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AN EQUATION WITH LEFT AND RIGHT FRACTIONAL DERIVATIVES

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Abstract. We consider an equation with left and right fractional derivatives and with the boundary condition $y(0) = \lim_{x\to 0^+} y(x) = 0$, $y(b) = \lim_{x\to b^-} y(x) = 0$ in the space $\mathcal{L}^1(0,b)$ and in the subspace of tempered distributions. The asymptotic behavior of solutions in the end points 0 and b have been specially analyzed by using Karamata's regularly varying functions.

1. Introduction

In the last years differential equations of fractional orders have been used in many branches of mechanics and physics. Many results have been published with concrete problems solved in classical spaces of functions and in the spaces of generalized functions. We cite only some of them, recently published or with a new approach: [2]–[4], [7], [8], [13], [15], [17], [19], [20], [22], [23] and with Karamata's regularly varying functions: [11], [24]. In this paper we treat such an equation with the boundary condition $y(0) = y(b) = 0$ in the space $\mathcal{L}^1(0,b)$ and in a subspace of tempered distributions constructed for this problem. We specially discussed asymptotic behavior of solutions in the end points 0 and b using Karamata's regularly varying functions and quasi-asymptotics in the space of tempered distributions.

As far as we are aware the equation treated in this paper has been solved only in [1] and [18] in some very special cases.

2. Preliminaries

2.1. Regular variation. A positive measurable function f, defined on a neighborhood $(0,\varepsilon)$ is called regularly varying at zero of index r if $f(1/x)$ is regularly varying at infinity of index $-r$; we write $f \in R_r$. A function $f \in R_r$ if and only if $f(x) = x^r\ell(x)$, $x \in (0,\varepsilon)$, where ℓ is slowly varying at zero (cf. [5], [12]).

We need to measure the behavior of a function not only at the points zero and infinity but also at a point $b \in \mathbf{R}_+$.

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Definition 1. A function f such that $f(b-t) \equiv g(t) \in R_r$ is called regularly varying at the point $b \in \mathbf{R}_+$ of index r. ($g(t) = t^r \ell(t)$, $t \in (0, \varepsilon)$ and $f(t) = (b-t)^r \ell(b-x)$, for an $\varepsilon > 0$).

Definition 2. [5, p. 436]. Let I be an interval in $\mathbf{R}$. The class $BV_{\mathrm{loc}} I$ is the class of all right-continuous functions $f : I \to \mathbf{R}$ that are locally of bounded variation on I, i.e., $V(f; J) < \infty$ for each compact set $J \subseteq I$.

Definition 3. [5, p. 104]. Let $f \in BV_{\mathrm{loc}}([0, \infty))$ be positive; f is quasi-monotone if for some $\delta > 0$

$$\int_0^x t^\delta |df(t)| = O(x^\delta f(x)), \quad x \to \infty.$$

2.2. Fractional integrals and derivatives on the interval $(0, b)$, $0 < b < \infty$. Let $\varphi \in \mathcal{L}^1(0, b)$ and $\alpha \in (0, 1)$. The integrals

$$(I^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau\,,$$

$$(I_\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}}\, d\tau\,,$$

are called fractional integrals of order α (Riemann–Liouville fractional integrals).

The fractional derivatives of order α are defined as:

$$(D^\alpha \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^\alpha}\, d\tau\,,$$

$$(D_\alpha \varphi)(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^\alpha}\, d\tau\,.$$

For any function $\varphi \in \mathcal{L}^1(0, b)$ we have $D^\alpha \circ I^\alpha \varphi = \varphi$ and $D_\alpha \circ I_\alpha = \varphi$. This follows from Theorem 2.4, p. 44 in [21] and the connection: $(I^\alpha \varphi)(b-t) = (I_\alpha \varphi(b-\tau))(t)$.

3. Behavior of fractional integrals at 0 and b

3.1. Elementary access. The asymptotic expansions of the fractional integrals $I^\alpha \varphi$ as $x \to 0$ or $x \to \infty$ are known only in the case the expansions involved the power, logarithmic and exponential terms (cf. [21]).

In [11] the following is proved.

Theorem A. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function with $\lim\limits_{x \to 0} f(x) = f(0) \neq 0$ and let $0 < \alpha < 1$. Then $\lim\limits_{x \to 0} (I^\alpha f)(\lambda x)/(I^\alpha f)(x) = \lambda^\alpha$.*

We use here the asymptotic behavior not only of $I^\alpha\varphi$, but also of $I_\alpha\varphi$ and not only at $x=0$ but also at $x=b>0$. Also, the Karamata regularly varying functions contribute to the preciseness of the asymptotic behavior found.

PROPOSITION 1. *Let* $\alpha\in(0,1)$.
1) *If* $g\in\mathcal{L}^1(0,b)$, *then* $I_\alpha g\in\mathcal{L}^1(0,b)$.
2) *If* $g\in\mathcal{L}^1(0,b)$, $g\in R_\gamma$ *and* $g(t)=t^\gamma\ell_1(t)$, $t\in(0,\varepsilon)$, $\varepsilon>0$, $\gamma+\alpha>0$, *then*

$$\lim_{t\to0^+}(I_\alpha g)(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^b\frac{g(\tau)}{\tau^{1-\alpha}}\,d\tau\,.$$

3) *If* $g\in\mathcal{L}^1(0,b)$, g *is regularly varying at* b *and* $g(b-t)=t^\beta\ell_2(t)$, $t\in(0,\varepsilon)$, $\beta>-1$, *where* $\ell_2(1/t)$ *is slowly varying quasi-monotone at infinity, then* $(I_\alpha g)(t)$ *is regularly varying at* b,

$$(I_\alpha g)(b-t)=t^{\alpha+\beta}\,\frac{\Gamma(1+\beta)}{\Gamma(\alpha+\beta+1)}\,\ell_2(t),\quad t\in(0,\varepsilon/2)\,.$$

PROOF. 1) follows from the property that the set $\{I_\alpha g,\ \alpha>0\}$ is a semigroup (cf. [21], p. 48).

2) Let $t\in(0,\varepsilon/2)$. Then:

$$\int\limits_t^b\frac{g(\tau)}{(\tau-t)^{1-\alpha}}d\tau=\int\limits_t^\varepsilon+\int\limits_\varepsilon^b\frac{g(\tau)}{(\tau-t)^{1-\alpha}}d\tau\,. \tag{1}$$

For the first integral, where ε is fixed so that $g(t)=t^\gamma\ell_1(t)$ we have:

$$\Big|\int\limits_t^\varepsilon\frac{g(\tau)}{(\tau-t)^{1-\alpha}}d\tau\Big|=\Big|\int\limits_t^\varepsilon\frac{\tau^\gamma\ell_1(\tau)}{(\tau-t)^{1-\alpha}}\Big|\leqslant\int\limits_t^\varepsilon\frac{\tau^{\gamma-\eta}}{(\tau-t)^{1-\alpha}}d\tau$$

$$\leqslant\varepsilon^{\gamma-\eta}\int\limits_t^\varepsilon\frac{d\tau}{(\tau-t)^{1-\alpha}}\leqslant\varepsilon^{\gamma-\eta}\frac{(\tau-t)^\alpha}{\alpha}\Big|_t^\varepsilon\leqslant\frac{\varepsilon^{\gamma+\alpha-\eta}}{\alpha}\,,\quad 0\leqslant t\leqslant\varepsilon/2, \tag{2}$$

where η is a positive number such that $\alpha+\gamma-\eta>0$.

For the second integral in (1) the following properties hold:

$$\Big|\frac{g(\tau)}{(\tau-t)^{1-\alpha}}\Big|\leqslant\frac{|g(\tau)}{(\tau-\varepsilon/2)^{1-\alpha}}\,,\ \ \varepsilon\leqslant\tau\leqslant b,\ \ t\in(0,\varepsilon/2) \tag{3}$$

and

$$\lim_{t\to0^+}\frac{g(\tau)}{(\tau-t)^{1-\alpha}}=\frac{g(\tau)}{\tau^{1-\alpha}}\,,\quad\varepsilon\leqslant\tau\leqslant b\,. \tag{4}$$

With the properties (3) and (4) we can use Lebesgue's theorem:

$$\lim_{t\to0^+}\int\limits_\varepsilon^b\frac{g(\tau)}{(\tau-t)^{1-\alpha}}d\tau=\int\limits_\varepsilon^b\frac{g(\tau)}{\tau^{1-\alpha}}\,.$$

Hence for every $\varepsilon > 0$ we have:

$$\int_{\varepsilon}^{b} \frac{g(\tau)}{(\tau - t)^{1-\alpha}} d\tau = \int_{\varepsilon}^{b} \frac{g(\tau)}{\tau^{1-\alpha}} d\tau + O(\varepsilon^{\gamma+\alpha-\eta}),\ t \to 0^+ .$$

Since by (2)

$$\left| \int_{0}^{\varepsilon} \frac{g(\tau)}{\tau^{1-\alpha}} d\tau \right| = O(\varepsilon^{\alpha+\gamma-\eta}) .$$

we have:

$$\int_{\varepsilon}^{b} \frac{g(\tau)}{(\tau - y)^{1-\alpha}} d\tau = \int_{0}^{b} \frac{g(\tau)}{\tau^{1-\alpha}} d\tau + O(\varepsilon^{\alpha+\gamma-\eta}),\ \varepsilon \to 0 ,$$

which proves assertion 2).

3) Let us consider now $(I_\alpha g)(b - t)$.

$$(I_\alpha g)(b - t) = \frac{1}{\Gamma(\alpha)} \int_{b-t}^{b} \frac{g(\tau)}{(\tau - b + t)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha)} \int_{b-t}^{b} \frac{(b-\tau)^\beta \ell_2(b-\tau)}{(\tau - b + t)^{1-\alpha}} d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{x^\beta \ell_2(x)}{(t-x)^{1-\alpha}} dx, \quad t \in (0, \varepsilon/2).$$

Hence

$$(I_\alpha g)\Big(b - \frac{1}{y}\Big) = \frac{1}{\Gamma(\alpha)} \int_{0}^{1/y} \frac{x^\beta \ell_2(x)}{(1/y - x)^{1-\alpha}}\, dx, \quad \frac{1}{y} \in (0, \varepsilon/2)$$

$$= \frac{y^{1-\alpha}}{\Gamma(\alpha)} \int_{y}^{\infty} \frac{u^{-1-(\beta+\alpha)} \ell_2(1/u)}{(u-y)^{1-\alpha}}\, du = \frac{y^{-(\beta+\alpha)}}{\Gamma(\alpha)} \int_{1}^{\infty} \frac{v^{-1-(\beta+\alpha)} \ell_2(1/vy)}{(v-1)^{1-\alpha}}\, dv .$$

It only remains to apply Theorem 4.1.5 in [5] (cf. also [6]), which gives

$$(I_\alpha g)\Big(b - \frac{1}{y}\Big) = y^{-(\beta+\alpha)} \frac{\Gamma(1+\beta)}{\Gamma(\alpha+\beta+1)} \ell_2\Big(\frac{1}{y}\Big), \quad \frac{1}{y} \in (0, \varepsilon/2) ,$$

or

$$(I_\alpha g)(b - t) = t^{\alpha+\beta} \frac{\Gamma(1+\beta)}{\Gamma(\alpha+\beta+1)} \ell_2(t), \quad t \in (0, \varepsilon/2) .$$

This proves assertion 3). □

Proposition 2. *Let* $\alpha \in (0, 1)$.

1) *If* $h \in \mathcal{L}^1(0, b)$, *then* $I^\alpha h \in \mathcal{L}^1(0, b)$.

2) *If* $h \in \mathcal{L}^1(0, b)$ *and* h *is regularly varying at* b, $h(b - t) = t^\gamma \ell_1(t)$, $t \in (0, \varepsilon)$, $\varepsilon > 0$, *then*

$$\lim_{t \to 0^+} (I^\alpha h)(b - t) = (I^\alpha h)(b) .$$

3) *If* $h \in \mathcal{L}^1(0,b)$ *and* $h \in R_\beta$, $h(t) = t^\beta \ell_2(t)$, $t \in (0,\varepsilon)$, $\beta > -1$, *where* $\ell_2(1/t)$ *is quasi-monotone regularly varying, then*

$$(I^\alpha h)(t) = t^{\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \ell_2(t)\,.$$

4) *If* $h \in \mathcal{L}^1(0,b)$ *and additionally* $\lim\limits_{t\to 0^+} h(t) = A$, *then*

$$(I^\alpha h)(t) = \frac{A}{\Gamma(\alpha+1)} t^\alpha + o(1),\ t \to 0.$$

Proof. The proof for 1) is the same as the proof for 1) in Proposition 1.

2) We have

$$(I^\alpha h)(b-t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{b-t} \frac{h(\tau)}{(b-t-\tau)^{1-\alpha}}\, d\tau = \frac{1}{\Gamma(\alpha)} \int\limits_t^{b} \frac{h(b-x)}{(x-t)^{1-\alpha}}\, dx\,.$$

We denote by $g(t) = h(b-t)$. Then g satisfies condition 2) in Proposition 1. Therefore

$$\lim_{t\to 0^+} (I^\alpha h)(b-t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{b} \frac{h(b-x)}{x^{1-\alpha}}\, dx = \frac{1}{\Gamma(\alpha)} \int\limits_0^{b} \frac{h(\tau)}{(b-\tau)^{1-\alpha}}\, d\tau = (I^\alpha h)(b),$$

which proves 2).

3) Let $t \in (0, \varepsilon/2)$. Then

$$(I^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{t} \frac{\tau^\beta \ell_2(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau\,.$$

Introducing $t = 1/y$, we have:

$$(I^\alpha h)\Big(\frac{1}{y}\Big) = \frac{y^{1-\alpha}}{\Gamma(\alpha)} \int\limits_0^{1/y} \frac{\tau^\beta \ell_2(\tau)}{(1-y\tau)^{1-\alpha}}\, d\tau = \frac{y^{-\alpha-\beta}}{\Gamma(\alpha)} \int\limits_0^{1} \frac{x^\beta \ell_2(x/y)}{(1-x)^{1-\alpha}}\, dx\,.$$

Hence, by Theorem 4.1.5 in [5] and by [6]

$$I^\alpha(h)\Big(\frac{1}{y}\Big) = \frac{1}{\Gamma(\alpha)} y^{-\alpha-\beta} \ell_2\Big(\frac{1}{y}\Big) \int\limits_0^{1} \frac{x^\beta}{(1-x)^{1-\alpha}}\, dx = y^{-\alpha-\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \ell_2\Big(\frac{1}{y}\Big)\,.$$

Thus

$$(I^\alpha h)(t) = t^{\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, \ell_2(t)\,.$$

It only remains to prove 4). For $t \in (0, \varepsilon/2)$, $h(t) = A + r(t)$, $r(t) \to 0$, $t \to 0$. Then

$$\Big|(I^\alpha h)(t) - \frac{A}{\Gamma(\alpha+1)} t^\alpha\Big|$$
$$= \left|\frac{1}{\Gamma(\alpha)} \int_0^t \frac{A + r(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{A}{(t-\tau)^{1-\alpha}}\right| = \left|\frac{1}{\Gamma(\alpha)} \int_0^t \frac{r(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau\right|.$$

We fix ε in such a way that $|r(t)| < \delta$, $t \in (0, \varepsilon/2)$, then for any $\delta > 0$ there is $\varepsilon > 0$ such that

$$\Big|(I^\alpha h)(t) - A\,\frac{t^\alpha}{\Gamma(\alpha+1)}\Big| \leqslant \frac{1}{\Gamma(\alpha+1)} \delta(\varepsilon/2)^\alpha, \quad 0 < t < \varepsilon/2\,.$$

This concludes the proof of Proposition 2. $\square$

REMARK. The quoted Theorem A is a consequence of Proposition 2.4).

3.2. Application of Abel–Tauberian type theorems. We point at the possibility to use Abel–Tauberian type theorems to find asymptotic behavior of fractional integrals.

If the function $g \in \mathcal{L}^1(0, b)$, then it can be always extended by a function $\overline{g} \in \mathcal{L}^1(0, \infty)$, $\overline{g}(x) = g(x)$, $x \in (0, b)$. Then the Laplace transform of $\overline{g}$ exists and of $I^\alpha \overline{g}$, too. Let $\mathcal{L}$ denote the Laplace transform; then

$$(\mathcal{L} I^\alpha \overline{g})(s) = \frac{1}{s^\alpha}\,(\mathcal{L}\overline{g})(s)\,.$$

If we suppose: 1) $g(t) \sim t^\gamma \ell_1(t)$, $t \to 0$, then by Karamata's Tauberian theorem (cf. [5, p. 233])

$$(\mathcal{L}\overline{g})(s) \sim \frac{\Gamma(\gamma+1)}{s^{\gamma+1}} \ell_1\Big(\frac{1}{s}\Big), \quad s \to \infty \quad (s \text{ real}).$$

Consequently

$$(\mathcal{L} I^\alpha \overline{g})(s) = \frac{1}{s^\alpha}\,(\mathcal{L}\overline{g})(s) \sim \frac{\Gamma(\gamma+1)}{s^{\alpha+\gamma+1}} \ell_1\Big(\frac{1}{s}\Big)\; s \to \infty. \tag{5}$$

If in addition we suppose: 2) for some $\sigma \in (-1, \gamma)$ $t^{-\sigma}\overline{g}(t)$ is bounded on every $[a, \infty)$ and $g(t)/t^\gamma \ell(t)$ is slowly decreasing, then from (5) if follows that

$$g(t) \sim t^\gamma \ell_1(t), \quad t \to 0\,.$$

Using Tauberian type theorem we introduce an additional Tauberian condition. However this approach can be useful if we look for conditions on the function f to make sure the existence of a solution to equation

$$(I^\alpha g)(t) = f(t), \quad t \in [0, \infty)\,.$$

As it is shown above, if f has it Laplace transform $(\mathcal{L}f)(s)$, $s > s_0 \geqslant 0$, g can belong to the class of functions which have $g(t) \sim t^\gamma \ell_1(t)$, $t \to 0$ only if

$$(\mathcal{L}f)(s) \sim \frac{\Gamma(\gamma+1)}{s^{\alpha+\gamma+1}} \ell_1(s), \quad s \to \infty.$$

Conversely, if

$$(\mathcal{L}f)(s) \sim \frac{\Gamma(\gamma+1)}{s^{\alpha+\gamma+1}}\ell_1(t)$$

and g satisfies the additional condition 2), then $g(t) \sim t^\gamma \ell_1(t)$, $t \to 0$.

4. Behavior of solutions to equation $(D_\alpha D^\alpha y)(t) = g(t)$, $0 < t < b$

THEOREM 1. *Let $\alpha \in (0,1)$ and $g \in \mathcal{L}^1(0,b)$, then the family of functions*

$$f(t) = (I^\alpha I_\alpha g)(t) + C_1(I^\alpha (b-\tau)^{\alpha-1})(t) + C_2 t^{\alpha-1}, \quad t \in (0,b) \tag{6}$$

satisfies the equation

$$(D_\alpha D^\alpha f)(t) = g(t), \quad t \in (0,b)\,, \tag{7}$$

and belongs to $\mathcal{L}^1(0,b)$.

If in addition $\alpha \in (1/2,1)$ and the function g has the properties:

1) $g(t) = t^\gamma \ell_1(t)$, $t \in (0,\varepsilon)$, $\varepsilon > 0$,

2) $g(b-t) = t^\beta \ell_2(t)$, $t \in (0,\varepsilon)$, $\beta > -1$,

where $\ell_2(1/y)$ is quasi-monotone slowly varying at infinity, then there exists $f_0(t)$ belonging to the family $f(t)$, given by (6) *which satisfies boundary condition*

$$f_0(0) = f_0(b) = 0 \tag{8}$$

and with the properties

1) $f_0(t) \in \mathcal{L}^1(0,b)$,

2) $f_0(t) = Bt^\alpha + o(1)$, $t \to 0^+$; $B = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^b \frac{g(\tau)}{\tau^{1-\alpha}}\,d\tau + \frac{b^{\alpha-1}}{\Gamma(\alpha+1)}$,

3) $\displaystyle\lim_{t\to 0^+} f_0(b-t) = (I^\alpha I_\alpha g)(b) + \frac{b^{2\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha-1)} \overset{\mathrm{def}}{=} f_0(b)$,

4) $f_0(t) = (I^\alpha I_\alpha g)(t) + C_1(I^\alpha(b-\tau)^{\alpha-1})(t)$, where $C_1 = \dfrac{(I^\alpha I_\alpha g)(b)}{(I^\alpha(b-\tau)^{\alpha-1})(b)}$.

PROOF. By the properties of D_α, D^α, I_α and I^α, we cited in the Preliminaries, it is easily seen that $f \in \mathcal{L}^1(0,b)$ and:

$$D_\alpha D^\alpha (I^\alpha I_\alpha g) = D_\alpha (D^\alpha I^\alpha) I_\alpha g = D_\alpha I_\alpha g = g\,.$$

It is well known (cf. [21, p. 36]) that $(D^\alpha h)(t) = 0$ if and only if $h(t) = Ct^{\alpha-1}$ and $(D_\alpha h)(t) = 0$ if and only if $h(t) = C(b-t)^{\alpha-1}$. Hence, it follows that the family f, given by (6) is a solution to (7).

We can take that $f_0(t)$ has the analytical form

$$f_0(t) = (I^\alpha I_\alpha g)(t) + C_1(I^\alpha(b-\tau)^{\alpha-1})(t), \quad t \in (0,b).$$

We took that $C_2 = 0$ in $f(t)$ because of the boundary condition (8). Since $f \in \mathcal{L}^1(0,b)$ (cf. Propositions 1 and 2), then $f_0 \in \mathcal{L}^1(0,b)$, as well. This proves the property 1) of f_0.

With the supposed properties of g, by Proposition 1 we have

$$\lim_{t\to 0^+} (I_\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^b \frac{g(\tau)}{\tau^{1-\alpha}}\,d\tau$$

$$(I_\alpha g)(b-t) = \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\ell_2(t), \quad t \in (0, \varepsilon/2)\,. \tag{9}$$

If we apply Proposition 2 to $h = I_\alpha g$, then we obtain from (9)

$$(I^\alpha I_\alpha g)(t) = A\,\frac{t^\alpha}{\Gamma(\alpha+1)} + o(1), \quad t \to 0\,, \tag{10}$$

where

$$A = \frac{1}{\Gamma(\alpha)} \int_0^b \frac{g(\tau)}{\tau^{1-\alpha}}\,d\tau$$

and

$$\lim_{t\to 0^+}(I^\alpha I_\alpha g)(b-t) = (I^\alpha I_\alpha g)(b) = \frac{1}{\Gamma^2(\alpha)} \int_0^b \frac{d\tau}{(b-\tau)^{1-\alpha}} \int_\tau^b \frac{g(u)}{(u-\tau)^{1-\alpha}}\,. \tag{11}$$

With regard to the function $(I^\alpha(b-\tau)^{\alpha-1})(t)$ which appears in $f_0(t)$, we have by Proposition 2:

$$(I^\alpha(b-\tau)^{\alpha-1})(t) = \frac{b^{\alpha-1}}{\Gamma(\alpha+1)}t^\alpha + o(1), \quad t \to 0. \tag{12}$$

and

$$(I^\alpha(b-\tau)^{\alpha-1})(b) = \frac{1}{\Gamma(\alpha)}\int_0^b \frac{d\tau}{(b-\tau)^{2(1-\alpha)}} = \frac{-1}{\Gamma(\alpha)}\,\frac{b^{2\alpha-1}}{2\alpha-1}\bigg|_0^b = \frac{b^{2\alpha-1}}{\Gamma(\alpha)(2\alpha-1)}. \tag{13}$$

From (10)–(13) it follows 2) and 3) in Theorem 1. Now it is easy to satisfy the boundary condition taking that

$$C_1 = (I^\alpha I_\alpha g)(b)/(I^\alpha(b-\tau)^{\alpha-1})(b).$$

The proof of Theorem 1 is complete. □

5. Equation (7) in the subspace of tempered distributions $\mathcal{D}'_b$

5.1. Preliminaries. We use the following notation: $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$ for the space of tempered distributions, $\mathcal{S}'_+ = \{T \in \mathcal{S}',\ \operatorname{supp} T \subset [0,\infty)\}$.

The following class of distributions $\{f_\beta;\ \beta \in \mathbf{R}\}$:

$$f_\beta(t) = \begin{cases} H(t)t^{\beta-1}/\Gamma(\beta), & \beta > 0, \\ f^{(m)}_{\beta+m}(t), & \beta \leqslant 0,\ \beta+m > 0,\ m \in \mathbf{N}\,, \end{cases} \tag{14}$$

belonging to $\mathcal{S}'_+$, has an important role in definition of the asymptotic behavior of distributions; $f^{(m)}$ is the m-th derivative in the distributional sense and H is Heaviside's function. By $f^{(-\beta)}$ for $f \in \mathcal{S}'_+$ we denote $f_\beta * f$, where $*$ is the sign for the convolution and $\beta \in \mathbf{R}$. If $\beta > 0$, $f^{(-\beta)}$ is called the operator of fractional integral of order β, but if $\beta < 0$, $f^{(-\beta)}$ is operator of fractional derivative of order $-\beta$ (cf. [26, p. 36]).

The class $\{f_\beta;\ \beta \in \mathbf{R}\}$ with the operation convolution form an Abelian group: $f_{\beta_1} * f_{\beta_2} = f_{\beta_1+\beta_2}$, $f_0 = \delta$ (cf. [26, p. 36]).

If $T \in \mathcal{S}'_+$ is regular distribution defined by the function f, then we write $T = [f]$.

To measure the asymptotic behavior in $\mathcal{S}'_+$ we use the quasi-asymptotics. (cf. [9], [10], [16]).

Definition 4. Let $f \in \mathcal{S}'_+$ and $c(x)$, $x \in (0, a)$, $a > 0$, be a measurable positive function. It is said that f has the quasi-asymptotics at 0^+ related to $c(1/k)$ if there is a $g \in \mathcal{S}'_+$, $g \neq 0$ such that

$$\lim_{k\to\infty} \Big\langle \frac{f(x/k)}{c(1/k)}, \varphi(x) \Big\rangle = \big\langle g(x), \varphi(x) \big\rangle, \quad \varphi \in \mathcal{S}\,.$$

We write for short $f \overset{q}{\sim} g$ at 0^+ related to $c(1/k)$.

It has been proved (cf. [9], [16]) that $c(x) = x^\beta \ell(x)$, $x \in (0, \varepsilon)$, $\varepsilon > 0$, $\beta \in \mathbf{R}$, ℓ is slowly varying at zero and $g = C f_{\beta+1}$.

Let $f(b-x)$ denote the distribution which is obtained after exchange of variables in $f \in \mathcal{S}'$. If

$$\lim_{k\to\infty} \Big\langle \frac{f(b - x/k)}{(1/k)^\beta \ell(1/k)},\ \varphi(x) \Big\rangle = \big\langle g(x), \varphi(x) \big\rangle, \quad \beta \in \mathbf{R},\ \ \phi \in \mathcal{S}\,,$$

we say that f has quasi-asymptotics at b related to $(1/k)^\beta \ell(1/k)$ and write for short $f(b-t) \overset{q}{\sim} g$ at b related to $(1/k)^\beta \ell(1/k)$.

The quasi-asymptotics at b describes the distribution f in a neighborhood of the point b.

If $\beta = 0$, $\ell = 1$ and:

a) $f(b-t) \overset{q}{\sim} C < \infty$, $C \geqslant 0$, at b we say that the distribution f has C as its value at the point b. In this sense we write $f(b) = C$;

b) $f \in \mathcal{S}'_+$, $f(t) \overset{q}{\sim} C < \infty$, $C \geqslant 0$, at 0^+, we write $f(0) = C$ (cf. [14]).

The quasi-asymptotics at 0^+ is a local property.

Lemma 1. *Let $f \in \mathcal{S}'_+$. Then $f \overset{q}{\sim} C f_{\alpha+1}$ at 0^+ related to $c(1/k) = (1/k)^\alpha \ell(1/k)$ if and only if there exists $\gamma \in \mathbf{R}$ such that $f_\gamma * f \overset{q}{\sim} C f_{\alpha+\gamma+1}$ at 0^+ related to $(1/k)^\gamma c(1/k)$.*

For the proof cf. [9], [26]. For the applications of the quasi-asymptotics it is important to know:

Lemma 2. *Let $f \in \mathcal{S}'_+$ be regular distribution defined by the function $f(x)$ which is locally integrable on $[0, b)$, $0 < b$, $\beta > -1$.*

1) *If $f(t) \sim C t^\beta \ell(t)$, $t \to 0$, then $f \overset{q}{\sim} C t^\beta$ at 0^+ related to $t^\beta \ell(t)$.*

2) *If $f(t) \overset{q}{\sim} C t^\beta$ at 0^+ related to $(1/k)^\beta \ell(1/k)$ and $t^m f(t)$ is monotone for some $m \in \mathbf{N}$ on $(0, \varepsilon)$, $\varepsilon > 0$, then $f(t) \sim t^\beta \ell(t)$, $t \to 0$.*

For the proof cf. [9].

We need a special space of generalized functions and we are going to construct it. Let $\mathcal{A}$ be the subspace of $\mathcal{S}'_+$:

$$\mathcal{A} = \{T \in \mathcal{S}'_+;\ \operatorname{supp} T \subset [b, \infty)\}.$$

In $\mathcal{S}'_+$ we define the following equivalence relation: $f \sim g \Leftrightarrow f - g \in \mathcal{A}$. Let $\mathcal{B}$ denote the quotient space $\mathcal{B} = \mathcal{S}'_+/\mathcal{A}$. An element of $\mathcal{B}$ is a class defined by a $T \in \mathcal{S}'_+$.

Definition 5. By $\mathcal{D}'_b$ we denote the space:

$$\mathcal{D}'_b = \{T_b \in \mathcal{D}'([0,b));\ \exists\, T \in \mathcal{S}'_+,\ T|_{(-\infty,b)} = T_b\}.$$

($T|_{(-\infty,b)}$ is the restriction of T on $(-\infty,b)$).

Lemma 3. *$\mathcal{D}'_b$ is algebraically isomorphic to $\mathcal{B}$.*

Proof. Let $T_b \in \mathcal{D}'_b$; then there exists $T \in \mathcal{S}'_+$ such that $T|_{(-\infty,b)} = T_b$. The distribution T defines the class $c\ell(T) \in \mathcal{B}$. Then to $T_b \in \mathcal{D}'_b$ it corresponds $c\ell(T) \in \mathcal{B}$. Conversely, to the class $c\ell(T) \in \mathcal{B}$ there corresponds $T_b = T|_{(-\infty,b)}$, $T_b \in \mathcal{D}'_b$. Both correspondences are unique. $\square$

In $\mathcal{D}'_b$ we can define convolution. Let T_b and S_b belong to $\mathcal{D}'_b$ and let $c\ell(T)$ and $c\ell(S)$ be the elements from $\mathcal{B}$ corresponding to T_b and S_b respectively. Then by definition

$$T_b * S_b = (T * S)|_{(-\infty,b)}\,,$$

where $T \in c\ell(T)$ and $S \in c\ell(S)$. It is easily seen that this convolution does not depend on the elements we choose from $c\ell(T)$ and $c\ell(S)$. Let $T_1 = T_b + A_1 \in c\ell(T)$ and $T_2 = T_b + A_2 \in c\ell(T)$. Also, let $S_1 = S_b + A_3 \in c\ell(S)$ and $S_2 = S_b + A_4 \in c\ell(S)$, A_1, A_2, A_3 and A_4 belong to $\mathcal{A}$. Then

$$\begin{aligned} T_1 * S_1 - T_2 * S_2 &= T_1 * S_1 - T_1 * S_2 + T_1 * S_2 - T_2 * S_2 \\ &= T_1 * (S_1 - S_2) + (T_1 - T_2) * S_2 \\ &= T_1 * (A_3 - A_4) + (A_1 - A_2) * S_2 \in \mathcal{A}, \end{aligned}$$

by the properties of convolution in $\mathcal{S}'_+$. Hence, $(T_1 * S_1)|_{(-\infty,b)} = (T_2 * S_2)|_{(-\infty,b)}$.

We introduce another operator denoted by Q.

Definition 6. Let $T_b \in \mathcal{D}'_b$ such that there exists $T_b(b)$ or T_b is regular distribution $T_b = [f]$, $f \in \mathcal{L}^1(0,b)$. Then $QT_b(t) = T_b(b-t)$ ($T_b(b-t)$ is obtained by change of variable in T_b).

Now we can extend the operators D^β and D_β into $\mathcal{D}'_b$, $\beta > 0$.

Definition 7. Let $T_b \in \mathcal{D}'_b$ for which there exists $T_b(b)$ or $T_b = [f]$, $f \in \mathcal{L}^1(0,b)$. Then

$$D^\beta T_b = (f_{-\beta} * T)|_{(-\infty,b)},\quad \beta > 0\,, \tag{15}$$

where $c\ell(T) \in \mathcal{B}$ and T corresponds to T_b;

$$D_\beta T_b = Q(f_{-\beta} * QT_b)|_{(-\infty,b)}),\quad \beta > 0. \tag{16}$$

Remark. If $T_b = [f]$, and if $D^\beta f$ and $D_\beta f$ exist, then $D^\beta T_b = D^\beta[f] = [D^\beta f]$ and $D_\beta T_b = D_\beta[f] = [D_\beta f]$, which means that with Definition 7 we extended the operators D^β and D_β on $\mathcal{D}'_b$.

By Lemma 2 it is easy to obtain the quasi-asymptotic behavior of $D^\beta T_b$ if we know the quasi-asymptotic behavior of T_b. The same is with the fractional integral.

Also we can use Abel–Tauberian-type theorems in the space $\mathcal{S}'_+$ (cf. [26, p. 132]) or in the space of Modified Fourier Hyperfunctions (cf. [25]) to find the quasi-asymptotic behavior of f if we know the quasi-asymptotic behavior of $f^{(-\beta)}$ for $\beta > 0$ and $\beta < 0$.

5.2. The solution to equation (7) in $\mathcal{D}'_b$ with the initial condition $f(0) = 0$. To equation (7) there corresponds in $\mathcal{D}'_b$ the following equation (cf. (15) and (16)):

$$Q(f_{-\alpha} * Q((f_{-\alpha} * f)|_{(-\infty,b)}))|_{(-\infty,b)} = g, \quad g \in \mathcal{D}'_b \text{ and } c\ell(f) \in \mathcal{B}. \tag{17}$$

If for $g \in \mathcal{D}'_b$ there exists $G \in \mathcal{S}'_+$, $G|_{(-\infty,b)} = g$ such that there is $G(b)$, then $g(b) = G(b)$.

THEOREM 2. *Suppose:* 1) $g \in \mathcal{D}'_b$, 2) *there exists* $g(b)$, *and* 3) $0 < \alpha < 1$. *The general solution to equation* (17) *in* $\mathcal{D}'_b$ *is the restriction of* f *on* $(-\infty, b)$, *where*

$$f = f_\alpha * (Q(f_\alpha * (Qg))|_{(-\infty,b)}) + C_1 f_\alpha * (Qf_\alpha) + C_2 f_\alpha\,. \tag{18}$$

If:

1) $Q(f_\alpha * (Qg)) \overset{q}{\sim} C f_{\beta+1}$ *at* 0 *related to* $(1/k)^\beta \ell(1/k)$,

2) $\beta + \alpha > 0$, $C_2 = 0$ *and* $\gamma = \min(\beta + \alpha, \alpha)$,

then f *has the quasi-asymptotics at zero related to* $(1/k)^\gamma \ell(1/k)$ *and* $f(0) = 0$. (*For the meaning of* $f(0)$ *see* 5.1).

3) *If in addition* $\frac{1}{2} < \alpha < 1$ *and the first summand in the sum which defines* f *has its value in* b, *then* C_1 *can be found in such a way that* $f(b) = 0$.

PROOF. Since $\{f_\beta,\ \beta \in \mathbf{R}\}$ form an Abelian group with δ as the unit element, it is easy to construct a solution to (17) applying one after the other the inverse operators to those appearing in (17). In such a way we find as a solution to (17):

$$f_1 = (f_\alpha * (Q(f_\alpha * (Qg))_{(-\infty,b)}))|_{(-\infty,b)}\,. \tag{19}$$

To find a solution f_2 of the homogeneous part of (17) we start with

$$Q((f_{-\alpha} * f_2)|_{(-\infty,b)}) = 0, \text{ or } (f_{-\alpha} * f_2)|_{(-\infty,b)} = 0 \tag{20}$$

which is equivalent to

$$(f_{1-\alpha} * f_2)^{(1)}|_{(-\infty,b)} = 0, \ \text{ or } f_{1-\alpha} * f_2|_{(-\infty,b)} = C_2\,.$$

This gives the solution f_2 to the homogeneous part of (17):

$$f_2 = f_{\alpha-1} * C_2|_{(-\infty,b)} = C_2 f_\alpha|_{(-\infty,b)}\,. \tag{21}$$

But if for f_3:

$$(f_{-\alpha} * Q(f_{-\alpha} * f_3)|_{(-\infty,b)})|_{(-\infty,b)} = 0,$$

then by (20) $f_{-\alpha} * f_3|_{(-\infty,b)} = C_1 Q f_\alpha|_{(-\infty,b)}$ and f_3 is the restriction on $(-\infty, b)$ of the distribution:

(22)

$$C_1 f_\alpha * Q f_\alpha = C_1 f_\alpha * \left[H(b-x)H(x)\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}\right] = C_1\left[H(x)\frac{1}{\Gamma^2(\alpha)}\int_0^x \frac{(b-t)^{\alpha-1}}{(x-t)^{1-\alpha}}dt\right]$$

f_3 is another solution to the homogeneous part of (17).

The general solution f to (17) is $f = f_1 + f_2 + f_3$, where f_1, f_2 and f_3 are given by (19), (20) and (22) respectively.

It remains to prove that f satisfies the boundary conditions $f(0) = f(b) = 0$.

The first summand f_1 in the sum which determined f, because of Lemma 1 and suppositions 1) and 2), has the quasi-asymptotics at zero related to $(1/k)^{\alpha+\beta}\ell(1/k)$.

Since $Qf_\alpha = [H(x)H(b-x)f_\alpha(b-x)]$, then $(Qf_\alpha)(0) = f_\alpha(b)$. By Lemma 1, the second summand f_3 of the mentioned sum has the quasi-asymptotics at zero related to $(1/k)^\alpha$. Now, it is easily seen that f has the quasi-asymptotics at zero related to $(1/k)^\gamma\ell(1/k)$ and consequently $f(0) = 0$.

We have only to prove that $f(b) = 0$. Let us consider first the summand f_3 in the sum which defines f. It is easily seen that

$$Q((f_\alpha * [Qf])|_{(-\infty,b)}) = \left[H(x)H(b-x)\frac{1}{\Gamma^2(\alpha)}\int_0^{b-t}\frac{(b-\tau)^{\alpha-1}}{(b-t-\tau)^{1-\alpha}}\,d\tau\right].$$

Since

$$\lim_{t\to 0^+}\int_0^{b-t}\frac{(b-\tau)^{\alpha-1}}{(b-t-\tau)^{1-\alpha}}\,d\tau = \frac{b^{2\alpha-1}}{2\alpha-1}, \quad \frac{1}{2} < \alpha < 1,$$

there exists $(f_\alpha * Qf_\alpha)(b) = \dfrac{b^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}$. Now by supposition 3) we can find C_1 in such a way that $f(b) = 0$. This completes the proof of Theorem 2. □

Example. Let $g(x) = \delta(x-h)$, $0 < h < b$. Then by the property of δ-distribution: $\delta(-x) = \delta(x)$ we have $(Q\delta(x-h)) = \delta(b-x-h) = \delta(x-(b-h))$ and

$$f_\alpha * (Q\delta(x-h)) = f_\alpha * \delta(x-(b-h)) = f_\alpha(x-(b-h)).$$

Hence

$$\begin{aligned}
(f_\alpha * (Q\delta(x-h)))|_{(-\infty,b)} &= [H(b-x)H(x-(b-h))f_\alpha(x-(b-h))],\\
Q(f_\alpha * (Q\delta(x-h)))|_{(-\infty,b)} &= [H(x)H(b-x-(b-h))f_\alpha(b-x-(b-h))]\\
&= [H(x)H(h-x)f_\alpha(h-x)].
\end{aligned}$$

$$\begin{aligned}
f_\alpha * (Qf_\alpha * (Q\delta(x-h)))|_{(-\infty,b)} &= f_\alpha * H(x)H(h-x)f_\alpha(h-x)\\
&= \frac{H(x)}{\Gamma^2(\alpha)}\int_0^x \frac{H(h-t)(h-t)^{\alpha-1}}{(x-t)^{1-\alpha}}\,dt. \qquad (23)
\end{aligned}$$

Hence, f_1 is the regular distribution defined by the function (23).

For f_3 we have

$$C_1 f_\alpha * Qf = C_1 f_\alpha * H(x)H(b-x)f_\alpha(b-x) = C_1\frac{H(x)}{\Gamma^2(\alpha)}\int_0^x\frac{(b-t)^{\alpha-1}}{(x-t)^{1-\alpha}}dt. \qquad (24)$$

Consequently, f_3 is also the regular distribution defined by the function (24). By (23) and (24) f is the regular distribution defined by the function

$$\frac{H(x)}{\Gamma^2(\alpha)}\int_0^x \frac{H(h-t)(h-t)^{\alpha-1}}{(x-t)^{1-\alpha}}\,dt + C_1\frac{H(x)}{\Gamma^2(\alpha)}\int_0^x \frac{(b-t)^{\alpha-1}}{(x-t)^{1-\alpha}}\,dt. \tag{25}$$

To find the asymptotic behavior of such an f at the end points, we analyze first

$$\int_0^{x/k} \frac{(b-t)^{\alpha-1}}{(x/k-t)^{1-\alpha}}\,dt = k^{1-\alpha}\int_0^{x/k} \frac{(b-t)^{\alpha-1}}{(x-kt)^{1-\alpha}}\,dt = k^{-\alpha}\int_0^{x} \frac{(b-u/k)^{\alpha-1}}{(x-u)^{1-\alpha}}\,du$$
$$\to k^{-\alpha}b^{\alpha-1}\frac{x^\alpha}{\alpha}, \quad k\to\infty.$$

This says that

$$f_3 \overset{q}{\sim} C_1\frac{b^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)} \text{ at 0 related to } (1/k)^\alpha.$$

Since the quasi-asymptotics is a local property, we have that

$$f_1 \overset{q}{\sim} \frac{b^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)}f_{\alpha+1} \text{ at zero related to } (1/k)^\alpha.$$

Consequently,

$$f \overset{q}{\sim} \frac{b^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)}(1+C_1)f_{\alpha+1} \text{ at 0 related to } (1/k)^\alpha.$$

Finally $f(b)$ satisfies the condition $f(b)=0$ if C_1 is defined by

$$\int_0^h \frac{(h-t)^{\alpha-1}}{(b-t)^{1-\alpha}}\,dt + C_1\frac{b^{2\alpha-1}}{(2\alpha-1)} = 0\,.$$

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