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CONTRAPUNCTUS OF THE CONTINUUM PROBLEM AND THE MEASURE PROBLEM

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Abstract. In the long history of CH (Continuum Hypothesis) and its extensions into CP (Continuum Problem) that determines cardinalities of all power sets, solutions of the later are related in a variety of ways with the solution of the measure problem.

1. Introduction

Without AC, the power set of the first infinite set is not bound by any of the alephs, so in a way it is extra large. Then, infinitary arithmetics is an open place from the calculation point of view, with a great amount of incomparable terms. The Axiom of Choice induces nice organization among infinite sets, where all cardinal arithmetics calculations have solutions within the most simply and beautifully well ordered cardinal line of alephs, which can be written as simply as

$$(\forall\alpha \in \mathrm{ORD})2^{\aleph_\alpha} = \aleph_{F(\alpha)},$$

which is referred as The Continuum Problem. This equation, after Cantor diagonal theorem can be written as

$$(\forall\alpha \in \mathrm{ORD})2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)},$$

where $f(\alpha) \geqslant 1$ for all ordinals α. When $f = 1$ the last equation becomes a formulation of the Generalized Continuum Hypothesis - GCH. The function f here we call CP displacement function. The major excitement about solution of CP maybe is contained in the following list of famous results:

- 1939 Gödel [**4, 5, 8**]: Con(ZF) implies Con(ZFC + GCH)
- 1963 Cohen [**2, 5, 8**]: Con(ZF) implies Con(ZFC + ¬GCH)
- 1964 Easton [**3, 5, 8**]: Con(ZF) implies Con(ZFC + "on regular $\aleph_\alpha, F(\alpha)$ could be arbitrary, respecting monotonicity and cofinality restrictions"
- 1974 Silver [**11, 5**]: (ZFC) If $\aleph_\alpha$ is a singular cardinal of uncountable cofinality, then if GCH holds below $\aleph_\alpha$, then it holds on $\aleph_\alpha$
- 1991 Shellah, Magidor, Gittik - the solution of the Singular Cardinal Hypothesis [**10, 5**]: In ZFC, if $(\forall n \in \omega)2^{\aleph_n} = \aleph_{n+1}$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$,

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or rewriting it as CP-displacement function solutions:

- $(\forall\alpha)f(\alpha)=1$
- $(\exists\alpha)f(\alpha)>1$
- For each regular $\aleph_\alpha$ $f(\alpha)$ could be arbitrary, respecting monotonicity and cofinality restrictions
- For each singular $\aleph_\alpha$ of uncountable cofinality, if $(\forall\beta\in\alpha)f(\beta)=1$, then[1] $f(\alpha)=1$
- If $(\forall n\in\omega)f(n)=1$, then $f(\omega)<\omega_4$.

The Theorem of Silver had a predecessor – the Theorem of Magidor with the same formulation but with one hypothesis: existence of nonregular ultrafilter over ω_1. Earlier Silver proved consistency of GCH with the existence of measurable cardinals; Scott proved that if GCH is true almost everywhere (mod normal measure over a measurable cardinal κ), then GCH is true at κ as well; Kunen proved that if $2^\kappa>\kappa^+$ for measurable κ, then there is a model of ZFC with many measurable cardinals; The Singular Cardinal Hypothesis – SCH is tightly related to the measure problem: Jensen proved if SCH fails then $0^\sharp$ exists.

In the extension of the list of beautiful theorems let us mention

Theorem 1 (Prikry). *Let $\kappa\leqslant 2^{\aleph_0}$ be a real valued measurable cardinal. Then $2^\lambda=2^{\aleph_0}$ for all infinite $\lambda<\kappa$.*

Theorem 2 (Solovay). *CH is true on singular strong limit cardinals $\geqslant$ first strongly compact cardinal.*

Theorem 3 (Bukovsky, Hechler). *Let κ be a singular cardinal such that the continuum function below κ is eventually constant. Then $2^\kappa=2^{\mathrm{cf}\,\kappa}$.*

In particular, constant CP-displacement is finite. More precisely, if there is an ordinal β such that for each ordinal α $2^{\aleph_\alpha}=\aleph_{\alpha+\beta}$ holds, then $\beta\in\omega$.

Theorem 4 (Hajnal). *Let $\aleph_\alpha$ be a singular cardinal of uncountable cofinality such that for each $\beta<\alpha$ $2^{\aleph_\beta}\leqslant\aleph_{\beta+\gamma}$ holds. Then $2^{\aleph_\alpha}\leqslant\aleph_{\alpha+\gamma}$.*

Theorem 5 (Galvin, Hajnal). *If $\aleph_\alpha$ is a strong limit singular cardinal with uncountable cofinality, then $2^{\aleph_\alpha}<\aleph_{(|\alpha|^{\mathrm{cf}\,\alpha})^+}$.*

For the proofs and additional references we refer the reader to [**5, 7**].

2. Two voice interplay

Usual considerations of continuum problem usually do not go far from Gödel flat solution for CP-displacement f. That is illustrated with formulations of majority of listed results. Solutions for f that are further away from the Gödel constant $\langle 1\mid\alpha\in\mathrm{ORD}\rangle$ did not attract equal-opportunity attention in the history of CP, probably because "simpler" solutions are somehow preferred by nonmathematical criteria and probably because those further-away solutions would involve more hard technics in the proofs that are already inaccessible to majority of mathematicians.

[1]Silver result is somewhat stronger: $f(\alpha)=1$ if $f(\beta)=1$ almost everywhere (mod stationary sets) below α.

Anyway, by the result of Easton all solutions have the same mathematical right, and our attention here will be devoted to some of those distant solutions as well. Back to the CP formulation, displacement f : ORD $\to$ ORD which is nondecreasing and satisfies the cofinality condition, by definition of V_α is trivially fulfilling the rank limitation: for all $x \in V_\alpha$, $P(x)$ does not leak out of $V_{\alpha+1}$. Thus, $1 \leqslant f(\alpha) \leqslant 2^{\aleph_\alpha}$ for all ordinals α.

As mentioned above, we are curious how far to the right end can f reach. With $f(0) = 2^{\aleph_0}$, we get $2^{\aleph_0} = \aleph_{2^{\aleph_0}}$, thus making this cardinal a fixed point of the enumeration of cardinals. When regular, it becomes weakly inaccessible. If $2^{\aleph_\alpha}$ is weakly inaccessible, we have $2^{\aleph_\alpha} = \aleph_{2^{\aleph_\alpha}} = \aleph_{\alpha+2^{\aleph_\alpha}}$, then $f(\alpha) = 2^{\aleph_\alpha}$.

For the other voice for the contrapunctus this time we will take minimal unbounded functions in nonregular (uniform) ultrafilters, i.e. those that would press the functions bellow them to be bound by some constant. These characterize normality conditions for nonregular ultrafilters.

An ultrafilter D over infinite cardinal κ is:

- κ-complete, if it is closed under $< \kappa$ intersections;
- normal, if it is κ-complete and every pressing down function $f(\alpha) < \alpha$ almost everywhere mod D is equal mod D to a constant function;
- weakly normal, if each function $f : \kappa \to \kappa$ such that $\{\alpha \in \kappa \mid f(\alpha) < \alpha\} \in D$ is bounded by some constant in $\prod_D \langle \kappa, < \rangle$, i.e., there is $\beta \in \kappa$ such that $\{\alpha \in \kappa \mid f(\alpha) < \beta\} \in D$;
- λ-weakly normal, if $\prod_D \langle \lambda, < \rangle$ has a minimal unbounded function in λ.

As Magidor has shown in [**9**], nonregular ultrafilters give rise to ultrapowers with jumping cardinalities over smallest cardinals, which were hardest to obtain. For an ultrafilter D over infinite cardinal κ we define it's cardinal trace by

$$\mathrm{ct}(D) = \left\{ \left| \prod_D \lambda \right| \mid \lambda < \kappa \right\}$$

and call D jumping if $|\,\mathrm{ct}(D)| > 1$. For example, if D is an uniform κ-complete ultrafilter over a measurable cardinal κ, then $|\,\mathrm{ct}(D)| = 2^\kappa$.

In order to relate the normality conditions of ultrafilters to the CP displacement we introduce some notion from model theory:

A theory T of a language L with an unary predicate symbol U admits pair $\langle \kappa, \lambda \rangle$ if it has a model of cardinality κ in which U has cardinality λ. Further, the pair $\langle \kappa, \lambda \rangle$ is:

- LLG (left large gap), if T admits $\langle \kappa, \lambda \rangle$ but does not admit $\langle \kappa^+, \lambda \rangle$;
- RLG (right large gap), if T admits $\langle \kappa, \lambda \rangle$ but does not admit $\langle \kappa, \lambda' \rangle$ for $\lambda' < \lambda$;
- LG (large gap), if it is both LLG and RLG.

We will show how behavior of the CP-displacement f is related to the existence of jumping ultrafilters in the next theorem.

Theorem 6. *Let f be the displacement function in the continuum problem, $2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}$. Let T be a theory with $\langle \aleph_\xi(\kappa), \kappa \rangle = \langle \aleph_\sigma, \kappa \rangle$ as LLG. Let $\aleph_\sigma^{<\aleph_\sigma} = \aleph_\sigma$*

and let D be a uniform nonregular ultrafilter over $\aleph_\sigma$ with jumps after κ:

$$\aleph_\eta = \Big|\prod_D \kappa\Big| < \Big|\prod_D \aleph_\sigma\Big|.$$

Then $\eta < \sigma + f(\sigma) \leqslant \eta+\xi \leqslant \eta+\sigma$, binding CP-jump with the ultrapower cardinality jump and the diameter of the gap.

PROOF. We will first list two lemmas whose proofs are straightforward.

LEMMA 1. *If T admits pairs $\langle \kappa_i, \lambda_i\rangle$, $i \in \kappa$ and D is an ultrafilter over κ, then T admits $\big\langle |\prod_D \kappa_i|, |\prod_D \lambda_i|\big\rangle$.*

LEMMA 2. *Let $\langle \Lambda(\kappa), \kappa\rangle$ be LLG for T_1 for all (many) κ and let $\langle \kappa, \Gamma(\kappa)\rangle$ be RLG for T_2 for all (many) κ. Then there is a theory T for which $\langle \Lambda(\kappa), \Gamma(\kappa)\rangle$ is LG for all (many) κ.*

COROLLARY 1. *For any ultrafilter D over κ*

$$\Gamma\Big(\Big|\prod_D \kappa\Big|\Big) \leqslant \Big|\prod_D \Gamma(\kappa)\Big| \leqslant \Big|\prod_D \Lambda(\kappa)\Big| \leqslant \Lambda\Big(\Big|\prod_D \kappa\Big|\Big).$$

There are theories T_1 and T_2 with $\langle \kappa^+, \kappa\rangle$ and $\langle 2^\kappa, \kappa\rangle$ as LLG respectively for all κ and this iterates to $\langle \aleph_n(\kappa), \kappa\rangle$ and $\langle \beth_n(\kappa), \kappa\rangle$ for all κ, giving (from the corollary):

$$\Big|\prod_D \aleph_n(\kappa)\Big| \leqslant \aleph_n\Big(\Big|\prod_D \kappa\Big|\Big) \quad \text{and} \quad \Big|\prod_D \beth_n(\kappa)\Big| \leqslant \beth_n\Big(\Big|\prod_D \kappa\Big|\Big).$$

We continue with the proof of Theorem 6. First let us observe that T admits $\big\langle |\prod_D \aleph_\sigma|, |\prod_D \kappa|\big\rangle$. Let $\aleph_\eta = |\prod_D \kappa|$ and let $\aleph_\sigma^{<\aleph_\sigma} = \aleph_\sigma$. Then

$$\aleph_\eta < \Big|\prod_D \aleph_\sigma\Big| = 2^{\aleph_\sigma} = \aleph_{\sigma+f(\sigma)} = \Big|\prod_D \aleph_\xi(\kappa)\Big| \leqslant \aleph_\xi\Big(\Big|\prod_D \kappa\Big|\Big) = \aleph_{\eta+\xi}.$$

Hence we get ordinal inequalities $\eta < \sigma + f(\sigma) \leqslant \eta + \xi \leqslant \eta + \sigma$. □

For the CP displacement f we can say that it is:

- well bounded at σ, if $f(\sigma) < \sigma$;
- bounded at σ, if $f(\sigma)$ is constrained by some expression not involving σ as exponent (something providing $f(\sigma) < 2^{\aleph_\sigma}$);
- unbounded at σ, if $f(\sigma) = 2^{\aleph_\sigma}$.

Here are some examples with above notation:

(1) Let D be an ultrafilter over $\aleph_\sigma$, with $\aleph_\sigma^{<\aleph_\sigma} = \aleph_\sigma$ and let D nicely separate κ and $\aleph_\sigma$ so that $|\prod_D \kappa| \leqslant \aleph_\sigma$. Then $f(\sigma) < \sigma$. Conversely, $f(\sigma) < \sigma$ reduces the possibility of ultrapower cardinality jump below $\aleph_\sigma$.

(2) The mentioned example gives $(\forall \xi < \omega)(\forall \kappa)\big(\langle \aleph_\xi(\kappa), \kappa\rangle$ is LLG$\big)$. So, let $\aleph_\xi(\kappa)^{<\aleph_\xi(\kappa)} = \aleph_\xi(\kappa)$. If there is an uniform ultrafilter over $\aleph_\xi(\kappa)$ jumping after κ, then $f(\sigma)$ is a successor ordinal and it is (well) bounded.

(3) More specifically, let D be a jumping ultrafilter over $\aleph_{17}$ and $\aleph_{17}^{<\aleph_{17}} = \aleph_{17}$.
   (a) If $|\prod_D \omega| \leqslant \aleph_{17}$, then $2^{\aleph_{17}} \leqslant \aleph_{34}$.
   (b) If $2^{\aleph_{17}} = \aleph_{\omega+1}$ then there is no jumping ultrafilter over $\aleph_{17}$.

(4) Let $2^{\aleph_{17}} = \aleph_{\omega_1+1}$ and let $\aleph_{17}^{\aleph_{16}} = \aleph_{17}$. Then if there is a jumping ultrafilter over $\aleph_{17}$, there would have to be $|\prod_D \omega| = \aleph_{\omega_1}$ a singular cardinal (problem listed in [1]). Is it possible?

Problems: (1) Could $\lambda^{<\lambda} = \lambda$ be omitted from the above assumptions?
(2) Are there examples of LLG's $\langle \Gamma(\kappa), \kappa \rangle$ with $\Gamma(\kappa) \geqslant \aleph_\omega(\kappa)$ for sufficiently many κ?

Normal ultrafilters. Now we will briefly discuss a result for measurable cardinals from [**6**], quoting some preliminaries first. The corresponding proofs can be found in [**1**] and [**6**], respectively.

THEOREM 7. *If κ is a measurable cardinal with normal ultrafilter D, then* $\langle V_{\kappa+1}, \in \rangle \cong \prod_D \langle V_{\alpha+1}, \in \rangle$.

As a consequence we have that $\mathrm{Sinac} \cap \kappa \in D$ (Sinac – strongly inaccessible cardinals) and that $|\prod_D \beth_{\alpha+1}| = 2^\kappa$.

LEMMA 3. *Let D be an ultrafilter over κ. Let $\mathfrak{A} = \langle A, <_A \rangle = \prod_D \langle \kappa, < \rangle$ and let $f : \kappa \to \kappa$ with $f(\alpha) \neq 0$ for $\alpha \in \kappa$. Then* $\left|\prod_D f(\alpha)\right| = \left|\{g_d^{\mathfrak{A}} \in \mathfrak{A} \mid g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}\}\right|$.

THEOREM 8. *Let D be a normal ultrafilter over κ and f the CP-displacement. Then:*

(1) *$f(\kappa) \leqslant ot\big(\prod_D \langle f(\alpha), \in \rangle\big)$, specially $\left|f(\kappa)\right| \leqslant \left|\prod_D f(\alpha)\right|$;*
(2) *if $f(\alpha) <_D \alpha$, then $f(\alpha) =_D n$ for some $n < \omega$.*

PROOF. (1) Clearly, $f \restriction \kappa : \kappa \to \kappa$. Define $G_f = \{g_D \in \mathfrak{A} \mid g <_D f \restriction \kappa\}$ and

$$H = \big\{h_D \in \mathfrak{A} \mid \{\alpha \in \kappa \mid h(\alpha) \in [\omega_\alpha, \omega_{\alpha+f(\alpha)}) \cap \mathrm{Card}\} \in D\big\}.$$

Thus, for $h_D \in H$ there is some $g_D \in G_f$ such that

$$\big\{\alpha \in \kappa \mid f(\alpha) = \omega_{\alpha+g(\alpha)}\big\} \in D. \tag{1}$$

Define an embedding $\bar{n} : H \to G_f$ by $\bar{n}(h_D) = g_D$ iff (1). Consequently, $|H| \leqslant |G_f|$. For a cardinal λ such that $\kappa \leqslant \lambda < 2^\kappa$, there is an $f^\lambda : \kappa \to \kappa$ which is the λ-th element in $\mathfrak{A}$. Then $\left|\prod_D f^\lambda(\alpha)\right| = |G_{f^\lambda}| = \lambda$. For the function g with domain κ, define the function $|g|$ by $|g| = \big\langle |g(\alpha)| : \alpha \in \kappa \big\rangle$, we have $\left|\prod_D |f^\lambda(\alpha)|\right| = \lambda$, which means that $|f^\lambda|$ is the λ-th element, i.e. $|f^\lambda| =_D f^\lambda$ and

$$X = \big\{\alpha \in \kappa \mid f^\lambda(\alpha) \in \mathrm{Card}\big\} \in D.$$

Consequently, $\{\alpha \in \kappa \mid f^\lambda(\alpha) \geqslant \alpha\} \in D$.

Since $\mathrm{Sinac} \cap \kappa \in D$, we have that either $\big\{\alpha \in \kappa \mid f^\lambda(\alpha) \geqslant \omega_{\alpha+f(\alpha)}\big\} \in D$ or $\big\{\alpha \in \kappa \mid f^\lambda(\alpha) < \omega_{\alpha+f(\alpha)}\big\} \in D$. In the first case we would have

$$\big\{\alpha \in \kappa \cap \mathrm{Sinac} \mid f^\lambda(\alpha) \geqslant \omega_{\alpha+f(\alpha)=\beth_{\alpha+1}}\big\} \in D,$$

therefore $2^\kappa \leqslant \left|\prod_D f^\lambda(\alpha)\right|$, contrary to the assumption for λ.

Thus

$$\big\{\alpha \in \kappa \mid f^\lambda(\alpha) \in [\omega_\alpha, \omega\alpha + f(\alpha)) \cap \mathrm{Card}\big\} \in D.$$

It follows that there is some $h_D \in H$ such that $f^\lambda =_D h$, or equivalently $f^\lambda \in H$. Since $\lambda \neq \lambda'$ implies $f^\lambda \neq f^{\lambda'}$, we have

$$\big|[\kappa, 2^\kappa) \cap \mathrm{Card}\big| = \big|(\kappa, 2^\kappa] \cap \mathrm{Card}\big| = |f(\kappa)| \leqslant |H| \leqslant |G_f| = \Big|\prod_D f(\alpha)\Big|.$$

(2) This follows from normality and Theorem 3. □

Corollary 2. *If CP-displacement is bounded $f(\alpha) < \alpha$ almost everywhere below κ, then the same is true at κ: $2^\kappa \leqslant \aleph_n(\kappa)$, for some $n < \omega$. Thus, for measurable κ, $f(\kappa) = \mathrm{const} < \omega$ or $f(\kappa) \geqslant \kappa$, i.e., $f(\kappa)$ is unbounded.*

Specially, if CH is true almost everywhere below κ, then it is true at κ as well. This was proved earlier by Scott.

Note that the above proof also proves that $2^\kappa \leqslant \aleph_{\kappa + ot(\prod_D \langle f(\alpha), < \rangle)}$. In the above proof notice that:

- $\mathrm{Sinac} \cap \kappa \in D$;
- $\mathrm{ct}(D) = \mathrm{Card} \cap (2^\kappa)^+$;
- $\mathrm{ct}(D) \cap \kappa = \mathrm{rng}(|\,\mathrm{id}\,|_d)$;
- $|\,\mathrm{id}\,| =_D \mathrm{id}$ and id is the minimal unbounded function.

$|\prod_D \lambda|$ for $\lambda < \kappa$ is smoothly increasing, covering all cardinals $\leqslant 2^\kappa$. That is, $\prod_D \lambda$, $\lambda \in \kappa$ jumps at every cardinal, but these jumps are smallest possible. On the other hand, when D is weakly normal is where the chance of cardinality jumps occurs, but it is harder to provide the jumping. The key feature in both are the minimal unbounded functions.

Let $g : \mathrm{ORD} \to \mathrm{ORD}$ be nondecreasing, respecting cofinality condition like CP-displacement and let $g(\alpha) \leqslant 2^{\aleph_\alpha}$, for all ordinals α. For the CP-displacement f we say that it is bounded by g on κ iff $f(\alpha) \leqslant g(\alpha)$ for all $\alpha \leqslant \kappa$. We say that it is bounded at κ iff $f(\kappa) < g(\kappa)$. CP-displacement f is bounded on κ (at κ) if it is bounded on κ (at κ) by some g.

Now we are ready to formulate some problems.

(1) Can Theorem 6 sort of dependence of $f(\kappa)$ on f on smaller values generalize involving some extra conditions to some large cardinal smaller than measurable?
(2) How large can $\mathrm{ct}(D)$ be for cardinals smaller than measurable?
(3) In the theorem of Silver is it possible that for some singular cardinal $\aleph_\alpha$ of uncountable cofinality there is a nondecreasing ordinal function g satisfying cofinality restriction for the continuum function such that $f(\beta) < g(\beta)$ AE, and f is unbounded at α?

3. CH and complexity of sets

The purpose of this section is to give a kind of incremental approach to the negation of CH in a way similar to the beginnings of the continuum problem, when mathematicians tried to "prove" CH in steps—to prove that CH holds for certain types of sets, where each new type is broader then the older one.

The starting point is the following well known reformulation of the negation of CH:

A$_1$ For each function $f : \mathbb{R} \to [\mathbb{R}]^{\leqslant\omega}$ there are real numbers x and y such that $x \notin f(y)$ and $y \notin f(x)$.

The intuition behind the $\mathbf{A}_1$ is that if we randomly chose x and y independent from each other, then the probability of $x \notin f(y)$ and $y \notin f(x)$ is 1, since both $f(x)$ and $f(y)$ are countable sets.

The fact that $\mathbf{A}_1$ is equivalent to $\neg$CH can be easily shown by the contraposition argument. Here we will prove slightly generalized statement.

THEOREM 9. *Let κ and λ be infinite cardinals such that $\kappa > \lambda$. Then $\kappa > \lambda^+$ iff for each function $f : \kappa \to [\kappa]^{\leqslant\lambda}$ there are ordinals $\alpha, \beta \in \kappa$ such that $\alpha \notin f(\beta)$ and $\beta \notin f(\alpha)$.*

PROOF. . A contraposition argument. If $\kappa = \lambda^+$, then the successor function $\alpha \mapsto \alpha+1$ is the witness. Conversely, let $f : \kappa \to [\kappa]^{\leqslant\lambda}$ be such that for all $\alpha, \beta \in \kappa$ we have that $\alpha \in f(\beta)$ or $\beta \in f(\alpha)$. Since $\kappa > \lambda$, we also have $\lambda^+ \subseteq \kappa$.

Now we claim that the set $X = \bigcup_{\alpha\in\lambda^+} f(\alpha)$ is equal to κ and thus $\kappa = \lambda^+$. Otherwise, if $\beta \in \kappa \smallsetminus X$, then $\lambda^+ \subseteq f(\beta)$, contradicting the fact that $|f(\beta)| \leqslant \lambda$. $\square$

In particular, if $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_0$, then immediate consequence of Theorem 6 is the equivalence between $\neg$CH and $\mathbf{A}_1$. Of course, in the formulation of $\mathbf{A}_1$ we could replace $\mathbb{R}$ by an arbitrary set of cardinality $2^{\aleph_0}$, and the resulted statement would be equivalent to $\mathbf{A}_1$.

A similar argument to the one described in the proof of theorem 1 yields the following

LEMMA 4. *Let κ and λ be infinite cardinals, $\kappa > \lambda$ and let $f : \kappa \to [\kappa]^{<\lambda}$. Then there are $\alpha, \beta \in \kappa$ such that $\alpha \notin f(\beta)$ and $\beta \notin f(\alpha)$.*

Now we can state the general question:

For which type of functions $f : \mathbb{R} \to [\mathbb{R}]^{\leqslant\omega}$ we can directly prove (in ZFC formalism) that there are reals x and y such that $x \notin f(y)$ and $y \notin f(x)$?

Since CH is independent from ZFC, we cannot produce the ZFC-proof for all functions, so the incremental approach naturally arise. Before we proceed, we should have another, more operational reformulation of $\mathbf{A}_1$.

For the given subset A of the set $[0,1]^2_{\mathbb{R}}$ let $A^{\mathrm{T}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in A\}$ and let $A_x = \{y \mid \langle x, y\rangle \in a\}$, $x \in [0,1]_{\mathbb{R}}$.

A$_2$ Let A be a subset of $[0,1]^2_{\mathbb{R}}$ such that for all $x \in [0,1]_{\mathbb{R}}$ the set A_x is at most countable. Then

$$A \cup A^{\mathrm{T}} \subsetneqq [0,1]^2_{\mathbb{R}}.$$

LEMMA 5. *The following are equivalent:* $\neg$CH, $\mathbf{A}_1$, $\mathbf{A}_2$.

PROOF. The equivalence between $\neg$CH and $\mathbf{A}_1$ is already proven, and the equivalence between $\mathbf{A}_1$ and $\mathbf{A}_2$ one can prove straightforwardly. $\square$

THEOREM 10. *Let A be a Lebesgue measurable subset of $[0,1]^2_{\mathbb{R}}$ such that for each $x \in [0,1]_{\mathbb{R}}$ the set A_x is at most countable. Then $A \cup A^T \subsetneqq [0,1]^2_{\mathbb{R}}$.*

PROOF. Let A be a Lebesgue mesurable. Then by Fubini's theorem for the Lebesgue integral

$$m(A) = m(A^T) = \iint_{[0,1]^2_{\mathbb{R}}} \chi(A)\, dm = \int_0^1 \left(\int_0^1 \chi(A)(x,y)\, dy \right) dx = \int_0^1 0 \cdot dx = 0,$$

where $\chi(A)$ is the characteristic function of the set A. Since $m([0,1]^2_{\mathbb{R}}) = 1$, we have that

$$A \cup A^{\mathrm{T}} \subsetneqq [0,1]^2_{\mathbb{R}}. \qquad \square$$

References

[1] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland, 1990.
[2] P. J. Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. USA 50 (1963), 1143–1148; 51 (1964), 105–110.
[3] W. B. Easton, *Powers of regular cardinals*, Ann. Math. Logic 1 (1970), 139–178.
[4] K. Gödel, *The consistency of the axiom of choice and the generalized continuum hypothesis*, Proc. Nat. Acad. Sci. USA 24 (1938), 556–557.
[5] T. Jech, Set Theory*, The Third Millennium Edition, revised and expanded Series*, Springer Monographs in Mathematics, 3rd rev. ed., 2003.
[6] A. Jovanović, *Continuum problem at measurable cardinals*, Proc. Set Theory. Foundation of Mathematics, Belgrade 1977, 57–62.
[7] A. Kanamori, *The Higher Infinite*, Springer–Verlag 1994.
[8] K. Kunen *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, 102. North Holland, Amsterdam–New York, 1980.
[9] M. Magidor, *Nonregular ultrafilters*, Technical report, 1977.
[10] S. Shelah, *Cardinal Arithmetics*, Oxford University Press, 1994.
[11] J. H. Silver, *On the singular cardinals problem*, Proc. Int. Congr. Math. Vancouver 1974, 265–268.

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