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AN ENUMERATIVE PROBLEM IN THRESHOLD LOGIC

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Abstract. The number of Boolean threshold functions is investigated. A new lower bound on the number of n-dimensional threshold functions on a set $\{0, 1, \dots, K-1\}$ is given.

1. Introduction

Let $K \in N$ be positive integer and $E_K = \{0, 1, \dots, K-1\}$. An n-dimensional threshold function on E_K is a function $f : E_K^n \to \{-1, 1\}$ such that there exists a hyperplane π separating the pre-images $f^{-1}(-1)$ and $f^{-1}(1)$. The question is: *what is the number $P(K, n)$ of n-dimensional threshold functions on E_K?*

The bounds for these numbers have been well-studied only for the case $K = 2$. Nevertheless, the asymptotic even for $P(2, n)$ is still open. The case $K = 2$ has an application in switching theory.

A Boolean (switching) function $f : \{-1, +1\}^n \to \{-1, +1\}$ is a threshold function when there exist real numbers $a_0, a_1, \dots, a_n$ so that

$$f(x) = \operatorname{sgn}\left(a_0 + \sum_{i=1}^{n} a_i x_i\right) \tag{1}$$

i.e., hyperplane that separates vertices of n-dim cube in which f takes value -1 from the vertices in which it takes value 1. The number of all switching functions is obvious, but the basic problem in the study of threshold functions, their enumeration for each n, is still open.

Clearly, two sets of weights $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$ generate different functions f and g by rule (1) iff two points $a, b \in R^{n+1}$ are separated by one of 2^n hyperplanes $1 \pm x_1 \pm \cdots \pm x_n = 0$ in R^{n+1}. Thus, each distinct hyperplane partition of a cube, or each threshold function defined on that cube, corresponds to one of regions in R^{n+1} defined by the arrangement of the upper 2^n hyperplanes. This connection with the number of cells in central hyperplane arrangement yields that the best upper bound of the number $P(2, n)$ of threshold functions given by Schläfli [**3**] in 1850 is:

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$$P(2,n) < 2\sum_{i=0}^{n}\binom{2^n-1}{i} \sim \exp_2\big(n^2 - n\log n - O(n)\big), \quad n\to\infty$$

By direct application of Odlyzko's [**2**] and Winder's [**6**] results, Zuev [**4**] in 1989 obtained the asymptotics $\log_2 P(2,n) \sim n^2$, $n\to\infty$. More precisely, he obtained a lower bound

$$P(2,n) > \exp_2\Big(n^2 - \frac{10n^2}{\ln n} - O(n\ln n)\Big) \tag{2}$$

The interpretation in the terms of hyperplane arrangements permits us to obtain an upper bound $P(K,n) \leqslant 2\sum_{i=0}^{n}\binom{K^n-1}{i}$. For the lower bound it is necessary to develop much more sophisticated methods. Here we sketch the proof for the next lower bound:

$$P(K,n+1) \geqslant \frac{1}{2}\binom{K^n}{\lfloor n - 2\frac{n}{\log_K n} - 4\rfloor}$$
$$\times\left[P\Big(K,\Big\lfloor 2\frac{n}{\log_K n}+3\Big\rfloor\Big) - 1 + \Big\lfloor 2\frac{n}{\log_K n}+4\Big\rfloor\Big(P\Big(K,\Big\lfloor 2\frac{n}{\log_K n}+2\Big\rfloor\Big)-1\Big)\right]$$

As far as we now, this is the best lower bound for $P(K,n)$.

2. Previous work

Without loss of generality we may suppose that

$$E_K = \begin{cases}\{\pm1,\pm3,\dots,\pm(2Q-1)\}, & K=2Q\\ \{0,\pm1,\pm2,\dots,\pm Q\}, & K=2Q+1\end{cases}$$

Hyperplane $H: a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ divides the cubical net E_K^n on the three parts:

$$A_H = E_K^n\cap H^+,\ B_H = E_K^n\cap H,\ C_H = E_K^n\cap H^-.$$

For arbitrary $\varepsilon > 0$ let

$$\begin{aligned} H_{-\varepsilon} &: a_1x_1 + \ldots + a_nx_n - \varepsilon = 0\\ H_{\varepsilon} &: a_1x_1 + \ldots + a_nx_n + \varepsilon = 0.\end{aligned}$$

To be definite, assume that $H^+_{-\varepsilon}\cap H^-_{\varepsilon} \neq \emptyset$ and ε is chosen such that $\big(H^+_{-\varepsilon}\cap H^-_{\varepsilon}\big)\cap E_K^n = B_H$. Then, $A_H = E_K^n \cap H^+_{\varepsilon}$ and $C_H = E_K^n\cap H^-_{-\varepsilon}$. On the other hand, each triplet (A_H, B_H, C_H) defines a hyperplane partition of the $(n+1)$-dim cubic net E_K^{n+1} on the following way:

- If K is even, i.e., $K = 2Q$, the partition is defined by hyperplane

$$\overline{H}: a_1x_1 + \ldots + a_nx_n + \frac{\varepsilon}{2Q-1}x_{n+1} = 0$$

- If K is odd, i.e., $K = 2Q + 1$, the partition is defined by hyperplane
$$\overline{H} : a_1x_1 + \ldots + a_nx_n + \frac{\varepsilon}{Q}x_{n+1} = \delta,$$
where δ is sufficiently small such that $\overline{H} \cap (E_K)^{n+1} = \emptyset$.

Proof of our main result is based on the following simple observation (see [**1**]):

LEMMA 1. *Let H and G be two hyperplanes in R^n so that $B_H \neq B_G$. Then, associated hyperplanes $\overline{H}$ and $\overline{G}$ generate different partitions of the $(n+1)$-dim cubical net.*

It follows from the above that the lower bound of the number $P(K, n)$ can be obtained by estimation of the number of sets B_H appearing in triplets (A_H, B_H, C_H). Let us take the vectors $v_1, v_2, \ldots, v_p \in E_K^n$ in whose linear cover there is no "new" vector from E_K^n. Sets $\{v_1, v_2, \ldots, v_p\}$ will play the role of the B_H!

The most important argument in the construction of the sets B_H is the next theorem, proved in [**1**]. It is a generalization of Odlyzko's result [**2**] on subspaces spanned by random selections of ± 1 vectors.

THEOREM 1. *For any $K \in N$ and any nonnegative integer $p \leqslant n - 2\frac{n}{\log_K n} - 4$ probability P that in the linear cover of p vectors $v_1, v_2, \ldots, v_p$ chosen at random from the set E_K^n there is at least one vector from $E_K^n \setminus \bigcup_{i=1}^{p} \langle v_i \rangle$ tends to zero, as n tends to infinity.*

Let $p = \left\lfloor n - 2\frac{n}{\log_K n} - 4 \right\rfloor$ be the value from Theorem 1 and let $\mathcal{M}_n$ denotes the family of $p \times n$ matrices with elements from set E_K. Let $\mathcal{M}'_n$ be subset of $\mathcal{M}_n$ such that any two rows of the matrix $M \in \mathcal{M}'_n$ are linearly independent. In that case, $\left\| \mathcal{M}'^C_n \right\| \leqslant K^n K \binom{p}{2} K^{n(p-2)}$ i.e., $\|\mathcal{M}'_n\| \sim \|\mathcal{M}_n\|$, $n \to \infty$.

Over the family $\mathcal{M}'_n$ we define the relation $\sim$ on the next way: $A \sim B$ *iff A is obtainable from B by permutation of the rows or by replacement of one row with the row that is collinear to that one.* $\sim$ is equivalence relation and each equivalence class has $p!K^p$ elements. Two matrices from the same class of equivalence generate the same linear subspace. By Theorem1, linear covers of the row-vectors of almost all K-valued matrices $M \in \mathcal{M}_n$ do not contain K-valued vectors $v \in E_K^n$ different from that row-vectors and vectors collinear with any of them. It follows that the number of sets B_H from Lemma 1 is greater than or equal with
$$\frac{1}{2}\frac{K^{np}}{p!K^p}$$
Hence $\log_K P(K, n) \sim n^2$, $n \to \infty$.

The biggest $r \in N$ with the property that any r vectors of the system $\mathcal{S} = \{s_1, \ldots, s_n\}$ are linearly independent is called the *strong rank* of $\mathcal{S}$. It is denoted by $r_{st}(s_1, \ldots, s_n)$.

In [**5**] we have proved that the probability that a random n by n K-valued matrix is singular tends to zero. The next theorem can be proved by a little modification of that one.

THEOREM 2. *Let* $p = \lfloor n - 2\frac{n}{\log_K n} - 4 \rfloor$ *and let* $\overline{a}_1, \dots, \overline{a}_n$ *be at random independently chosen from* E_K^p. *The probability that* $r_{st}(\overline{a}_1, \dots, \overline{a}_n) = p$ *tends to* 1, *as* n *tends to infinity.*

3. The main result

Let $p = \lfloor n - 2\frac{n}{\log_K n} - 4 \rfloor$ and let A_n denotes the event: the rows $v_1, v_2, \dots, v_p \in E_K^n$ and the columns $c_1, c_2, \dots, c_n \in E_K^p$ of the random matrix $M_{p\times n} \in E_K^{pn}$ have the next properties:

(1) in the linear cover of the vectors $v_1, v_2, \dots, v_p$ there is no "new" vector of the same type, i.e., vector from $E_K^n \setminus \bigcup_{i=1}^p \langle v_i \rangle$,

(2) $r_{st}(c_1, \dots, c_n) = p$.

On the basis of theorems 1 and 2, we have that $P(A_n^c) \to 0$ as $n \to \infty$. Therefore, starting from some enough large $n_0 \in N$, the number of p-sets $\{v_1, v_2, \dots, v_p\} \subset E_K^n$, $n \geqslant n_0$, that satisfy the upper two conditions is bigger than $\frac{1}{2}\binom{K^n}{p}$. Let $B = \{v_1, v_2, \dots, v_p\}$ be one of them. Because of the first property, B is one of the sets B_H introduced in previous section. Denote by $\mathcal{B}$ p-dimensional linear subspace spanned by B. In what follows, with different chose of hyperplanes that expand subspace $\mathcal{B}$, we are going to get a different hyperplane partitions of the net E_K^n, with the same set B. Because of the simplicity of presentation, instead of the net E_K^n, only the cube $C = \{-1, +1\}^n$ will be considered on. Generalization on arbitrary K will be obvious.

Let $\mathcal{D}$ be the orthogonal complement of the space $\mathcal{B}$. By $g_1, \dots, g_n$ denote the images of the basis vectors $e_1, \dots, e_n$ of the space R^n under the orthogonal projection $\mathrm{pr}_D : [-1, 1]^n \to \mathcal{D}$ and by G_i, $i = 1, \dots, n$ linear segments $\mathrm{conv}\{-g_i, g_i\}$. First, let us prove that any $d = n - p$ vectors of the set $g_1, \dots, g_n$ are linearly independent. If it would not be true, there would be d vectors, for instance $g_1, \dots, g_d$, and their linear combination $\alpha_1 g_1 + \dots + \alpha_d g_d = 0$, with some nonzero coefficient. This is equivalent with $\alpha_1 e_1 + \dots + \alpha_d e_d \in \mathcal{B} - \{0\}$, i.e.,

$$\det(v_1, \dots, v_p, e_1, \dots, v_d) = \begin{vmatrix} v_1^1 & v_1^2 & \dots & v_1^d & v_1^{d+1} & \dots & v_1^n \\ v_2^1 & v_2^2 & \dots & v_2^d & v_2^{d+1} & \dots & v_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_p^1 & v_p^2 & \dots & v_p^d & v_p^{d+1} & \dots & v_p^n \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{vmatrix} = 0$$

It follows that

$$\begin{vmatrix} v_1^{d+1} & v_1^{d+2} & \dots & v_1^n \\ v_2^{d+1} & v_2^{d+2} & \dots & v_2^n \\ \vdots & \vdots & \ddots & \vdots \\ v_p^{d+1} & v_p^{d+2} & \dots & v_p^n \end{vmatrix} = 0.$$

This is contrary to the assumption $r_{st}(c_1, \dots, c_n) = p$, where $c_i = (v_1^i, v_2^i, \dots, v_p^i)$ are the column vectors of the matrix defined by row vectors $v_1, \dots, v_p \in \{-1, 1\}^n$.

We conclude that the image of cube C under orthogonal projection on to d-dimensional plane $\mathcal{D}$, is a cubical zonotop $Z = G_1 + \ldots + G_n$ and any $r \leqslant d-1$ vectors $g_{i_1}, \dots, g_{i_r} \in \{g_1, \dots, g_n\}$ define r-dimensional facies

$$F = G_{i_1} + \cdots + G_{i_r} + \sum_{j \neq i_1, \dots, i_r} \delta_j g_j, \quad \delta_j \in \{\pm 1\}.$$

Line segments $G_{i_1}, \dots, G_{i_r}$ will be called the components and vector $\sum_{j \neq i_1, \dots, i_r} \delta_j g_j$ the moving vector of F.

We shall now prove that different central partitions of the set of vertices

$$p(Z) = \{\delta_1 g_1 + \delta_2 g_2 + \cdots + \delta_n g_n \mid \delta_1 = \pm 1,\ i = \overline{1, n}\}$$

of zonotope Z yield to the different central partitions of cube C (the partition is central if it is defined by hyperplane that contains the origin; the points of $P(Z)$ are not necessary all distinct).

Let us take hyperplane $H_{d-1} = \langle h_1, \dots, h_{d-1} \rangle$ that define a partition of the set $P(Z)$. Let $h \in \mathcal{D}$ be its normal vector. Than, $H_{n-1} = \langle h_1, \dots, h_{d-1}, v_{i_1}, \dots, v_{i_p} \rangle = \mathcal{V} + H_{d-1}$ is hyperplane in R^n and h is its normal vector. Thus, for any $v \in R^n$:

$$\langle v, h \rangle < 0 \quad \text{iff} \quad \langle \mathrm{pr}_D v, h \rangle < 0$$

Let cube $F_0 = G_1 + \cdots + G_{d-1} + \sum_{j=d}^n \delta_j g_j$ be a facet (maximal or $(d-1)$-dimensional face) of zonotope Z and $B_0 = G_1 + \cdots + G_{d-2} + \sum_{j=d-1}^n \delta_j g_j$ a facet of cube F_0. Denote by F_1 the facet of Z such that $F_0 \cap F_1 = B_0$. The components of the face F_1 are $G_1, \dots, G_{d-2}$ and G_i for some $i \in \{1, \dots, n\} \setminus \{1, \dots, d-1\}$. Without loose of generality it can be assumed that $i = d$. Let B_1 be a facet of F_1 that is the reflection of B_0 in the center of cube F_1. Its components are $G_1, \dots, G_{d-2}$, too. If we continue this procedure, we obtain the sequence of $(d-1)$-dimensional faces $F_0, F_1, \dots, F_{p+2}$ and the sequence of $(d-2)$-dimensional faces $B_0, B_1, \dots, B_{p+2}$ such that $F_i \cap F_{i+1} = B_i$; B_i and B_{i+1} are mutually symmetric faces of the cube F_{i+1}, $i = \overline{0, p}$, F_{p+2} is the reflection of F_0 in the origin, $G_1, \dots, G_{d-2}$ are the components of each $(d-2)$-dimensional face B_i, $i = \overline{0, p+1}$ and the components of the face F_i are $G_1, \dots, G_{d-2}, G_{d-1+i}$, for each $i = \overline{0, p+2}$.

Let $\mathcal{A}$ be $(d-1)$-dimensional affine cover of the cube F_0. Each of $P(2, d-1)$ hyperplane partitions of F_0 can be uniquely expanded to central hyperplane partitions of the zonotope Z. Let us consider the number of hyperplane partitions of Z whose restriction on $\mathcal{A}$ is negative-empty partition of the face F_0.

The number of all hyperplane partitions of $(d-2)$-dimensional cube B_1 is $P(2, d-2)$. $P(2, d-2) - 1$ of them are proper or positive-empty. Let H_{d-3}^1 be hyperplane in $\mathrm{Aff}(B_1)$ that generates one of them. Denote by H_{d-2}^1 $(d-2)$-dimensional subspace that linearly spans H_{d-3}^1. In $\mathcal{D}$ there is a hyperplane H_{d-1}^1 such that

(1) $H_{d-2}^1 \subset H_{d-1}^1$
(2) F_0 is contained in the positive halfspace H_{d-1}^{1+},
(3) B_1 is not contained in the positive halfspace H_{d-1}^{1+}.

If we continue the same procedure for the faces B_i, $i = 2, \dots, p$, in each of p steps we construct $P(2, d-2)$ new partitions of Z with the next properties:

(1) the partitions obtained in i-th step are defined by the the proper or positive-empty partitions of $(d-2)$-dimensional cube B_i in the affine plane $\mathrm{Aff}(B_i)$,
(2) the faces $F_0, B_1, B_2, \dots, B_{i-1}$ are contained in the positive halfspace H_{d-1}^{i+},
(3) the face B_i is not contained in the positive halfspace H_{d-1}^{i+}.

Hence, the number of hyperplane partitions of Z whose restriction on $\mathcal{A}$ is negative-empty partition of the face F_0 (i.e., F_0 is contained in the positive halfspace) is $p\,(P(2, d-2) - 1)$.

The lower bound

$$P(2, n+1) \geqslant \frac{1}{2}\binom{2^n}{\lfloor n - 2\frac{n}{\log_2 n} - 4 \rfloor}$$
$$\times \left[P\Big(2, \Big\lfloor 2\frac{n}{\log_2 n} + 3\Big\rfloor\Big) - 1 + \Big\lfloor 2\frac{n}{\log_2 n} + 4\Big\rfloor \left(P\Big(2, \Big\lfloor 2\frac{n}{\log_2 n} + 2\Big\rfloor\Big) - 1\right)\right]$$

follows from $d = n - p = 2\frac{n}{\log_K n} - 4$ and the above estimates.

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