] that the necessary and sufficient condition for f to have the property of good decomposition in the multiplicative sense in the class $\mathcal{F}_0$ is that, for all $s > 1$,

$$f(st) - f(t) = O(1), \quad t \to +\infty.$$

The class of positive and nondecreasing to infinity functions satisfying the condition above was denoted $\mathrm{O}\Pi^+$. From the mentioned relationship between multiplicatively and additively slowly varying slowly varying functions, it is easy to deduce that a necessary and sufficient condition for a function F to have the property of good decomposition in the additive sense in the class $\mathcal{F}_0$, is the following one: for every $h \in \mathbb{R}$,

$$F(x+h) - F(x) = O(1) \quad x \to +\infty. \tag{1.1}$$

The class of positive and nondecreasing to infinity functions, satisfying (1.1), we denote by $\mathrm{O}\Pi_1^+$.

In [3], the class $\mathcal{F}_1$ consisted of positive, increasing and convex functions. It was proved there that a function $F \in \mathcal{F}_1$ has the property of good decomposition in the additive sense in the class $\mathcal{F}_1$ if and only if the following condition is satisfied: for every $h \in \mathbb{R}$,

$$F(x+2h) - 2F(x+h) + F(x) = O(x), \quad x \to +\infty. \tag{1.2}$$

The class of positive, increasing and convex functions, defined by the condition (1.2) is denoted by $\mathrm{O}\Pi_2^+$. The classes $\mathrm{O}\Pi_1^+$ and $\mathrm{O}\Pi_2^+$ are subclasses of the class of additively slowly varying functions.

In the sequel we investigate the property of good decomposition for the class $\mathcal{F}_{k-1}$ of positive, nondecreasing functions, which are i-convex, $i = 0, 1, \dots, k-1$, $k \geqslant 1$.

2. Convex functions of higher order

In this section we summarize some facts about convex functions of higher order, which will be used in the sequel, and can be found in [6]. Apart from this, we also prove a theorem concerning the remainder term in Taylor formula for $(k-1)$-convex functions.

Let $F : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, and let $x_0, x_1, \dots x_n$ be points from D. The *divided differences* of the function F are defined recurrently in the following way: the divided difference of order 1 at the point $x_0 \in D$ is the function in x defined by

$$[x_0, x; F] = \frac{F(x) - F(x_0)}{x - x_0}$$

on the set $D \setminus \{x_0\}$. The divided difference of order $k+1$ at different points $x_0, x_1, \dots x_k \in D$ is the function in x defined by

$$[x_0, x_1, \dots x_k, x; F] = \frac{[x_0, x_1, \dots x_{k-1}, x; F] - [x_0, x_1, \dots x_{k-1}, x_k; F]}{x - x_k},$$

on the set $D \setminus \{x_0, x_1, \dots x_k\}$. If we see the divided difference of order k as a function of $k+1$ variables, then this function is symmetric.

Monotonicity properties of the function F can be expressed in terms of divided differences. We have that the function F is nondecreasing if and only if its divided difference of the first order is nonnegative. The function F is convex if and only if its divided difference of the first order is monotone nondecreasing in each variable. Actually, since the divided difference is a symmetric function, it is enough to assume that the monotonicity holds for one variable only. From the above facts it follows that a function is convex if and only if its divided difference of second order is nonnegative. This inspired the idea to define convex functions of higher order in the following way: a function $F : D \to \mathbb{R}$, $D \subset \mathbb{R}$, is said to be $(k-1)$-*convex* if its divided difference of order k is nonnegative. Accordingly, 0-convex functions are nondecreasing functions, and 1-convex functions are classical convex functions.

If F is a $(k-1)$-convex function ($k \geqslant 2$), defined on an open interval I, then it is continuous, it has continuous derivatives of orders $1, 2, \dots, k-2$ (if $k \geqslant 3$), and it has the left and the right derivatives of order $k-1$. Also, its i-th derivative is a $(k-i-1)$-convex function, for $0 \leqslant i \leqslant k-1$.

If F is $(k-1)$-convex, defined on an open interval I, and if $a, b \in I$, then Newton–Leibniz formula holds:

$$\int_a^b F^{(i)}(x)dx = F^{(i-1)}(b) - F^{(i-1)}(a),$$

for $1 \leqslant i \leqslant k-1$. For $i < k-1$, the formula above follows from a theorem which can be found in every textbook on calculus, and for $i = k-1$ this formula follows from Theorem A, Section 1.2, in [6]. Here $F^{(k-1)}$ - the $(k-1)$-th derivative of F is either the left, or the right first derivative of the convex function $F^{(k-2)}$.

THEOREM 2.1. *Let $F : I \to \mathbb{R}$, where I is an open interval on the real line, be a $(k-1)$-convex function, $k > 1$. Let x, $x+h$ be points from I. Then $(k-1)$-th remainder term in Taylor formula*

$$\text{(2.1)}\quad \mathcal{R}_h^{k-1}F(x)=F(x+h)-F(x)-hF'(x)-\frac{h^2}{2!}F''(x)-\dots-\frac{h^{k-1}}{(k-1)!}F^{(k-1)}(x)$$

can be written as

$$\mathcal{R}_h^{k-1}F(x)=\int_0^h\frac{(h-t)^{k-2}}{(k-2)!}\Big(F^{(k-1)}(x+t)-F^{(k-1)}(x)\Big)\,dt. \tag{2.2}$$

PROOF. First we prove by induction in s that the remainder of order s in Taylor formula

$$\mathcal{R}_h^{s}F(x)=F(x+h)-F(x)-hF'(x)-\frac{h^2}{2!}F''(x)-\dots-\frac{h^{s}}{s!}F^{(s)}(x)$$

can be represented as

$$\mathcal{R}_h^{s}F(x)=\int_0^h\frac{(h-t)^{s}}{s!}F^{(s+1)}(x+t)\,dt, \tag{2.3}$$

for $0\leqslant s\leqslant k-2$. For $s=0$ we have

$$\mathcal{R}_h^{0}F(x)=F(x+h)-F(x)=\int_0^h F'(x+t)\,dt=\int_0^h\frac{(h-t)^0}{0!}F^{(0+1)}(x+t)\,dt.$$

Suppose that $0<s\leqslant k-2$ and suppose that the formula obtained when in (2.3) we replace s by sa $s-1$ is true. We have to prove that the formula (2.3) is correct. Since, by the induction hypothesis,

$$\mathcal{R}_h^{s-1}F(x)=\int_0^h\frac{(h-t)^{s-1}}{(s-1)!}F^{(s)}(x+t)\,dt,$$

and since

$$\mathcal{R}_h^{s}F(x)=\mathcal{R}_h^{s-1}F(x)-\frac{h^s}{s!}F^{(s)}(x), \tag{2.4}$$

we have

$$\begin{aligned}\mathcal{R}_h^{s}F(x)&=\int_0^h\frac{(h-t)^{s-1}}{(s-1)!}F^{(s)}(x+t)\,dt-\frac{h^s}{s!}F^{(s)}(x)\\&=-\frac{(h-t)^s}{s!}F^{(s)}(x+t)\Big|_{t=0}^{t=h}+\int_0^h\frac{(h-t)^{s}}{s!}F^{(s+1)}(x+t)\,dt-\frac{h^s}{s!}F^{(s)}(x)\\&=\frac{h^s}{s!}F^{(s)}(x)+\int_0^h\frac{(h-t)^{s}}{s!}F^{(s+1)}(x+t)\,dt-\frac{h^s}{s!}F^{(s)}(x)\\&=\int_0^h\frac{(h-t)^{s}}{s!}F^{(s+1)}(x+t)\,dt,\end{aligned}$$

which had to be proved. From (2.4), (2.3) and

$$\int_0^h\frac{(h-t)^{k-2}}{(k-2)!}dt=\frac{(h-t)^{k-1}}{(k-1)!}\Big|_{t=0}^{t=h}=\frac{h^{k-1}}{(k-1)!},$$

we obtain that

$$\begin{aligned}\mathcal{R}_h^{k-1}F(x) &= \mathcal{R}_h^{k-2}F(x) - \frac{h^{k-1}}{(k-1)!}F^{(k-1)}(x)\\ &= \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}F^{(k-1)}(x+t)\,dt - \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}F^{(k-1)}(x)\,dt\\ &= \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}\Big(F^{(k-1)}(x+t) - F^{(k-1)}(x)\Big)\,dt,\end{aligned}$$

which ends the proof. □

3. The class $\mathbf{O\Pi_k^+}$

The conditions (1.1) and (1.2) put restriction on the growth of functions. It is natural to expect that, since the class $\mathcal{F}_{k-1}$ consists of functions satisfying convexity conditions of higher order, the restriction on the growth of functions in order to obtain necessary and sufficient conditions for the property of good decomposition will be expressed in terms of differences of higher order.

Let $F : [a, +\infty) \to \mathbb{R}$ and let $h > 0$. The difference of the function F with the step h is defined by $\Delta_h F(x) = F(x+h) - F(x)$. As usual, $\Delta_h^k = \Delta_h \circ \Delta_h \cdots \circ \Delta_h$, where Δ_h appears k times on the right-hand side. Usually, in the case when $h = 1$, the index h can be omitted; in particular, $\Delta_h = \Delta$

In the sequel we shall deal with functions defined on the half-line $[a, +\infty)$, which are positive, nondecreasing and i-convex, for $i = 0, 1, \ldots, k-1$, $k \geqslant 1$. This will be the class of functions $\mathcal{F}_{k-1}$, on which we shall further investigate the property of good decomposition.

Definition 3.1. If k is a positive integer, $O\Pi_k^+$ is the class of positive nondecreasing real functions, defined on the infinite interval $[a, +\infty)$, for some $a > 0$, which are i-convex, for $i = 0, 1, \ldots, k-1$, and satisfy the following condition:

$$\Delta_h^k F(x) = O\left(x^{k-1}\right) \quad x \to +\infty \tag{3.1}$$

for every $h \in \mathbb{R}^+$.

Obviously, (1.1) and (1.2) are special cases of (3.1). It will be proved that the condition (3.1) is sufficient to guarantee that functions from $O\Pi_k^+$ are additively slowly varying.

Lemma 3.1. *For any fixed integer $k \geqslant 1$, the solution of the difference equation*

$$\Delta^k y_n = b_n, \quad n = 0, 1, 2, \ldots \tag{3.2}$$

is given by

$$y_n = \sum_{i=0}^{k-1} \binom{n}{i-s} \Delta^i y_0 + \sum_{j=0}^{n-1} \binom{n-j-1}{k-1} b_j. \tag{3.3}$$

Proof. The solution (3.3) of the difference equation (3.2) can be obtained by using the procedure of reducing the order of the difference equation. This procedure is based on the fact that the solution of the difference equation

$$\Delta z_n = c_n, \quad n = 0, 1, 2, \dots \tag{3.4}$$

is given by

$$z_n = z_0 + \sum_{r=0}^{n-1} c_r. \tag{3.5}$$

The formula (3.5) can be proved in the following way:

$$z_n = z_0 + \sum_{r=0}^{n-1}(z_{r+1} - z_r) = z_0 + \sum_{r=0}^{n-1} \Delta z_r = z_0 + \sum_{r=0}^{n-1} c_r.$$

Coming back to the equation (3.2), we have that, since the order of the difference equation (3.2) is equal to k, the procedure of reducing the order has to be applied k times. This way we obtain, consecutively, formulas in which

$$\Delta^{k-1} y_n,\ \Delta^{k-2} y_n,\ \dots\ \Delta^0 y_n (= y_n) \tag{3.6}$$

are expressed through members of the sequence b_n. By computing several members of the sequence (3.6), one can verify that the formula

$$\Delta^s y_n = \sum_{i=0}^{k-1} \binom{n}{i-s} \Delta^i y_0 + \sum_{j=0}^{n-1} \binom{n-j-1}{k-s-1} b_j + \binom{0}{k-s} b_n, \tag{3.7}$$

is valid for them. That the formula (3.7) is indeed correct can be proved using the method of mathematical induction in s, $s = k,\ k-1,\ \dots\ 0$.

For $s = k$ the correctness of the formula (3.7) follows from the fact that in this case it reduces to (3.2). Suppose that (3.7) is valid for some integer s, $k \geqslant s > 0$, and let us prove that it is valid for the integer $s - 1$. Put

$$z_n = \Delta^{s-1} y_n, \quad c_n = \sum_{i=0}^{k-1} \binom{n}{i-s} \Delta^i y_0 + \sum_{j=0}^{n-1} \binom{n-j-1}{k-s-1} b_j + \binom{0}{k-s} b_n.$$

Since $\Delta^s y_n = \Delta\Delta^{s-1} y_n = \Delta z_n$, the inductive hypothesis reduces to

$$\Delta z_n = c_n, n = 0, 1, 2, \dots$$

By applying the formula (4) we obtain

$$\begin{aligned}
\Delta^{s-1} y_n &= z_n = z_0 + \sum_{r=0}^{n-1} c_r \\
&= \Delta^{s-1} y_0 + \sum_{r=0}^{n-1} \left(\sum_{i=0}^{k-1} \binom{r}{i-s} \Delta^i y_0 + \sum_{j=0}^{r-1} \binom{r-j-1}{k-s-1} b_j + \binom{0}{k-s} b_r \right) \\
&= \Delta^{s-1} y_0 + \sum_{i=0}^{k-1} \left(\sum_{r=0}^{n-1} \binom{r}{i-s} \right) \Delta^i y_0 + \sum_{j=0}^{n-1} \left(\sum_{r=j+1}^{n-1} \binom{r-j-1}{k-s-1} \right) b_j + \sum_{r=0}^{n-1} \binom{0}{k-s} b_r.
\end{aligned}$$

Since

$$\sum_{r=0}^{n-1}\binom{r}{i-s}=\sum_{r=0}^{n-1}\left(\binom{r+1}{i-s+1}-\binom{r}{i-s+1}\right)=\binom{n}{i-s+1}-\binom{0}{i-s+1},$$

$$\sum_{r=j+1}^{n-1}\binom{r-j-1}{k-s-1}=\sum_{r=j+1}^{n-1}\left(\binom{r-j}{k-s}-\binom{r-j-1}{k-s}\right)=\binom{n-j-1}{k-s}-\binom{0}{k-s},$$

$$\binom{0}{i-s+1}=\begin{cases}1, & i=s-1\\ 0, & i\neq s-1\end{cases},$$

we have

$$\begin{aligned}\Delta^{s-1}y_n&=\Delta^{s-1}y_0+\sum_{i=0}^{k-1}\binom{n}{i-s+1}\Delta^i y_0-\Delta^{s-1}y_0\\&\quad+\sum_{j=0}^{n-1}\left(\binom{n-j-1}{k-s}-\binom{0}{k-s}\right)b_j+\sum_{r=0}^{n-1}\binom{0}{k-s}b_r\\&=\sum_{i=0}^{k-1}\binom{n}{i-s+1}\Delta^i y_0+\sum_{j=0}^{n-1}\binom{n-j-1}{k-s}b_j\\&=\sum_{i=0}^{k-1}\binom{n}{i-s+1}\Delta^i y_0+\sum_{j=0}^{n-1}\binom{n-j-1}{k-s}b_j+\binom{0}{k-s+1}b_n,\end{aligned}$$

which finishes the proof. □

Remark. We used that the value of the binomial coefficient $\binom{a}{m}=0$ if m is a negative integer. In this way the binomial coefficient $\binom{a}{m}$ is defined for every integer m, while the basic recurrent relation for binomial coefficients $\binom{a}{m}+\binom{a}{m+1}=\binom{a+1}{m+1}$ is preserved.

Theorem 3.1. *If $F\in \mathrm{O}\Pi_k^+$, $k\geqslant 1$, then F is additively slowly varying.*

Proof. Let $[a,+\infty)$ be the interval on which the function F is defined. It is sufficient to prove that $y_{n+1}/y_n\to 1$, as $n\to\infty$, where $y_n=F(a+n)$, $n=0,1,2,\dots$. Really, if for $x\geqslant a$ we denote by n the integer part of $x-a$, then we have since F is nondecreasing,

$$1\leqslant\frac{F(x+1)}{F(x)}\leqslant\frac{y_{n+2}}{y_n}=\frac{y_{n+2}}{y_{n+1}}\cdot\frac{y_{n+1}}{y_n}\to 1,\quad x\to+\infty,$$

(here n depends on x, and $n\to\infty$ as $x\to+\infty$), and by an additive form of Lemma 1.15 of Seneta [7] this is sufficient to deduce the additive slow variation of F.

Define the sequence $b_n:=\Delta^k y_n$. Condition (3.1) gives $b_n=a_n n^{k-1}$, where a_n is a bounded sequence of nonnegative real numbers. Using Lemma 3.1 we obtain

$$\Delta y_n=\sum_{i=1}^{k-1}\binom{n}{i-1}\Delta^i y_0+\sum_{j=0}^{n-1}\binom{n-1-j}{k-2}b_j\quad(:=A_n+B_n),$$

$$y_n = \sum_{i=0}^{k-1} \binom{n}{i} \Delta^i y_0 + \sum_{j=0}^{n-1} \binom{n-1-j}{k-1} b_j \quad (:= C_n + D_n).$$

It is enough to prove that

$$\frac{\Delta y_n}{y_n} = \frac{y_{n+1} - y_n}{y_n} \to 0, \quad n \to \infty,$$

i.e. that

$$\frac{A_n + B_n}{C_n + D_n} \to 0, \quad n \to \infty.$$

Since

$$\frac{A_n + B_n}{C_n + D_n} \leqslant \max\left\{ \frac{A_n}{C_n}, \frac{B_n}{D_n} \right\},$$

we shall do it by proving that

$$\frac{A_n}{C_n}, \frac{B_n}{D_n} \to 0, \quad n \to \infty$$

Since A_n and C_n are polynomials in n, and $\deg(A_n) < \deg(C_n)$, we have that $A_n/C_n \to 0$, $n \to \infty$. So it remains to prove that $B_n/D_n \to 0$, $n \to \infty$. We shall prove that, for every positive integer m, the inequality

$$\frac{B_n}{D_n} \leqslant \frac{1}{m},$$

holds for n large enough. The inequality above is equivalent to $D_n - mB_n \geqslant 0$, which, in turn is equivalent to

$$\sum_{j=0}^{n-1} b_j \binom{n-1-j}{k-2} \frac{n-k+1-km+m-j}{k-1} \geqslant 0. \tag{3.8}$$

We shall split the sum from (3.8) into two sums

$$\sum_{j=0}^{n-k-mk+m} b_j \binom{n-1-j}{k-2} \frac{n-k+1-km+m-j}{k-1}$$
$$- \sum_{j=n-k-mk+m+2}^{n-k+1} b_j \binom{n-1-j}{k-2} \frac{j-n+k-1+km-m}{k-1} \quad (:= S_n - T_n),$$

and we only have to prove that $S_n \geqslant T_n$ for n large enough. The summands from (3.8), which do not appear in either S_n or T_n, are equal to zero. The summands from S_n and T_n are nonnegative. We have

$$S_n \geqslant \sum_{j=1}^{n-k-mk+m} a_j j^{k-1} \binom{n-1-j}{k-2} \frac{n-k+1-km+m-j}{k-1}.$$

We can write the coefficient in the preceding sum in the form

$$j^{k-1} \binom{n-1-j}{k-2} \frac{n-k+1-km+m-j}{k-1} =$$

$$= j^{k-1}\frac{(n-1-j)(n-2-j)\cdots(n-k+2-j)}{(k-2)!}\frac{n-k+1-km+m-j}{k-1}$$
$$= \frac{1}{(k-1)!}[j(n-1-j)][j(n-2-j)]\cdots[j(n-k+2-j)][j(n-k+1-km+m-j)].$$

Every term in the square brackets achieves its minimum in j, $1 \leqslant j \leqslant n-k-mk+m$, in the case when $j=1$. Having that in mind, we obtain

$$S_n \geqslant \binom{n-2}{k-2}\frac{n-k-km+m}{k-1}\sum_{j=1}^{n-k-mk+m} a_j.$$

We have

$$T_n = \sum_{j=n-k-mk+m+2}^{n-k+1} a_j j^{k-1}\binom{n-1-j}{k-2}\frac{j-n+k-1+km-m}{k-1}.$$

As

$$j^{k-1}\binom{n-1-j}{k-2}\frac{j-n+k-1+km-m}{k-1} \leqslant (n-k+1)^{k-1}\binom{mk-m+k-3}{k-2}m,$$

for $n-k-mk+m+2 \leqslant j \leqslant n-k+1$, we obtain

$$T_n \leqslant (n-k+1)^{k-1}\binom{mk-m+k-3}{k-2}m \sum_{j=n-k-mk+m+2}^{n-k+1} a_j.$$

We have

$$\sum_{j=1}^{n-k-mk+m} a_j \succ\!\succ \sum_{j=n-k-mk+m+2}^{n-k+1} a_j.$$

Really, if the series $\sum_{j=1}^{\infty} a_j$ converges, then the left-hand side of the preceding formula has a positive limit and the right-hand side tends to zero. And if this series diverges, then the left-hand side tends to plus infinity and the right-hand side is bounded, because the sequence a_n is bounded and the number of summands on the right-hand side $mk-m-1$ does not depend on n. We have also

$$\binom{n-2}{k-2}\frac{n-k-km+m}{k-1} \asymp (n-k+1)^{k-1}\binom{mk-m+k-3}{k-2}m \quad (\asymp n^{k-1}).$$

From these two facts and from inequalities for S_n and T_n proven above, it follows that $S_n \geqslant T_n$ for n large enough. □

4. The property of good decomposition in the class $\mathcal{F}_{k-1}$

Let $\mathcal{F}_{k-1}$ be a class of positive, nondecreasing functions, defined on the interval $[a,+\infty)$, which are i-convex for $i=0,1,\ldots,k-1$, $k \geqslant 1$. In the sequel we shall prove that the class $\mathrm{O\Pi}_k^+$ is the class of functions having the property of good decomposition in the additive sense in the class $\mathcal{F}_{k-1}$.

First we need the following lemma which will establish the relationship between the condition (3.1) with a condition on the operator

$$\mathcal{R}_h^{k-1}F(x) = F(x+h) - F(x) - hF'(x) - \frac{h^2}{2!}F''(x) - \cdots - \frac{h^{k-1}}{(k-1)!}F^{(k-1)}(x).$$

LEMMA 4.1. *Let F be a positive $(k-1)$-convex function, $k \geqslant 1$. If*

$$(\forall h > 0) \quad \mathcal{R}_h^{k-1}F(x) = O\left(x^{k-1}\right), \quad x \to +\infty, \tag{4.1}$$

then the condition (3.1) *is valid too.*

PROOF. Since $F^{(k-1)}$ is a nondecreasing function, we obtain from (2.2)

$$\begin{aligned} \mathcal{R}_h^{k-1}F(x) &\leqslant \Big(F^{(k-1)}(x+h) - F^{(k-1)}(x)\Big) \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}\,dt \\ &= \frac{h^{k-1}}{(k-1)!}\,\Delta_h F^{(k-1)}(x), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \mathcal{R}_h^{k-1}F(x) &\geqslant \int_{h/2}^h \frac{(h-t)^{k-2}}{(k-2)!} \Big(F^{(k-1)}(x+t) - F^{(k-1)}(x)\Big)\,dt \\ &\geqslant \Big(F^{(k-1)}(x+h/2) - F^{(k-1)}(x)\Big) \int_{h/2}^h \frac{(h-t)^{k-2}}{(k-2)!}\,dt \\ &= \frac{h^{k-1}}{2^{k-1}(k-1)!}\,\Delta_{h/2}F^{(k-1)}. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) we obtain that the condition (4.1) is equivalent to the condition

$$(\forall h > 0) \quad \Delta_h F^{(k-1)}(x) = O\left(x^{k-1}\right), \quad x \to +\infty. \tag{4.4}$$

We shall prove by induction in i that conditions

$$(\forall h > 0) \quad \Delta_h^i F^{(k-i)}(x) = O\left(x^{k-1}\right), \quad x \to +\infty, \tag{4.5}$$

hold, for $i = 1, 2, \dots, k$. For $i = 1$, the condition (4.5) reduces to (4.4). Suppose that $1 \leqslant i < k$ and that (4.5) is satisfied. Bearing in mind that

$$\begin{aligned} \Delta_h^{i+1}F^{(k-i-1)}(x) &= \Delta_h^i \Delta_h F^{(k-i-1)}(x) = \\ &= \Delta_h^i \int_0^h F^{(k-i)}(x+t)\,dt = \int_0^h \Delta_h^i F^{(k-i)}(x+t)\,dt, \end{aligned}$$

we conclude that the condition (4.5) holds also in the case when we substitute i by $i+1$ in it. The induction is done.

If we put $i = k$ in (4.5), we obtain the condition (3.1) from the definition of the class $\mathrm{O\Pi}_k^+$. □

THEOREM 4.1. *A function L has the property of good decomposition in the additive sense in the class $\mathcal{F}_{k-1}$ if and only if it belongs to the class $\mathrm{O\Pi}_k^+$.*

PROOF. Let $L \in \mathrm{O\Pi}_k^+$ and let F, G be functions from $\mathcal{F}^{k-1}$ such that $F+G=L$. For $h>0$, we have that $\Delta_h^k F(x)+\Delta_h^k G(x)=\Delta_h^k L(x)$. Since F and G are $(k-1)$-convex, then $\Delta_h^k F(x) \geqslant 0$ and $\Delta_h^k G(x) \geqslant 0$. From this and from $\Delta_h^k L(x)=O(x^{k-1})$, $x \to +\infty$, it follows that the same holds for F and for G, i.e., functions F and G also belong to $\mathrm{O\Pi}_k^+$. According to Theorem 3.1 we conclude that F and G are additively slowly varying.

For the converse, suppose that $L \in \mathcal{F}_{k-1}$, but $L \notin \mathrm{O\Pi}_k^+$. According to Lemma 4.1, it follows that there exists an increasing sequence of points x_n, $x_0=a$, $x_n \to +\infty$ as $n \to \infty$, and $h>0$ such that the following conditions are fulfilled: $x_n > x_{n-1}+h$, and

$$\frac{\mathcal{R}_h^{k-1} L(x_n)}{x_n^{k-1}} \to +\infty, \quad n \to \infty,$$

holds. We select a subsequence of x_n in such a way that when the point x_{n-1} is chosen, we pick x_n which satisfies

$$\begin{aligned}(4.6)\quad L(x_{n-1}) + \frac{L'(x_{n-1})}{1!}(x_n - x_{n-1}) + \frac{L''(x_{n-1})}{2!}(x_n - x_{n-1})^2 + \cdots \\ + \frac{L^{(k-1)}(x_{n-1})}{(k-1)!}(x_n - x_{n-1})^{k-1} < \mathcal{R}_h^{k-1} L(x_n),\end{aligned}$$

for $n = 1, 2, \dots$.

We shall construct functions F and G which belong to $\mathcal{F}_{k-1}$, $F+G=L$, and which are not additively slowly varying. We start this by defining functions f and g in the following way: $f(a)=g(a)=0$ and, for $n=0,1,2,\dots$,

$$\begin{aligned}
f(x) &= f(x_{2n}) + L^{(k-1)}(x) - L^{(k-1)}(x_{2n}), && x \in [x_{2n}, x_{2n+1}]\\
g(x) &= g(x_{2n}), && x \in [x_{2n}, x_{2n+1}]\\
f(x) &= f(x_{2n+1}), && x \in [x_{2n+1}, x_{2n+2}]\\
g(x) &= g(x_{2n+1}) + L^{(k-1)}(x) - L^{(k-1)}(x_{2n+1}), && x \in [x_{2n+1}, x_{2n+2}].
\end{aligned}$$

Functions f and g defined above are nonnegative, nondecreasing and they satisfy $f(x)+g(x)=L^{(k-1)}(x)-L^{(k-1)}(a)$. Put

$$F(x) = L(a) + \frac{L'(a)}{1!}(x-a) + \cdots + \frac{L^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} + \int_a^x \frac{(x-t)^{k-2}}{(k-2)!} f(t)\, dt,$$

$$G(x) = \int_a^x \frac{(x-t)^{k-2}}{(k-2)!} g(t)\, dt.$$

We have

$$\begin{aligned}F(x)+G(x) = L(a) + \frac{L'(a)}{1!}(x-a) + \cdots + \frac{L^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} \\ + \int_a^x \frac{(x-t)^{k-2}}{(k-2)!} \Big(L^{(k-1)}(x) - L^{(k-1)}(a)\Big)\, dt = L(x).\end{aligned}$$

Functions F and G obviously belong to $\mathcal{F}_{k-1}$; it remains to prove that they are not additively slowly varying. We have that

$$\begin{aligned}\mathcal{R}_h^{k-1}F(x_{2n}) &= \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}\Big(F^{(k-1)}(x_{2n}+t)-F^{(k-1)}(x_{2n})\Big)\,dt\\ &= \int_0^h \frac{(h-t)^{k-2}}{(k-2)!}\Big(L^{(k-1)}(x_{2n}+t)-L^{(k-1)}(x_{2n})\Big)\,dt = \mathcal{R}_h^{k-1}L(x_{2n}).\end{aligned}$$

We obtain that

$$F(x_{2n}+h)-F(x_{2n}) \geqslant \mathcal{R}_h^{k-1}F(x_{2n}) = \mathcal{R}_h^{k-1}L(x_{2n}). \tag{4.7}$$

On the other hand, using (2.1) and (2.2), we have

$$\begin{aligned}F(x_{2n}) &= F(x_{2n-1}) + \frac{F'(x_{2n-1})}{1!}(x_{2n}-x_{2n-1})\\ &+ \frac{F''(x_{2n-1})}{2!}(x_{2n}-x_{2n-1})^2+\cdots+\frac{F^{(k-1)}(x_{2n-1})}{(k-1)!}(x_{2n}-x_{2n-1})^{k-1}\\ &+\int_{x_{2n-1}}^{x_{2n}}\frac{(x_{2n-t})^{k-2}}{(k-2)!}\Big(F^{(k-1)}(t)-F^{(k-1)}(x_{2n-1})\Big)\,dt.\end{aligned} \tag{4.8}$$

From the construction of F it follows that, for $t\in[x_{2n-1},x_{2n}]$, $F^{(k-1)}(t)-F^{(k-1)}(x_{2n-1})=0$, so that the integral in the formula (4.8) is equal to zero. Since $F^{(i)}(x)\leqslant L^{(i)}(x)$, for $i=0,1,\dots,k-1$, we have

$$\begin{aligned}F(x_{2n}) &\leqslant L(x_{2n-1}) + \frac{L'(x_{2n-1})}{1!}(x_{2n}-x_{2n-1})\\ &+ \frac{L''(x_{2n-1})}{2!}(x_{2n}-x_{2n-1})^2+\cdots+\frac{L^{(k-1)}(x_{2n-1})}{(k-1)!}(x_{2n}-x_{2n-1})^{k-1}.\end{aligned} \tag{4.9}$$

From (4.7) and (4.9) and by (4.6) we have

$$\frac{F(x_{2n}+h)-F(x_{2n})}{F(x_{2n})} > 1,$$

which proves that F is not additively slowly varying. By similar argument on intervals of the form $[x_{2n+1},x_{2n+2}]$ it can be proved that G is not additively slowly varying either. □

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