PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 83(97) (2008), 15–25 DOI: 10.2298/PIM0897015K

COMPLEX POWERS OF OPERATORS

Marko Kostić

Communicated by Stevan Pilipović

Abstract. We define the complex powers of a densely defined operator A whose resolvent exists in a suitable region of the complex plane. Generally, this region is strictly contained in an angle and there exists $\alpha \in [0, \infty)$ such that the resolvent of A is bounded by $O((1+|\lambda|)^{\alpha})$ there. We prove that for some particular choices of a fractional number b, the negative of the fractional power $(-A)^b$ is the c.i.g. of an analytic semigroup of growth order $r > 0$.

1. Introduction

Chronologically, the theory of fractional powers of operators dates from a paper of S. Bochner who constructed the fractional powers of $-\Delta$ in 1949. From then on, many different techniques have been established in the framework of this theory. Let us mention the papers of Balakrishnan [**4**], Komatsu [**11**], Balabane [**3**], Martínez, Sanz and Marco [**15**], Straub [**20**] and deLaubenfels, Yao and Wang [**6**]. Especially, we refer to the monograph [**14**] where the interested reader can find a great part of the theory of fractional powers of non-negative operators including topics related to extensions of Hirsch functional calculus, fractional powers of operators in locally convex spaces, interpolation spaces and the famous Dore-Venni theorem.

This paper is motivated by the work of Straub [**20**] who defined the complex powers of a closed, densely defined operator A satisfying:

(1.1) $\Sigma(\gamma) := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leqslant \gamma\} \cup \{0\} \subset \rho(A)$, for some $\gamma \in (0, \frac{\pi}{2})$;
(1.2) $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^n$, $\lambda \in \Sigma(\gamma)$, for some $M > 0$ and $n \in \mathbb{N}_0$.

For such an operator A, Straub defined in [**20**] the fractional powers $(-A)^b$, for all $b \in \mathbb{C}$. If A fulfills (1.1) and (1.2), then one can employ the construction given in [**6**] to obtain the definition of the fractional operator of $-A$, but only for $b \in [0, \infty)$. In general, the definitions given in [**6**] and [**20**] do not coincide; see [**6**] and [**19**] for further information.

In this paper, we show how ideas developed in [**20**] can be applied to an essentially larger class of closed, densely defined operators. It is worth noting that

2000 *Mathematics Subject Classification*: Primary 47A99; Secondary 47D03, 47D09, 47D62.

the spectrum of an operator belonging this class and $\Sigma(\gamma)$ may have non-empty intersection, for every $\gamma \in (0, \frac{\pi}{2})$. Condition (1.2) in our analysis is replaced by $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\alpha}$, for some $M > 0$ and $\alpha \in [0, \infty)$. This implies that the operators J^b (cf. Section 3), which are fundamental in the construction of fractional powers given in [**9**], [**20**] and this paper, remain bounded for all $b \in \mathbb{C}$ with $\operatorname{Re} b < -(\alpha+1)$. The fractional power $(-A)^b$, $b \in \mathbb{C}$, is defined to be the closure of J^b. If $b \in (0, \frac{1}{2})$, then the negative of the fractional power $(-A)^b$ is the c.i.g. of an analytic semigroup of growth order $r > 0$. This allows one to consider the incomplete higher order abstract Cauchy problems; in this paper, it is necessary that the order of such a problem is strictly greater than two.

2. Basic concepts

Throughout this paper, E denotes a complex Banach space and A a closed, densely defined operator in E. The space of all bounded linear operators from E into E is denoted by $L(E)$; $\rho(A)$ stands for the resolvent set of A while $[D(A)]$ designates the Banach space $D(A)$ equipped with the graph norm. Let $a \in (0,1)$, $C \in (0,1]$ and $d \in (0,1]$; $B_d := \{z \in \mathbb{C} : |z| \leqslant d\}$ and

$$P_{a,C} := \{\xi + i\eta : \xi \in (0,\infty),\ \eta \in \mathbb{R}, \||\eta| \leqslant C\xi^{a}\}.$$

We assume that A satisfies the following conditions:

($*$) $P_{a,C} \cup B_d \subset \rho(A)$,

($**$) $||R(\lambda : A)|| \leqslant M(1+|\lambda|)^{\alpha}$, $\lambda \in P_{a,C} \cup B_d$, for some $M > 0$ and $\alpha \geqslant 0$.

Note, if $P_{a,C} \cup \{0\} \subset \rho(A)$ and $\|R(\lambda : A)\| = O((1+|\lambda|)^{\alpha})$, $\lambda \in P_{a,C} \cup \{0\}$, then there exists a $d \in (0,1]$ and an appropriate $M > 0$ such that ($*$) and ($**$) are fulfilled.

Example 2.1. (a) Let $\alpha > 0$ and $0 < \tau \leqslant \infty$. It is said that the abstract Cauchy problem

$$C_{\alpha+1}(\tau) : \begin{cases} u \in C\big([0,\tau) : [D(A)]\big) \cap C^1([0,\tau) : E), \\ u'(t) = Au(t) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\, x,\ 0 \leqslant t < \tau, \\ u(0) = x, \end{cases}$$

is well posed if it has a unique solution for every $x \in E$, cf. [**1**] if $\alpha \in \mathbb{N}$ and [**13**] if $\alpha > 0$. If $u(t,x)$ is a solution of $C_{\alpha+1}(\tau)$, then the operators $S(t)x := \frac{d}{dt}u(t,x)$, $t \in [0,\tau)$, $x \in E$, are bounded and form an α-times integrated semigroup generated by A. By [**13**, Theorem 2.1], the well-posedness of the problem $C_{\alpha+1}(\tau)$ implies that for every $c \in (0, \frac{\tau}{\alpha})$, there exist constants $c_1 > 0$ and $M > 0$ such that the exponential region $E(c,c_1) := \{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \xi \geqslant c_1,\ |\eta| \leqslant e^{c\xi}\} \subset \rho(A)$ and that $\|R(\lambda : A)\| \leqslant M|\lambda|^{\alpha}$, $\lambda \in E(c,c_1)$. If, additionally, $\overline{D(A)} = E$, then there exists a sufficiently large $\omega > 0$ such that $A - \omega$ satisfies the assumptions ($*$) and ($**$) given above. Further on, if A is the densely defined generator of an α-times integrated semigroup $(S(t))_{t \geqslant 0}$ satisfying $\|S(t)\| \leqslant Mt^{\beta}e^{\omega t}$, for all $t \geqslant 0$, where $\omega \geqslant 0$, $\beta \geqslant 0$, $\alpha > 0$, then for every $a \in (0,1)$, $C \in (0,1]$ and $d \in (0,1]$, there exists a sufficiently large $\omega > 0$ such that $A - \omega$ satisfies ($*$). Furthermore, it can be proved that $A-\omega$ satisfies $(**)$ with $\alpha-\beta-1$ instead of α, see [**17**, page 158] for this refinement.

(b) If the operator A satisfies the assumptions (1.1) and (1.2), then it can be easily proved that, for every $a \in (0,1)$, there exist $C \in (0,1]$ and $b \in (0,1]$ such that $(*)$ and $(**)$ are valid (with $\alpha = n$). It is clear that there exist a great number of multiplication, differential and pseudo-differential operators acting on L^p type spaces which fulfill $(*)$ and $(**)$, but not (1.1). Especially, the construction given in [**20**] cannot be applied even if $E := L^2(\mathbb{R})$ and A is chosen to be the operator $\Delta^2 - i\Delta - I$ with maximal distributional domain. Then the spectrum of A is $\{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \eta^2 = \xi + 1\}$ and, for every $b \in \mathbb{C}$, our construction gives the definition of $(I + i\Delta - \Delta^2)^b$.

We recall the basic fasts about semigroups of growth order $r > 0$, see [**5**], [**18**], [**19**] and [**21**] for this notion. An operator family $(T(t))_{t>0}$ in $L(E)$ is a semigroup of growth order r if it satisfies:

(i) $T(t+s) = T(t)T(s),\ t,\ s > 0$,
(ii) for every $x \in E$, the mapping $t \mapsto T(t)x$ is continuous,
(iii) $\|t^r T(t)\| = O(1),\ t \to 0+$,
(iv) $T(t)x = 0$ for all $t > 0$ implies $x = 0$, and
(v) $E_0 = \bigcup_{t>0} T(t)E$ is dense in E.

The infinitesimal generator of $(T(t))_{t>0}$ is defined by

$$A_0 := \Big\{(x,y) \in E^2 : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y\Big\}.$$

It is a closable linear operator and its closure $A = \overline{A_0}$ is called the complete infinitesimal generator (c.i.g.) of $(T(t))_{t>0}$. Following Tanaka [**21**], if the semigroup $(T(t))_{t>0}$ of growth order $r > 0$ has an analytic extension to $\Sigma_\gamma := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| < \gamma\}$, for some $\gamma \in (0, \frac{\pi}{2})$, denoted by the same symbol, and if additionally there exists an $\omega \in \mathbb{R}$ such that, for every $\delta \in (0,\gamma)$, there exists a suitable constant $M_\delta > 0$ with $\|z^r T(z)\| \leqslant M_\delta e^{\omega \operatorname{Re} z},\ z \in \Sigma_\delta$, then the family $(T(t))_{t\in\Sigma_\gamma}$ is called an analytic semigroup of growth order r. We will use the following notations. For given $a \in (0,1)$, $C \in (0,1]$ and $d \in (0,1]$, put $\Gamma_1(a,C,d) := \{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \eta = -C\xi^a,\ \xi^2 + \eta^2 \geqslant d^2\}$. It is clear that there exists a unique $\varepsilon(a,C,d) \in (0,d)$ such that $(\varepsilon(a,C,d), -C\varepsilon(a,C,d)^a) \in \partial B_d$. We define $\Gamma_2(a,C,d) := \{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \xi^2 + \eta^2 = d^2,\ \xi \leqslant \varepsilon(a,C,d)\}$ and $\Gamma_3(a,C,d) := \{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in \mathbb{R},\ \eta = C\xi^a,\ \xi^2 + \eta^2 \geqslant d^2\}$. The upwards oriented curve $\Gamma(a,C,d)$ is defined by $\Gamma(a,C,d) := \Gamma_1(a,C,d) \cup \Gamma_2(a,C,d) \cup \Gamma_3(a,C,d)$; put now $H(a,C,d) := \{\xi + i\eta : \xi \in (0,\infty),\ \eta \in \mathbb{R},\ |\eta| \leqslant C\xi^a\} \cup B_d$. For given $\tilde{d} \in (0,d]$ and $\tilde{a} \in (0,a]$, one can find a suitable constant $\tilde{C}$ so that $\Gamma(\tilde{a},\tilde{C},\tilde{d}) \subset H(a,C,d)$, where we define $\Gamma(\tilde{a},\tilde{C},\tilde{d})$ in the same way as $\Gamma(a,C,d)$.

3. Operators J^b, $b \in \mathbb{C}$

In order to construct the fractional powers $(-A)^b$, for every $b \in \mathbb{C}$, we define a closable linear operator J^b. As in [**20**], if $\alpha \in \mathbb{N}_0$ and [**9**], if $\alpha = -1$, the construction is based on improper integrals of the form

$$\frac{1}{2\pi i}\int\limits_{\Gamma}(-\lambda)^b R(\lambda:A)x\ d\lambda.$$

Proposition 3.1. *Let $b\in\mathbb{C}$ satisfy $\operatorname{Re} b<-(\alpha+1)$ and let $x\in E$. Then the integral*

$$I(b)x:=\frac{1}{2\pi i}\int\limits_{\Gamma(a,C,d)}(-\lambda)^b R(\lambda:A)x\,d\lambda$$

exists and defines a bounded linear operator $I(b)\in L(E)$. Moreover, if for some $\tilde a\in(0,a]$, $\tilde C\in(0,C]$ and $\tilde d\in(0,d]$: $\Gamma(\tilde a,\tilde C,\tilde d)\subset H(a,C,d)$, then

$$I(b)x=\frac{1}{2\pi i}\int\limits_{\Gamma(\tilde a,\tilde C,\tilde d)}(-\lambda)^b R(\lambda:A)x\,d\lambda.$$

Proof. The proof is essentially contained in that of Lemma 1.1 in [**20**]. We sketch it for the sake of completeness. Note that the function $\lambda\mapsto(-\lambda)^b$ $(1^b=1)$ is analytic in $\mathbb{C}\smallsetminus[0,\infty)$ and that

$$|(-\lambda)^b|\leqslant|\lambda|^{\operatorname{Re} b}e^{\pi|\operatorname{Im} b|},\ \lambda\in\mathbb{C}\smallsetminus\{0\}. \tag{3.1}$$

The integral over $\Gamma_2(a,C,d)$ exists since $\Gamma_2(a,C,d)$ is a finite path. The estimate $(**)$ implies

$$\begin{aligned}&\Bigg\|\frac{1}{2\pi i}\int\limits_{\Gamma_3(a,C,d)}(-\lambda)^b R(\lambda:A)x\,d\lambda\Bigg\|\\ &\leqslant\frac{M}{2\pi}\int_{\varepsilon(a,C,d)}^{\infty}\Big(\sqrt{t^2+C^2t^{2a}}\Big)^{\operatorname{Re} b}e^{\pi|\operatorname{Im} b|}\Big(1+\sqrt{t^2+C^2t^{2a}}\Big)^{\alpha}\big(1+Ca\varepsilon(a,C,d)^{a-1}\big)\,dt\\ &\leqslant\frac{M\big(1+Ca\varepsilon(a,C,d)\big)^{a-1}e^{\pi|\operatorname{Im} b|}}{2\pi}\int\limits_{\varepsilon(a,C,d)}^{\infty}\Big(\sqrt{t^2+C^2t^{2a}}\Big)^{\operatorname{Re} b}t^{\alpha}dt.\end{aligned}$$

Since $(t^2+C^2t^{2a})^{\operatorname{Re} b/2}t^{\alpha}\sim t^{\alpha+\operatorname{Re} b}$, $t\to+\infty$, the integral over $\Gamma_3(a,C,d)$ exists. Similarly, the integral over $\Gamma_1(a,C,d)$ exists. It remains to be shown that the integral $I(b)$ is independent of the choice of a curve $\Gamma(a,C,d)$. Let R be sufficiently large and let the curve $\Gamma_R=\{Re^{it}:t\in[\arctan(\tilde C R^{\tilde a-1}),\arctan(CR^{a-1})]\}$ be upwards oriented. Then

$$\Bigg\|\int\limits_{\Gamma_R}(-\lambda)^b R(\lambda:A)x\,d\lambda\Bigg\|\leqslant 2\pi e^{\pi|\operatorname{Im} b|}R^{\operatorname{Re} b}(1+R)^{\alpha}R\to 0,\ R\to+\infty.$$

The proof completes an elementary application of Cauchy's theorem. $\square$

If no confusion seems likely, we shall simply denote $\Gamma(a,C,d)$, $H(a,C,d)$ and $\varepsilon(a,C,d)$ by Γ, H and ε, respectively.

Let $t\in\mathbb{R}$. Denote by $\lfloor t\rfloor$ and $\lceil t\rceil$ the largest integer $\leqslant t$ and the smaller integer $\geqslant t$, respectively. Put $\{t\}:=t-\lfloor t\rfloor$. Note, if $b\in\mathbb{C}$, then $\operatorname{Re}(b-\lfloor\operatorname{Re} b+\alpha\rfloor-2)\in$

$[-(\alpha+2), -(\alpha+1))$. Hence, the following definition of the operator J^b makes a sense.

DEFINITION 3.1. Let $b \in \mathbb{C}$. The operator J^b is defined as follows: $D(J^b) := D(A^{\lfloor \operatorname{Re} b+\alpha \rfloor +2})$ and

$$J^b x := \begin{cases} I(b)x, & -(\alpha+2) \leqslant \operatorname{Re} b < -(\alpha+1), \\ I(b - \lfloor \operatorname{Re} b + \alpha \rfloor - 2)(-A)^{\lfloor \operatorname{Re} b+\alpha \rfloor +2} x, & \text{otherwise.} \end{cases}$$

REMARK 3.1. If a densely defined operator A satisfies (1.1) and (1.2), then we have already seen that, for every $a \in (0,1)$, there exist $C \in (0,1]$ and $d \in (0,1]$ such that $(*)$ and $(**)$ are fulfilled. In this case, the definition of J^b is equivalent to the corresponding one given in Definition 1.2 of [**20**].

In what follows, we will use the generalized resolvent equation

$$(-\lambda)^{-n-1} R(\lambda : A)(-A)^{n+1} x = R(\lambda : A)x + \sum_{i=0}^{n} (-\lambda)^{-i-1}(-A)^i x, \tag{3.2}$$

if $\lambda \in \rho(A)$, $\lambda \neq 0$, $n \in \mathbb{N}_0$, $x \in D(A^{n+1})$, and the simple equality $\int_\Gamma (-\lambda)^b \, d\lambda = 0$, if $\operatorname{Re} b < -1$.

PROPOSITION 3.2. *Suppose $x \in D(A^{\lfloor \operatorname{Re} b+\alpha \rfloor +2})$. Then*

$$J^b x = \begin{cases} \frac{1}{2\pi i} \int\limits_\Gamma (-\lambda)^b R(\lambda : A)x \, d\lambda, & \operatorname{Re} b < 0, \\ \frac{1}{2\pi i} \int\limits_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b \rfloor -1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor +1} x \, d\lambda, & \operatorname{Re} b \geqslant 0. \end{cases}$$

PROOF. Suppose $\operatorname{Re} b < 0$. If $\operatorname{Re} b \in [-(\alpha+2), -(\alpha+1))$, the conclusion follows directly from the definition of J^b. If $\operatorname{Re} b \notin [-(\alpha+2), -(\alpha+1))$, then by Definition 3.1 and (3.2):

$$\begin{aligned} J^b x &= J^{b - \lfloor \operatorname{Re} b+\alpha \rfloor -2}(-A)^{\lfloor \operatorname{Re} b+\alpha \rfloor +2} x \\ &= \frac{1}{2\pi i} \int\limits_\Gamma (-\lambda)^{b - \lfloor \operatorname{Re} b+\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b+\alpha \rfloor +2} x \, d\lambda \\ &= \frac{1}{2\pi i} \int\limits_\Gamma (-\lambda)^b (R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b+\alpha \rfloor +1} (-\lambda)^{-i-1}(-A)^i x) \, d\lambda. \end{aligned}$$

If $\operatorname{Re} b < 0$ and $i = 0, 1, \ldots, \lfloor \operatorname{Re} b + \alpha \rfloor + 1$, then $\operatorname{Re} b - i - 1 < -1$ and the last term equals

$$\frac{1}{2\pi i} \int\limits_\Gamma (-\lambda)^b R(\lambda : A)x \, d\lambda$$

as claimed. Suppose now $\operatorname{Re} b \geqslant 0$. Then (3.2) implies

$$(-\lambda)^{b - \lfloor \operatorname{Re} b \rfloor -1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b \rfloor +1} x = (-\lambda)^b (R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b \rfloor} (-\lambda)^{-i-1}(-A)^i x).$$

By Definition 3.1,

$$
\begin{aligned}
J^b x &= J^{b-\lfloor \operatorname{Re} b+\alpha\rfloor-2}(-A)^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}x \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \operatorname{Re} b+\alpha\rfloor-2}R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}x\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b}\Big(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b+\alpha+1\rfloor}(-\lambda)^{-i-1}(-A)^i x\Big)\,d\lambda,
\end{aligned}
$$

and since for $j = \lfloor \operatorname{Re} b + 1\rfloor, \dots, \lfloor \operatorname{Re} b + \alpha + 1\rfloor$, $\operatorname{Re} b - i - 1 < -1$, we obtain

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b}\Big(R(\lambda : A)x + \sum_{i=0}^{\lfloor \operatorname{Re} b\rfloor}(-\lambda)^{-i-1}(-A)^i x\Big)\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \operatorname{Re} b\rfloor-1}R(\lambda : A)(-A)^{\lfloor \operatorname{Re} b+1\rfloor}x\,d\lambda.
\end{aligned}
$$

The proof is completed. □

Put $C^b := (-A)^{\lfloor \operatorname{Re} b+\alpha\rfloor+2}J^{b-\lfloor \operatorname{Re} b+\alpha\rfloor-2}$. Then, for every $b \in \mathbb{C}$, C^b is a closed linear operator. Proceeding similarly as in the proof of [**20**, Proposition 1.3], one obtains that, for every $b \in \mathbb{C}$ with $\operatorname{Re} b \geqslant -(\alpha+1)$, $J^b \subset C^b$ and, consequently, J^b is a closable operator. Clearly, $J^b \in L(E)$ for every $b \in \mathbb{C}$ with $\operatorname{Re} b < -(\alpha+1)$.

LEMMA 3.1. *Let $b \in \mathbb{C}$. Then J^b is a closable operator.*

LEMMA 3.2. *Let $b \in \mathbb{C}$. Then*
(i′) $J^b x = J^{b+k}(-A)^{-k}x$, $k \in \mathbb{N}_0$, $x \in D(J^b)$, *and*
(ii′) $J^b x = J^{b+k}(-A)^{-k}x$, *if* $-k \in \mathbb{N}$ *and* $x \in D(A^{\max(-k,\lfloor \operatorname{Re} b+\alpha+2\rfloor)})$.

PROOF. (i′) If $k = 0$, the proof is trivial. Suppose now $k = 1$. If $-(\alpha+2) \leqslant \operatorname{Re} b < -(\alpha+1)$, then $\operatorname{Re} b + 1 < 0$ and by Proposition 3.2, we obtain

$$
\begin{aligned}
J^{b+1}(-A)^{-1}x &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b+1}R(\lambda : A)(-A)^{-1}x\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b+1}\frac{R(\lambda : A)x - (-A)^{-1}x}{-\lambda}\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b}R(\lambda : A)x\,d\lambda.
\end{aligned}
$$

If $\operatorname{Re} b \notin [-(\alpha+2), -(\alpha+1))$, the assertion follows from Definition 3.1. Now (i′) follows by induction; (ii′) can be proved by the use of (i′). □

For $b \in \mathbb{C}$, denote $\langle b\rangle := \max(0, \lfloor \operatorname{Re} b + \alpha\rfloor + 2)$. Note that $\langle b + c\rangle \leqslant \langle b\rangle + \langle c\rangle$. The expected semigroup property of the family $(J^b)_{b\in\mathbb{C}}$ can be proved similarly as in [**20**, Lemma 1.4]. More precisely, we have:

Proposition 3.3. *Let* $b,\ c \in \mathbb{C}$. *Then* $J^b J^c x = J^{b+c}x$, $x \in D(A^{\langle b\rangle + \langle c\rangle})$.

If $k \in \mathbb{N}$ and $x \in D(A^k)$, put $\|x\|_k := \|x\| + \|Ax\| + \cdots + \|A^k x\|$. We prove the following lemma which naturally corresponds to [**20**, Lemma 1.5].

Proposition 3.4. *Let* $b \in \mathbb{Z}$ *and* $x \in D(A^{\lfloor b+\alpha\rfloor+2})$. *Then* $J^b x = (-A)^b x$.

Proof. By Lemma 3.2, it is enough to prove the statement in the case $b = 1$. We have to prove (see Proposition 3.2) that

$$\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-1}R(\lambda : A)(-A)^2 x\, d\lambda = -Ax.$$

By the resolvent equation, if $x \in D(A^{\lfloor \alpha+1\rfloor})$, then there exists a suitable constant $M > 0$ such that

$$\|R(\lambda : A)x\| \leqslant M|\lambda|^{\alpha-\lfloor\alpha\rfloor-1}\|x\|_{\lfloor\alpha+1\rfloor},\ \lambda \in H \cup \Gamma,\ |\lambda| \geqslant d.$$

Let $R > d$. Then there exists a unique number $\kappa(R) \in (0, R)$ such that $\kappa(R)^2 + C^2\kappa(R)^{2a} = R^2$. Denote $\Gamma_R = \{Re^{i\theta} : |\theta| \leqslant \arctan(C\kappa(R)^{a-1})\}$; we assume that Γ_R is upwards oriented. If $x \in D(A^{\lfloor\alpha\rfloor+3})$, then $A^2x \in D(A^{\lfloor\alpha\rfloor+1})$ and the previous inequality implies

$$\bigg\|\int_{\Gamma_R}(-\lambda)^{-1}R(\lambda : A)(-A)^2 x\, d\lambda\bigg\| \leqslant 2\pi\frac{M}{R}R^{\alpha-\lfloor\alpha\rfloor-1}\|x\|_{\lfloor\alpha+3\rfloor}R \to 0,\ R \to +\infty.$$

The rest of proof follows by an application of Cauchy's integral formula. $\square$

Proceeding as in [**20**], one can prove the next proposition.

Proposition 3.5. *The next assertions are valid*

(i) *If* $b \in \mathbb{Z}$, *then* $\overline{J^b} = (-A)^b$.
(ii) *If* $\operatorname{Re} b > \alpha + 1$, *then* $\overline{J^b} = C^b$.

4. Fractional powers and semigroups generated by them

Definition 4.1. Let $b \in \mathbb{C}$. Then the fractional power $(-A)^b$ of the operator $-A$ is defined by $(-A)^b := \overline{J^b}$.

The next theorem clarifies the basic structural properties of fractional powers. See [**20**] for the proof.

Theorem 4.1. *Let* $b,\ c \in \mathbb{C}$ *and* $k \in \mathbb{N}_0$. *Then*

(a) $D(A^{\lfloor\operatorname{Re} b+\alpha\rfloor+2+k})$ *is a core for* $(-A)^b$.
(b) $(-A)^{b+c} \subset \overline{(-A)^b(-A)^c}$.
(c) $(-A)^{b+c} = \overline{(-A)^b(-A)^c}$, *if* $(-A)^{b+c} = C^{b+c}$.
(d) $\overline{(-A)^{-b}(-A)^b} = I$; $(-A)^{-b}(-A)^b x = x$, $x \in D((-A)^b)$.
(e) $(-A)^b$ *is injective.*

In the rest of this section, we consider fractional powers as generators of analytic semigroups of growth order $r > 0$.

THEOREM 4.2. *Let $b \in (0, \frac{1}{2})$. Then the operator $-(-A)^b$ is the c.i.g. of an analytic semigroup $(T_b(z))_{z \in \Sigma_{\arctan(\cos \pi b)}}$ of growth order $\frac{\alpha+1}{b}$, where*

$$T_b(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda,\ \ z \in \Sigma_{\arctan(\cos \pi b)}. \tag{4.1}$$

PROOF. Our choice of b implies $b\pi < \frac{\pi}{2}$. Put $\gamma := \arctan(\cos \pi b)$. Then $\gamma \in (0, \frac{\pi}{2})$ and, for every $z = \xi + i\eta \in \Sigma_\gamma$, we have $\xi \cos(b\pi) - |\eta| > 0$. Furthermore,

$$\left|e^{-z(-\lambda)^b}\right| = e^{-\xi|\lambda|^b \cos(b \arg(-\lambda)) + \eta |\lambda|^b \sin(b \arg(-\lambda))} \leqslant e^{-(\xi \cos(b\pi) - |\eta|)|\lambda|^b}.$$

The convergence of the curve integral over Γ_1 and Γ_3 follows from the computation

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} \int\limits_{\Gamma_1} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda \right\| \\
&\leqslant \frac{M}{2\pi} \int\limits_{\varepsilon}^{\infty} e^{-(\xi\cos(b\pi)-|\eta|)\left(\sqrt{t^2+t^{2a}}\right)^b} \left(1 + \sqrt{t^2+t^{2a}}\right)^\alpha (1 + a\varepsilon^{a-1}) dt \\
&\leqslant \frac{M(1+a\varepsilon^{a-1})}{2\pi} \left[\int\limits_{\varepsilon}^{\infty} e^{-(\xi\cos(b\pi)-|\eta|)t^b} \left(1+\sqrt{2}\right)^\alpha dt + \int_1^\infty e^{-(\xi\cos(b\pi)-|\eta|)t^b} \left(1+\sqrt{2}\right)^\alpha t^\alpha dt \right] \\
&\leqslant \frac{M\left(1+\sqrt{2}\right)^\alpha (1+a\varepsilon^{a-1})}{2\pi} \left[(1-\varepsilon) e^{-(\xi\cos(b\pi)-|\eta|)\varepsilon^b} + \int_0^\infty e^{-(\xi\cos(b\pi)-|\eta|)t^b} t^\alpha dt \right] \\
&= \frac{M\left(1+\sqrt{2}\right)^\alpha (1+a\varepsilon^{a-1})}{2\pi} \left[(1-\varepsilon) e^{-(\xi\cos(b\pi)-|\eta|)\varepsilon^b} + \frac{1}{b}\Gamma\left(\frac{\alpha+1}{b}\right) \left(\xi\cos(b\pi) - |\eta|\right)^{-\frac{\alpha+1}{b}} \right],
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function in the last estimate. The convergence of the integral over Γ_2 is obvious and one obtains

$$\left\| \frac{1}{2\pi i} \int\limits_{\Gamma_2} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda \right\| \leqslant M e^{-(\xi\cos(b\pi)-|\eta|)d^b} (1+d)^{\alpha+1}.$$

Hence, for every $\delta \in (0, \gamma)$, we have $\|z^{\frac{\alpha+1}{b}} T_b(z)\| = O(1)$, $z \in \Sigma_\delta$. By an elementary application of Cauchy's formula, it follows that the integral in (4.1) does not depend on the choice of the curve $\Gamma(a, C, d)$. Fix a $\lambda_0 \in \rho(A) \smallsetminus H$. Here we would like to point out that $\rho(A) \smallsetminus H$ is a nonempty set since $\rho(A)$ is an open subset of $\mathbb{C}$. Using the same arguments as in [**20**, Propositions 2.3, 2.5, 2.6, 2.7 and 2.8], one obtains:

1. Let $m \in \{0, 1\}$. The improper integral

$$\int\limits_{\Gamma} \frac{-(-\lambda)^{mb} e^{-z(-\lambda)^b}}{(\lambda - \lambda_0)^{\lfloor b+\alpha \rfloor + 2}} R(\lambda : A)\, d\lambda$$

converges uniformly for $z \in \overline{\Sigma_\gamma}$.

2. The mapping $z \mapsto T_b(z)$, $z \in \Sigma_\gamma$, is analytic and

$$\frac{d^n}{dz^n} T_b(z) = \frac{(-1)^n}{2\pi i} \int\limits_{\Gamma} (-\lambda)^{nb} e^{-t(-\lambda)^b} R(\lambda : A)\, d\lambda, \quad n \in \mathbb{N}.$$

3. $T_b(z_1+z_2)=T_b(z_1)T_b(z_2),\ z_1,\ z_2\in\Sigma_\gamma$.

4. If $x\in D(A^{\lfloor b+\alpha\rfloor+2})=D(J^b)$, then $\lim_{z\to 0,\ z\in\Sigma_\gamma}\frac{T_b(z)x-x}{z}=-J^bx$. Especially, the continuity set $\Omega_b:=\{x\in E: T_b(t)x\to x \text{ as } t\to 0+\}$ is dense in E. Since the set $E_b:=\bigcup_{t>0}T_b(t)E$ is dense in Ω_b, it follows that $\overline{E_b}=E$.

5. For every $z\in\Sigma_\gamma$, $T_b(z)$ is an injective operator.

By the foregoing, we obtain that $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{\alpha+1}{b}$. Denote by A_b the generator of $(T_b(t))_{t>0}$. By 4, we obtain that $-J^b\subset A_b$. Consequently, $-(-A)^b\subset\overline{A_b}$. Since $\int_\Gamma e^{-t(-\lambda)^b}\lambda^n\,d\lambda=0,\ n\in\mathbb{N}_0$, one can repeat literally the arguments given in [**20**, Lemma 2.10] in order to obtain that, for every $x\in E$ and $t>0$, $T_b(t)x\in D(A^n)$ and that

$$A^nT_b(t)x=\frac{1}{2\pi i}\int\limits_\Gamma e^{-t(-\lambda)^b}\lambda^nR(\lambda:A)x\,d\lambda. \tag{4.2}$$

Particularly, $E_b\subset D_\infty(A)$. In order to prove that $\overline{A_b}\subset-(-A)^b$, one can proceed in the same manner as in [**20**]. Actually, it is sufficient to replace the natural number n in the proofs of [**20**, Propositions 2.11 and 2.12] with $\lfloor b+\alpha\rfloor$. □

The following theorem is the main result of this paper and possesses several natural consequences in the theory of partial differential equations.

Theorem 4.3. *Suppose that a closed, densely defined operator A satisfies $(*)$ and $(**)$. Then, for every $n\in\mathbb{N}$ with $n\geqslant 3$ and $x\in D(A^{\lfloor\frac{1}{n}+\alpha\rfloor+2})$, the abstract Cauchy problem*

$$\begin{cases} u\in C\big((0,\infty):[D(A)]\big)\cap C^n\big((0,\infty):E\big),\\ \dfrac{d^n}{dt^n}u(t)=(-1)^{n+1}Au(t),\ t>0,\\ \lim\limits_{t\to 0+}u(t)=x,\ \sup\limits_{t>0}\|u(t)\|<\infty,\end{cases}$$

has a solution $u(t)=T_{\frac{1}{n}}(t)x,\ t>0$. Further on, u can be analytically extended to $\Sigma_{\arctan(\cos\frac{\pi}{n})}$ and, for every $\delta\in(0,\arctan(\cos\frac{\pi}{n}))$ and $i\in\mathbb{N}_0$,

$$\sup_{z\in\Sigma_\delta}\Big\|z^{i+n\alpha+n}\frac{d^i}{dz^i}u(z)\Big\|<\infty.$$

Proof. One can use the assertion 2 given in the proof of the Theorem 4.2 and (4.2) to obtain that $\frac{d^n}{dt^n}u(t)=(-1)^{n+1}Au(t),\ t>0$. By Theorem 4.2, u can be analytically extended to $\Sigma_{\arctan(\cos\frac{\pi}{n})}$. Due to the proof of Theorem 4.2 (see the assertion 4), we conclude that $\lim_{t\to 0+}u(t)=x$. Let $\delta\in(0,\arctan(\cos\frac{\pi}{n}))$ and $z\in\Sigma_\delta$ be fixed. Since

$$\frac{d^i}{dz^i}u(z)=\frac{(-1)^n}{2\pi i}\int\limits_\Gamma(-\lambda)^{\frac{i}{n}}e^{-z(-\lambda)^{\frac{1}{n}}}R(\lambda:A)x\,d\lambda,$$

and $\|(-\lambda)^{\frac{i}{n}}R(\lambda:A)\| \leqslant M(1+|\lambda|)^{\alpha+\frac{i}{n}}$, it follows (see the proofs of Theorem 4.2 and [**20**, Proposition 2.2]) that

$$\Big\|\frac{d^i}{dz^i}u(z)\Big\| = O\Big(\big(\xi\cos(\frac{\pi}{n}) - |\eta|\big)^{-(\alpha+\frac{i}{n}+1)/\frac{i}{n}}\Big), \quad z \in \Sigma_\delta.$$

Hence, $\sup_{z\in\Sigma_\delta} \|z^{i+n\alpha+n}\frac{d^i}{dz^i}u(z)\| < \infty$. □

The author wants to thank B. Straub and F. Periago for sending him the copies of [**17**]–[**20**] and [**22**]. The genesis of the paper is partially supported by Ministry of Sciences and Technology of Republic Serbia and Faculty of Technical Sciences of Novi Sad.

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Faculty of Technical Sciences
University of Novi Sad
21000 Novi Sad
Serbia
marco.s@verat.net

(Received 14 06 2006)