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VARIANTS OF KARAMATA'S ITERATION THEOREM

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ABSTRACT. Karamata's Iteration Theorem is used to refine the asymptotic behavior of iterates of a function, under a more restrictive assumption than Karamata's, but still involving regular variation. A second result gives a necessary and sufficient integral condition for convergence of a series of iterates. Historical background to the idea of regularly varying sequence precedes a short concluding section on attribution of a probabilistic result.

1. Introduction

An International Conference on Karamata's Regular Variation was held in Dubrovnik–Kupari, Yugoslavia, June 1–10, 1989. The paper written for that conference by the present author and published with other papers presented there, appeared in the *Publications de l'Institut Mathématique* as [**23**]

I had met Tatjana (Tania) Ostrogorski (20.02.1950 –25.08.2005) in person only once. It was at that Kupari Conference, and I remember walks and pleasant conversations amidst small groups , which included her close friend and colleague Boba Janković, on the esplanade at Kupari. My sporadic contact by email continued over the years, and I had occasion to review for *Mathematical Reviews* several of her papers. Serious contact was reestablished with the proposal to publish an issue of the *Publications* dedicated to Karamata and edited by Tania. My contribution to that was [**24**].

In all I have had 3 publications before the present one in the *Publications.* The first was [**21**]. All 3 have been based on fundamental results due to Karamata, and the last two at least were handled editorially by Tania. In all 3 there is a subtheme of results going back to Cauchy. I thought it appropriate for this occasion to continue in the same vein.

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The present paper is a sequel to [**23**]. That paper showed how Karamata's Iteration Theorem can be used to express necessary and sufficient conditions for a sequence of functional iterates to be a regularly varying sequence of negative index.

In fact, the author's initial interest in regular variation arose out of iteration theory and functional equations, and was in sequential criteria for regular variation and in regularly varying sequences [**18**], [**19**], [**20**], [**3**], [**9**], [**4**], [**22**]. This seems to have given some impetus to continuing work on regular varying sequences, including [**11**], [**12**], [**27**], [**16**], [**8**].

Our main result is the following:

THEOREM 1.1. *Suppose* $0 < f(x) < x$, $0 < x \leqslant x_0$, $x_0 > 0$, *and that* $f(x)$ *is continuous on* $0 < x \leqslant x_0$. *Further suppose that:*

$$f(x) = x - ax^{1+\beta} + b(x)x^{1+\beta+\delta}L(x) \tag{1.1}$$

where $a > 0$, $\beta > \delta > 0$, $L(x)$ *is slowly varying in the neighborhood of* 0, *and* $b(x) \to b \neq 0$ *as* $x \to 0+$. *If* $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \cdots$ *then*

$$x_n = (n\beta a)^{-1/\beta}\Big\{1 + \frac{b(\beta a)^{-\delta/\beta-1}}{(1-\delta/\beta)}\, n^{-\delta/\beta}L(n^{-1/\beta})(1+o(1))\Big\} \tag{1.2}$$

Karamata's Iteration Theorem [**15**], [**23**] replaces (1.1) by

$$f(x) = x - a(x)x^k\mathcal{L}(x) \tag{1.3}$$

where $k > 1$, $\mathcal{L}(x)$ is slowly varying as $x \to 0+$, and $a(x) \to a > 0$, and concludes that as $n \to \infty$

$$x_n = a^* n^{-k^*}\mathcal{L}^*(n^{-1})(1+o(1)) \tag{1.4}$$

where $k^* = 1/(k-1)$, $a^* = (k^*/a)^{k^*}$ and $\mathcal{L}^*(x)$ is slowly varying at 0, x^{k^*} denoting the inverse function of $x^{k-1}\mathcal{L}(x)$ since within a neighborhood of zero $x^{k-1}\mathcal{L}(x)$ can be taken as continuous and strictly monotone increasing without loss of generality.

In our Theorem 1.1 we in effect take

$$a(x) = a, \quad k = 1+\beta, \quad \mathcal{L}(x) = 1 - (b(x)/a)x^\delta L(x),$$

and from Karamata's theorem can conclude, since $k^* = 1/\beta$, $a^* = (1/(a\beta))^{1/\beta}$ that

$$x_n = (n\beta a)^{-1/\beta}(1+o(1)), \; n \to \infty. \tag{1.5}$$

Slightly later than Karamata's paper Szekeres [**26**, pp. 223–224], treated the case where $b(x)x^\delta L(x)$ in (1.1) is replaced by the weaker $o(1)$, $x \to 0+$, to obtain the conclusion (1.5).

Stević [**25**, Theorem 1(c)], has made a detailed study of the case of (1.1) where $L(x) = 1 + o(1)$ and obtained (1.2) in this case.

We shall need (1.5) in our sequel.

Note that our preliminary assumptions on f imply that $x_n \to 0+$, $n \to \infty$.

Notice that a conjugate function L^* does not appear in the refined rate of convergence result (1.2), in contrast to (1.4).

The result (1.2) was announced in somewhat garbled form in the present author's proposed commentary on [**15**] for the book *Selected Papers of J. Karamata* which has yet to appear. Inasmuch as the proof of Theorem 1.1 uses Karamata's

Iteration Theorem, and several fundamental results of the theory of regularly varying functions [**22**], [**1**], the author hopes that it may serve as a suitable tribute to the memory of Tania Ostrogorski, whose research creativity and editorial activity was intimately associated with her countryman Karamata's creation of the theory of regular variation.

We shall also need the following auxiliary result.

LEMMA 1.1. *Suppose $\phi(y)$ is a positive continuous non-increasing function on $m \leqslant y < \infty$, $m > 0$ and suppose*

$$\int_m^\infty \phi(y)\,dy = \infty. \tag{1.6}$$

Then for integer n, as $n \to \infty$

$$\sum_{k=m}^{n} \phi(k) \sim \int_{y=m}^{n} \phi(y)\,dy. \tag{1.7}$$

PROOF. Write

$$A_n = \sum_{k=m}^{n} \phi(k),\ I_n = \int_m^n \phi(y)\,dy.$$

Then, in the manner of establishing Cauchy's Integral Test, $\phi(m) \geqslant A_n - I_n \geqslant \phi(n)$ so

$$1 + \frac{\phi(m)}{I_n} \geqslant \frac{A_n}{I_n} \geqslant 1 + \frac{\phi(n)}{I_n}. \tag{1.8}$$

Now, by the Mean Value Theorem and the non-increasing nature of ϕ

$$I_n = (n-m)\phi(\xi_n) \geqslant (n-m)\phi(n),\ m < \xi_n < n,$$

so that

$$\frac{\phi(n)}{I_n} \leqslant \frac{1}{n-m} \longrightarrow 0,\ n \to \infty.$$

Hence from (1.6) and (1.8), (1.7) follows. □

2. Proof of Theorem 1.1 and consequences

PROOF. First notice that since $b(x) \to b \neq 0$, $x \to 0+$, from the Representation Theorem for slowly varying functions, we can assume without loss of generality that for any $\gamma > 0$, $x^\gamma L(x)$ is strictly decreasing as $x \to 0+$ in the neighborhood of 0 [**22**, pp. 21–23]. Thus putting $\phi(y) = y^{-\delta/\beta} L(y^{-1/\beta})$, there is an m such that for $y \geqslant m$, ϕ is strictly decreasing as y increases. Further, since $0 < \delta < \beta$, it follows that the conditions of Lemma 1.1 are satisfied, and so

$$\sum_{k=m}^{n-1} k^{-\delta/\beta} L(k^{-1/\beta}) \sim \int_{y=m}^{n-1} y^{-\delta/\beta} L(y^{-1/\beta})\,dy$$

as $n \to \infty$. Applying a theorem due to Karamata [**14**] (see [**22**, Exercise 2.1, p. 86]), since $y^{-\delta/\beta}L(y^{-1/\beta})$ is regularly varying at infinity, with index $\rho = -\delta/\beta$, as $w \to \infty$

$$\int_{y=m}^{w} y^{-\delta/\beta} L(y^{-1/\beta})\, dy \sim \frac{w^{1-\delta/\beta}L(w^{-1/\beta})}{(1-\delta/\beta)}.$$

Hence, by the Uniform Convergence Theorem, as $n \to \infty$:

$$\sum_{k=1}^{n-1} k^{-\delta/\beta} L(k^{-1/\beta}) \sim n^{1-\delta/\beta} L(n^{-1/\beta})/(1-\delta/\beta). \tag{2.1}$$

We shall need this shortly.

Now define the sequence $\{\rho_n\}$ recursively by $x_n^{-\beta} = \rho_1 + \rho_2 + \cdots + \rho_n$, $n \geqslant 1$, and the sequence $\{a_n\}$ recursively by

$$a_n = a - b(x_n) x_n^{\delta} L(x_n). \tag{2.2}$$

Notice that since $x_n \to 0+$ as $n \to \infty$, it follows that $b(x_n) \to b$, and since $x^{\delta}L(x) \to 0$, $x \to 0+$, $a_n \to a$, $n \to \infty$. Thus

$$\begin{aligned} \rho_1 &= x_1^{-\beta} \\ \rho_{n+1} &= x_{n+1}^{-\beta} - x_n^{-\beta},\ n \geqslant 1. \end{aligned} \tag{2.3}$$

Then since from (1.1)

$$x_{n+1} = x_n(1 - a_n x_n^{\beta}) \tag{2.4}$$

and $x_n \to 0$ as $n \to \infty$:

$$\begin{aligned} a_n &= \frac{x_n - x_{n+1}}{x_n^{1+\beta}} = \frac{\rho_{n+1} x_{n+1}^{\beta}(x_n - x_{n+1})}{x_n(x_n^{\beta} - x_{n+1}^{\beta})} \\ &= \frac{\rho_{n+1}(1 - a_n x_n^{\beta})^{\beta}(a_n x_n^{\beta})}{(1 - (1 - a_n x_n^{\beta})^{\beta})} = \frac{\rho_{n+1}(1 - \beta a_n x_n^{\beta} + O(x_n^{2\beta}))}{\beta(1 + O(x_n^{\beta}))}. \end{aligned}$$

Therefore from (1.1)

$$\begin{aligned} \rho_{n+1} &= \beta(a - b(x_n)x_n^{\delta}L(x_n))(1 + O(x_n^{\beta})) \\ &= \beta a - \beta\{b(x_n) - b + b\}x_n^{\delta}L(x_n)(1 + O(x_n^{\beta})) + O(x_n^{\beta}) \\ &= \beta a - \beta b(1 + o(1))x_n^{\delta}L(x_n)(1 + O(x_n^{\beta})) + O(x_n^{\beta}) \end{aligned}$$

since $b(x_n) - b \to 0$. Now, using (1.5)

$$\begin{aligned} \rho_{n+1} &= \beta a - \beta b\{(n\beta a)^{-1/\beta}(1 + o(1))\}^{\delta} L((n\beta a)^{-1/\beta}(1 + o(1))) + O(n^{-1}) \\ &= \beta a - \beta b\{(n\beta a)^{-\delta/\beta}(1 + o(1))\} L(n^{-1/\beta})(1 + o(1)) + O(n^{-1}) \end{aligned}$$

by the Uniform Convergence Theorem of regularly varying functions. Thus

$$\beta a - \rho_{n+1} = \beta b (n\beta a)^{-\delta/\beta} L(n^{-1/\beta})(1 + o(1)) \tag{2.5}$$

as $n \to \infty$, since $0 < \delta/\beta < 1$.

Now assume that $b > 0$ (the argument will be similar for $b < 0$). Then using (2.5), for given small $\epsilon > 0$, for $n \geqslant m$ ($= m(\epsilon)$), where m is also sufficiently large to ensure the monotonicity of $\phi(y)$, $y \geqslant m$, as defined at the beginning of the proof:

$$\beta b(n\beta a)^{-\delta/\beta}L(n^{-1/\beta})(1-\epsilon) \leqslant \beta a - \rho_{n+1} \leqslant \beta b(n\beta a)^{-\delta/\beta}L(n^{-1/\beta})(1+\epsilon). \tag{2.6}$$

Using (2.3)

$$\sum_{k=m}^{n-1}(\beta a - \rho_{k+1}) = (n-m)\beta a - \sum_{k=m}^{n-1}\rho_{k+1} = (n-m)\beta a - x_n^{-\beta} + x_m^{-\beta}. \tag{2.7}$$

Write

$$\psi(m,n) = \beta b(\beta a)^{-\delta/\beta}\sum_{k=m}^{n-1}k^{-\delta/\beta}L(k^{-1/\beta}). \tag{2.8}$$

Then from (2.6)–(2.8)

$$\psi(m,n)(1-\epsilon) \leqslant (n-m)\beta a - x_n^{-\beta} + x_m^{-\beta} \leqslant \psi(m,n)(1+\epsilon). \tag{2.9}$$

Now using (2.1) and dividing through (2.9) by $\psi(m,n)$, as $n \to \infty$:

$$\begin{aligned} 1-\epsilon &\leqslant \liminf \frac{n\beta a - x_n^{-\beta}}{\beta b(\beta a)^{-\delta/\beta}n^{1-\delta/\beta}L(n^{-1/\beta})/(1-\delta/\beta)} \\ &\leqslant \limsup \frac{n\beta a - x_n^{-\beta}}{\beta b(\beta a)^{-\delta/\beta}n^{1-\delta/\beta}L(n^{-1/\beta})/(1-\delta/\beta)} \\ &\leqslant 1+\epsilon. \end{aligned}$$

Since ϵ is arbitrary

$$x_n^{-\beta} = n\beta a - \frac{\beta b(\beta a)^{-\delta/\beta}}{1-\delta/\beta}n^{1-\delta/\beta}L(n^{-1/\beta})(1+o(1)). \tag{2.10}$$

It follows that

$$\begin{aligned} x_n &= (n\beta a)^{-1/\beta}\Big\{1 - \frac{\beta b(\beta a)^{-\delta/\beta}}{(\beta a)(1-\delta/\beta)}n^{-\delta/\beta}L(n^{-1/\beta})(1+o(1))\Big\}^{-1/\beta} \\ &= (n\beta a)^{-1/\beta}\Big\{1 + \frac{b(\beta a)^{-\delta/\beta-1}}{(1-\delta/\beta)}n^{-\delta/\beta}L(n^{-1/\beta})(1+o(1))\Big\}. \end{aligned}$$

which is (1.2). □

The conclusion of Theorem 1.1 and the methodology of its proof above make possible a rate of convergence sharpening of the result:

$$n\beta\Big(1 - \frac{x_{n+1}}{x_n}\Big) = 1 + o(1) \quad \text{as} \quad n \to \infty. \tag{2.11}$$

The result (2.11) follows directly from Karamata's Iteration Theorem, and $\{x_n\}$ is a normalized regularly varying sequence [**23**, Section 2]. Further:

THEOREM 2.1. *Under the conditions of Theorem* 1.1,

$$n\beta\Big(1 - \frac{x_{n+1}}{x_n}\Big) = 1 + Kn^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1)) \quad as \quad n \to \infty, \tag{2.12}$$

where K is a constant which is positive multiple of b.

PROOF. From (2.4)

$$x_n^{-\beta}\Big(1 - \frac{x_{n+1}}{x_n}\Big) = a_n = a + (a_n - a) \tag{2.13}$$

from which, using (2.2) on the right hand side:

$$\begin{aligned} &= a - (b(x_n) - b)x_n^{\delta}L(x_n) - bx_n^{\delta}L(x_n) \\ &= a - bx_n^{\delta}L(x_n) + o(x_n^{\delta}L(x_n)). \end{aligned}$$

From (2.13) and (1.2) then:

$$na\beta\Big(1 - \frac{x_{n+1}}{x_n}\Big) = \frac{a - bx_n^{\delta}L(x_n) + o(x_n^{\delta}L(x_n))}{1 - c\beta n^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1))} \tag{2.14}$$

using the Uniform Convergence Theorem, where $c = b(\beta a)^{-\delta/\beta - 1}/(1 - \delta/\beta)$. Now put

$$D_n = 1 - (b/a)x_n^{\delta}L(x_n) + o(x_n^{\delta}L(x_n)), \tag{2.15}$$

so that

$$n\beta\Big(1 - \frac{x_{n+1}}{x_n}\Big) = D_n\big\{1 + c\beta n^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1))\big\}. \tag{2.16}$$

Using (1.2), and the Uniform Convergence Theorem:

$$x_n^{\delta}L(x_n) = (na\beta)^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1)), \tag{2.17}$$

whence from (2.15) and (2.17)

$$D_n = 1 - (b/a)(na\beta)^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1)). \tag{2.18}$$

Consequently, from (2.16) and (2.18)

$$n\beta\Big(1 - \frac{x_{n+1}}{x_n}\Big) = 1 + Kn^{-\delta/\beta}L(n^{-1/\beta})(1 + o(1))$$

where

$$K = c\beta - (\beta a)^{-\delta/\beta}\frac{b}{a} = \frac{b}{a}\,(\beta a)^{-\delta/\beta}\Big(\Big(1 - \frac{\delta}{\beta}\Big)^{-1} - 1\Big) = \delta c. \qquad \square$$

3. Series of iterates

We notice that under the conditions of Karamata's Iteration Theorem, using the conclusion (1.4), that $\sum x_n$ converges if $1 < k < 2$, and diverges if $k > 2$. By modifying the conditions we can obtain a necessary and sufficient condition for this convergence.

It may be of interest to note that under the conditions of Theorem 3.1 below, while still $x_n \downarrow 0$, $n \to \infty$, we have, in contrast to (2.11), only

$$\psi(x_n)(1 - x_{n+1}/x_n) \downarrow 0.$$

THEOREM 3.1. *Suppose* $0 < f(x) < x$, $0 < x \leqslant x_0$, $x_0 > 0$, *and that* $f(x)$ *is continuous on* $0 < x \leqslant x_0$. *Further suppose that* $f(x)/x \uparrow 1$ *as* $x \downarrow 0$, *and that* $f(x)$ *is concave on* $0 < x \leqslant x_0$. *Finally, suppose* $\psi(x)$ *is a positive monotone continuous function on* $0 < x \leqslant x_0$ *such that* $\psi(x) \downarrow$ *as* $x \downarrow$. *If* $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$, *then*

$$\int_0^{x_0} \frac{w}{\psi(w)(w - f(w))}\, dw < \infty \tag{3.1}$$

is necessary and sufficient for

$$\sum_n \frac{x_n}{\psi(x_n)} < \infty. \tag{3.2}$$

PROOF. Write

$$H(x) = \frac{1}{\psi(x)(1 - f(x)/x)},\ 0 < x < x_0.$$

Then $H(x)$ is positive, continuous and decreasing with increasing x on $0 < x < x_0$, and $H(x) \uparrow \infty$, $x \downarrow 0$. Now write

$$I = \int_0^{x_0} H(w)\, dw.$$

Hence in the manner of Cauchy's Integral Test bounds,

$$\sum_{n=0}^{\infty}(x_n - x_{n+1})H(x_n) \leqslant I \leqslant \sum_{n=0}^{\infty}(x_n - x_{n+1})H(x_{n+1}) = \sum_{n=1}^{\infty}(x_{n-1} - x_n)H(x_n)$$

so that

$$\sum_{n=1}^{\infty} \frac{x_n - x_{n+1}}{x_{n-1} - x_n}(x_{n-1} - x_n)H(x_n) \leqslant I \leqslant \sum_{n=1}^{\infty}(x_{n-1} - x_n)H(x_n). \tag{3.3}$$

Now

$$\frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} \leqslant \frac{f(x_n) - f(x_{n+1})}{x_n - x_{n+1}},\ \ n \geqslant 1, \tag{3.4}$$

using the concavity assumption about $f(x)$. Hence from (3.4)and (3.5):

$$\frac{x_1 - x_2}{x_0 - x_1}\sum_{n=1}^{\infty}(x_{n-1} - x_n)H(x_n) \leqslant I \leqslant \sum_{n=1}^{\infty}(x_{n-1} - x_n)H(x_n),$$

so that I and $\sum_{n=1}^{\infty}(x_{n-1} - x_n)H(x_n)$ are finite or infinite together; and hence from (3.3) and (3.4) I and $\sum_{n=0}^{\infty}(x_n - x_{n+1})H(x_n)$ are finite or infinite together. But using the form of $H(x)$:

$$\sum_{n=0}^{\infty}(x_n - x_{n+1})H(x_n) = \sum_{n=0}^{\infty} \frac{x_n}{\psi(x_n)}$$

which completes the proof. □

The choice of $\psi(x) = 1$ in (3.1) and (3.2) gives a necessary and sufficient integral condition for the convergence of $\sum x_n$.

Notice that one of the assumptions of the theorem could be relaxed by merely assuming that $f(x)/x \uparrow$ as $x \downarrow$, with almost no modification of the proof. In this situation, under the other assumptions of the theorem, $\lim_{x\to 0+} f(x)/x = c$, for some constant c, $0 < c \leqslant 1$. If $c < 1$ and $\psi(x) = 1$ then the integral (3.1) always converges. But it is then obvious that x_n decreases to 0 as $n \to \infty$ at a geometric rate c^n.

Theorem 3.1 is relevant even if its assumptions are extended to include Karamata's assumption (1.3), since in the case $k^* = 1$ (equivalently $k = 2$) the series $\sum x_n$, with x_n given by (1.4), may converge or diverge, depending on the nature of the function $\mathcal{L}$.

Thus

$$\sum \frac{1}{n(\log n)^h}$$

diverges for $0 \leqslant h \leqslant 1$ but converges for $h > 1$. Results like this were originally obtained from progressively generalizing the ratio test for convergence of series by using Cauchy's Integral Test. We pass onto this historical topic now.

4. Normalized regularly varying sequences

A sequence of positive-terms $\{\alpha(n)\}$, $n \geqslant 1$, satisfying

$$n\big(1 - \{\alpha(n-1)/\alpha(n)\}\big) \to \rho,\ \rho \text{ finite} \tag{4.1}$$

is called a normalized regularly varying sequence [**9**], [**23**], because of the structural analogy with a property of normalized regularly varying functions.

Now, (4.1) is a manifestation of Raabe's ratio test for convergence of a positive *series* formed from the sequence $\{\alpha(n)\}$, if the (primary) ratio test for convergence gives the result unity (i.e., if $\alpha(n-1)/\alpha(n) \to 1$). Then if (4.1) holds with $\rho > -1$, $\sum_{n=0}^{\infty} \alpha(n) = \infty$, while if $\rho < -1$, $\sum_{n=0}^{\infty} \alpha(n) < \infty$. If $\rho = -1$ in (4.1), in which case no decision on convergence on the basis of Raabe's test is possible, one may proceed to what is called Bertrand's test, but should be called De Morgan's test: if the limit of

$$(\log n)\left\{n\left(\frac{\alpha(n-1)}{\alpha(n)} - 1\right) - 1\right\} \tag{4.2}$$

exists as $n \to \infty$, and is > 1, the series converges; if < 1 the series diverges; and if $= 1$ the case is again indeterminate and one may go onto a more refined ratio test still. Indeed, these tests can be taken as the first three of an infinite sequence of tests, and this theme has previously been developed somewhat differently by Pakes [**16**].

Let $\{u_n\}$, $n \geqslant 0$ be a sequence of positive terms; and define $D_{-1}(n) = u_n/u_{n+1}$; and for $r = 0, 1, 2, \ldots$, and n sufficiently large:

$$D_r(n) = (\log_r n)(D_{r-1}(n) - 1) \tag{4.3}$$

defining $\log_0 n = n$ and $\log_r n$ as the r-th functional iterate of $\log n$. Existence of the limit of $D_{-1}(n)$ allows the application of the ratio test if the limit is $\neq 1$. If the limit is 1, suppose the limit of $D_r(n)$ as $n \to \infty$ exists for $-1 \leqslant r \leqslant r_0$, and is unity for $-1 \leqslant r \leqslant r_0 - 1$, and $\neq 1$ at $r = r_0$. Then the series $\sum u_n$ converges if the limit at $r = r_0$ is > 1 and diverges if < 1. This general rule is due to De Morgan [**7**, p. 326], to whom Buniakovsky [**5**, p. 393], ascribes it in his Notice [*Primechanie*] III and states it according to form (4.3) on his pp. 404–405. This *Primechanie* is on a favorite topic of his, the convergence of series, and takes up pp. 391–410. It has little to do with the main theme of the book, intended as the first monograph in Russian on probability theory.

In his careful expository synthesis of then-recent methodology on tests of convergence of series, with focus on De Morgan's result, Buniakovsky begins with a comparison lemma of Duhamel, which subsequently permits him to compare u_n/u_{n+1} with ν_n/ν_{n+1} of a "simpler" series $\sum \nu_n$, and thereby to deduce convergence or divergence of $\sum u_n$ on the basis of that of $\sum \nu_n$. He also derives Cauchy's Integral Test for an eventually monotone decreasing sequence, which permits him to deduce the result that

$$\sum_n \left\{ \left(\left(\prod_{r=0}^{h-1} \log_r n \right) (\log_h n)^k \right)^{-1} \right\} \tag{4.4}$$

converges for $k > 1$ and diverges for $k \leqslant 1$, for each $h \geqslant 0$, interpreting $\prod_{r=0}^{-1}$ as 1. (We are using modern notation.)

Buniakovsky's is a careful, leisurely, textbook exposition, focussed squarely on *sequences* $\{u_n\}$, in the form

$$\left(\prod_{r=0}^{r_0} \log_r n \right) \left(1 - \frac{u_n}{u_{n+1}} \right) \to (1 + \delta) \tag{4.5}$$

and so can be regarded as not only a forerunner of the theory of *regularly varying sequences* (the case $r_0 = 0$), but also of *very slowly varying sequences* on a logarithmic scale (for a fixed $r_0 \geqslant 1$). De Morgan's [**7**] derivation is very compressed, inviting by its nature a careful subsequent exposition and De Morgan gives no citations, not even the names of earlier authors. Cauchy's Integral Test plays a key role. De Morgan's theme is not primarily sequences but orders of growth of functions, and there is a remarkable resemblance to the later theory of *regularly varying functions*, and indeed to very slowly varying functions. We confine ourselves to indicating here that his Section 206 [**7**, p. 324] begins with:

"The critical value of n in $\phi x : x^n$, or the limit of $x\phi' x : \phi x$, being $a, \dots$."

5. Attribution and extension

The author first takes this opportunity to make some comments on his paper [**24**], written for the issue of *Publications* dedicated to Karamata, especially its Section 4: Regular Variation as Necessary and Sufficient. In the first place Bingham's paper [**2**] partly on closely related topics, written for the Kupari Conference and therefore appearing in the same issue as [**23**], should have been cited in [**24**]. Secondly, that the necessary and sufficient condition for a distribution F to be in the domain of attraction of a normal (Gaussian) Law, written as (12) in [**24**], amounts to

$$U(x) = \int_{|y| \leqslant x} y^2 dF(x)$$

being slowly varying at infinity in Karamata's sense (while pointed out by Feller in his 1966 monograph) was already contained in Sakovich's paper [**17**]. Indeed, this was within the more general context of domains of attraction of stable laws. Thus there was explicit citation, already in 1956, and also use of, within this probabilistic context, of Karamata's paper [**13**]. Sakovich thanks Gnedenko for his help with the publication of the note. Gnedenko's use in precisely the same context of regular variation conditions, but not explicit mention of Karamata, dates back at least to 1939 [**10**].

Secondly the author takes this opportunity to remark that in his first collaborative paper with Ranko Bojanić [**3**] the additional condition

$$\frac{L(xL^\alpha(x))}{L(x)} \to 1 \quad \text{as } x \to 0+ \tag{5.1}$$

where $\alpha = 1/\beta$ was used to give an explicit form to conclusion (1.4) of Karamata's Iteration Theorem. Remarkably, such a condition (with $\alpha = -1$ in (5.1)) occurs as *necessary and sufficient* for a version of the probabilistic Weak Law of Large Numbers, in a forthcoming paper of Csörgö and Simons [**6**], where it is called the Bojanić–Seneta condition.

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