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DRAZIN INVERSES OF OPERATORS WITH RATIONAL RESOLVENT

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Abstract. Let A be a bounded linear operator on a Banach space such that the resolvent of A is rational. If 0 is in the spectrum of A, then it is well known that A is Drazin invertible. We investigate spectral properties of the Drazin inverse of A. For example we show that the Drazin inverse of A is a polynomial in A.

1. Introduction and terminology

In this paper X is always a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. For $A \in \mathcal{L}(X)$ we write $N(A)$ for its kernel and $A(X)$ for its range. We write $\sigma(A)$, $\rho(A)$ and $R_\lambda(A)$ for the spectrum, the resolvent set and the resolvent operator $(A-\lambda)^{-1}$ $(\lambda \notin \sigma(A))$ of A, respectively. The ascent of A is denoted by $\alpha(A)$ and the descent of A is denoted by $\delta(A)$.

An operator $A \in \mathcal{L}(X)$ is *Drazin invertible* if there is $C \in \mathcal{L}(X)$ such that

(i) $CAC = C$, (ii) $AC = CA$ and (iii) $A^{\nu+1}C = A^\nu$ for nonnegative integer ν.

In this case C is uniquely determined (see [**2**]) and is called the *Drazin inverse* of A. The smallest nonnegative integer ν such that (iii) holds is called the *index* $i(A)$ of A. Observe that

$$0 \in \rho(A) \Leftrightarrow A \text{ is Drazin invertible and } i(A) = 0.$$

The following proposition tells us exactly which operators are Drazin invertible with index > 0:

1.1. Proposition. *Let $A \in \mathcal{L}(X)$ and let ν be a positive integer. Then the following assertions are equivalent:*

(1) *A is Drazin invertible and $i(A) = \nu$.*
(2) *$\alpha(A) = \delta(A) = \nu$.*
(3) *$R_\lambda(A)$ has a pole of order ν at $\lambda = 0$.*

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Proof. [**2**, § 5.2] and [**3**, Satz 101.2]. □

The next result we will use frequently in our investigations.

1.2. Proposition. *Suppose that $A \in \mathcal{L}(X)$ is Drazin invertible, $i(A) = \nu \geqslant 1$, P is the spectral projection of A associated with the spectral set $\{0\}$ and that C is the Drazin inverse of A. Then*

$$\begin{aligned}
&P = I - AC, \quad N(C) = N(A^\nu) = P(X),\\
&C(X) = N(P) = A^\nu(X),\\
&C \textit{ is Drazin invertible}, \quad i(C) = 1,\\
&ACA \textit{ is the Drazin inverse of } C,\\
&0 \in \sigma(C) \textit{ and } \sigma(C) \smallsetminus \{0\} = \{\tfrac{1}{\lambda} : \lambda \in \sigma(A) \smallsetminus \{0\}\}.
\end{aligned}$$

Proof. We have $P = I - AC$, $N(A^\nu) = P(X)$ and $\sigma(C) \smallsetminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma(A) \smallsetminus \{0\}\}$ by [**2**, § 52]. It is clear that $0 \in \sigma(C)$. From Proposition 1.1 and [**3**, Satz 101.2] we get $N(P) = A^\nu(X)$. If $x \in X$ then $Cx = 0 \Leftrightarrow Px = x$, hence $N(C) = P(X)$. From $P = I - AC = I - CA$ it is easily seen that $N(P) = C(X)$. Let $B = ACA$. Then

$$\begin{aligned}
C^2B &= CBC = CACAC = CAC = C,\\
CB &= CACA = ACAC = BC\\
BCB &= ACACACA = ACACA = ACA = B.
\end{aligned}$$

This shows that C is Drazin invertible, B is the Drazin inverse of C and that $i(C) = 1$. □

Now we introduce the class of operators which we will consider in this paper. We say that $A \in \mathcal{L}(X)$ has a *rational resolvent* if

$$R_\lambda(A) = \frac{P(\lambda)}{q(\lambda)}$$

where $P(\lambda)$ is a polynomial with coefficients in $\mathcal{L}(X)$, $q(\lambda)$ is polynomial with coefficients in $\mathbb{C}$ and where P and q have no common zeros. We use the symbol $\mathcal{F}(X)$ to denote the subclass of $\mathcal{L}(X)$ consisting of those operators whose resolvent is rational. For $A \in \mathcal{L}(X)$ let $\mathcal{H}(A)$ be the set of all functions $f : \triangle(f) \to \mathbb{C}$ such that $\triangle(f)$ is an open set in $\mathbb{C}$, $\sigma(A) \subseteq \triangle(f)$ and f is holomorphic on $\triangle(f)$. For $f \in \mathcal{H}(A)$ the operator $f(A) \in \mathcal{L}(X)$ is defined by the usual operational calculus (see [**3**] or [**4**]).

The following proposition collects some properties of operators in $\mathcal{F}(X)$. An operator $A \in \mathcal{L}(X)$ is called *algebraic* if $p(A) = 0$ for some nonzero polynomial p.

1.3. Proposition. *Let $A \in \mathcal{L}(X)$. Then*

(1) *$A \in \mathcal{F}(X)$ if and only if $\sigma(A)$ consists of a finite number of poles of $R_\lambda(A)$.*
(2) *$A \in \mathcal{F}(X)$ if and only if A is algebraic.*
(3) *If $\dim A(X) < \infty$, then $A \in \mathcal{F}(X)$.*

(4) *If* $A \in \mathcal{F}(X)$ *and* $f \in \mathcal{H}(A)$, *then* $f(A) = p(A)$ *for some polynomial* p.
(5) *If* $A \in \mathcal{F}(X)$, *the* $p(A) \in \mathcal{F}(X)$ *for every polynomial* p.

PROOF. [**4**, Chapter V.11] □

1.4. COROLLARY. *Suppose that* $A \in \mathcal{F}(X)$ *and* $0 \in \rho(A)$. *Then* $A^{-1} \in \mathcal{F}(X)$ *and* A^{-1} *is a polynomial in* A.

PROOF. Define the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(\lambda) = \lambda^{-1}$. Then $f \in \mathcal{H}(A)$ and $f(A) = A^{-1}$. Now apply Proposition 1.3 (4) and (5). □

REMARK. That $A \in \mathcal{F}(X)$ and $0 \in \rho(A)$ implies $A^{-1} \in \mathcal{F}(X)$ is also shown in [**1**, Theorem 2]. In Section 2 we will give a further proof of this fact.

2. Drazin inverses of operators in $\mathcal{F}(X)$

Throughout this section A will be an operator in $\mathcal{F}(X)$ and $\sigma(A) = \{\lambda_1, \dots, \lambda_k\}$, where $\lambda_1, \dots, \lambda_k$ are the distinct poles of $R_\lambda(A)$ of orders $m_1, \dots, m_k$ (see Proposition 1.3 (1)).

Recall that $m_j = \alpha(A - \lambda_j) = \delta(A - \lambda_j)$ $(j = 1, \dots, k)$. Let

$$m_A(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}. \tag{2.1}$$

By [**4**, Theorem V.10.7],

$$m_A(A) = 0.$$

The polynomial m_A is called the *minimal polynomial* of A. It follows from [**4**, Theorem V.10.7] that m_A divides any other polynomial p such that $p(A) = 0$. In what follows we always assume that m_A has degree n, thus $n = m_1 + \dots + m_k$ and that m_A has the representations (2.1) and

$$m_A(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_{n-1}\lambda^{n-1} + \lambda^n. \tag{2.2}$$

Observe that

$$0 \in \rho(A) \Leftrightarrow a_0 \neq 0$$

and that

$$0 \text{ is a pole of order } \nu \geqslant 1 \text{ of } R_\lambda(A) \Leftrightarrow a_0 = \dots = a_{\nu-1} = 0 \text{ and } a_\nu \neq 0.$$

Now we are in a position to state our first result. Recall from Proposition 1.1 that if $\lambda_0 \in \sigma(A)$, then $A - \lambda_0$ is Drazin invertible.

2.1. THEOREM. *If* $\lambda_0 \in \sigma(A)$ *and if* C *is the Drazin inverse of* $A - \lambda_0$, *then there is a scalar polynomial* p *such that* $C = p(A)$.

PROOF. Without loss of generality we can assume that $\lambda_0 = \lambda_1 = 0$. Let $\nu = m_1$. Then we have

$$m_A(\lambda) = a_\nu\lambda^\nu + a\nu + 1\lambda^{\nu+1} + \dots + \lambda^{n-1} + \lambda^n$$

and that $a_\nu \neq 0$. Let

$$q_1(\lambda) = -\frac{1}{a_\nu}\big(a_{\nu+1} + a_{\nu+2}\lambda + \dots + \lambda^{n-(\nu+1)}\big).$$

Then

$$A^{\nu+1}q_1(A) = -\frac{1}{a_\nu}\big(a_{\nu+1}A^{\nu+1} + a_{\nu+2}A^{\nu+2} + \cdots + A^n\big)$$
$$= -\frac{1}{a_\nu}\big(m_A(A) - a_\nu A^\nu\big) = A^\nu.$$

Let $B = q_1(A)$. Then $A^{\nu+1}B = A^\nu$ and $BA = AB$. For the Drazin inverse C we have

$$A^{\nu+1}C = A, \quad CAC = C \quad \text{and} \quad CA = AC.$$

Thus

$$A^{\nu+1}(B - C) = A^{\nu+1}B - A^{\nu+1}C = A^\nu - A^\nu = 0$$

This shows that $(B - C)(X) \subseteq N(A^{\nu+1})$. By Proposition 1.1, $\alpha(A) = \nu$, thus $(B - C)(X) \subseteq N(A^\nu)$, therefore $(B - C)(X) \subseteq P_1(X)$, where P_1 denotes the spectral projection of A associated with the spectral set $\{0\}$ (see Proposition 1.2). Since $P_1 = I - AC = I - CA$, it follows that

$$B - C = P_1(B - C) = P_1B - P_1C = P_1B - (I - CA)C$$
$$= P_1B - C + CAC = P_1B,$$

thus $C = B - P_1B$. We have $P_1 = f(A)$ for some $f \in \mathcal{H}(A)$. By Proposition 1.3 (4), $f(A) = q_2(A)$ for some polynomial q_2. Now define the polynomial p by $p = q_1 - q_2q_1$. It results that

$$p(A) = q_1(A) - q_2(A)q_1(A) = B - P_1B = C. \qquad \square$$

2.2. Corollary. *If $\lambda_0 \in \sigma(A)$ and if C is the Drazin inverse of $A - \lambda_0$, then $C \in \mathcal{F}(X)$.*

Proof. Theorem 2.1 and Proposition 1.3 (5). □

2.3. Corollary. *Let A be a complex square matrix and λ_0 a characteristic value of A. Then the Drazin inverse of $A - \lambda_0$ is a polynomial in A.*

Proof. Theorem 2.1 and Proposition 1.3 (3). □

Let $T \in \mathcal{L}(X)$. An operator $S \in \mathcal{L}(X)$ is called a *pseudo inverse* of T provided that $TST = T$. In general the set of all pseudo inverses of T is infinite and this set consists of all operators of the form $STS + U - STUTS$, where $U \in \mathcal{L}(X)$ is arbitrary (see [**2**, Theorem 2.3.2]). Observe that if T is Drazin invertible with $i(T) = 1$, then the Drazin inverse of T is a pseudo inverse of T.

2.4. Corollary. *If $\lambda_0 \in \sigma(A)$, then the following assertions are equivalent:*

(1) *λ_0 is a simple pole of $R_\lambda(A)$;*
(2) *there is a pseudo inverse B of $A - \lambda_0$ such that $B(A - \lambda_0) = (A - \lambda_0)B$;*
(3) *there is a polynomial p such that $p(A)$ is a pseudo inverse of $A - \lambda_0$.*

Proof. (1) $\Leftrightarrow$ (2): Proposition 1.1.

(1)$\Rightarrow$ (3): We can assume that $\lambda_0 = 0$. Let q_1 and B as in the proof of Theorem 2.1. Then $A^2B = A$ and $AB = BA$, hence $ABA = A$.

(3) $\Rightarrow$ (1): Again we can assume that $\lambda_0 = 0$. With $B = p(A)$ we have $ABA = A$ and $AB = BA$. Set $C = BAB$; then $ACA = A$, $CAC = C$ and $AC = CA$. This shows that C is the Drazin inverse of A and that $i(A) = 1$. By Proposition 1.1, $\lambda_0 = 0$ is a simple pole of $R_\lambda(A)$. $\square$

2.5. Corollary. *Let X be a complex Hilbert space and suppose that $N \in \mathcal{L}(X)$ is normal and that $\sigma(N)$ is finite. We have:*

(1) *$N \in \mathcal{F}(X)$,*

(2) *If $\lambda_0 \in \sigma(N)$, then there is a polynomial p such that*

$$(N - \lambda_0)p(N)(N - \lambda_0) = N - \lambda_0.$$

Proof. By [**3**, Satz 111.2], each point in $\sigma(N)$ is a simple pole of $R_\lambda(N)$, thus $N \in \mathcal{F}(X)$. Now apply Theorem 2.4. $\square$

Our results suggest the following.

Question. If $A \in \mathcal{F}(X)$ and if B is a pseudo inverse such that $AB = BA$, does there exist a polynomial p with $B = p(A)$?

The answer is negative:

Example. Consider the square matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that the minimal polynomial of A is given by $m_A(\lambda) = \lambda^2 - 3\lambda = \lambda(\lambda - 3)$, hence $\sigma(A) = \{0, 3\}$ and $A^2 = 3A$. Let

$$B = \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $AB = BA = \frac{1}{3}A$, thus $ABA = \frac{1}{3}A^2 = A$, hence B is a pseudo inverse of A. Since $A^2 = 3A$, any polynomial in A has the form $\alpha I + \beta A$ with $\alpha, \beta \in \mathbb{C}$. But there are no α and β such that $B = \alpha I + \beta A$. An easy computation shows that the Drazin inverse of A is given by $\frac{1}{9}A$ and that $i(A) = 1$.

If 0 is a simple pole of $R_\lambda(A)$, then we have seen in Theorem 2.4 that A has a pseudo inverse. If 0 is a pole of $R_\lambda(A)$ of order $\geqslant 2$, then, in general A does not have a pseudo inverse, as the following example shows.

Example. Let $T \in \mathcal{L}(X)$ be any operator with $T(X)$ not closed (of course X must be infinite dimensional). Define the operator $A \in \mathcal{L}(X \oplus X)$ by the matrix

$$A = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Then the range of A is not closed. By [**2**, Theorem 2.1], A has no pseudo inverse. From $A^2 = 0$ it follows that $A \in \mathcal{F}(X \oplus X)$ and that 0 is a pole of order 2 of $R_\lambda(A)$.

Now we return to the investigations of our operator $A \in \mathcal{F}(X)$. To this end we need the following propositions.

2.6. Proposition. *Suppose that $T \in \mathcal{L}(X)$, $0 \in \rho(T)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and that k is a nonnegative integer. Then:*

(1) $N(T-\lambda)^k) = N\big((T^{-1} - \frac{1}{\lambda})^k\big)$;
(2) $\alpha(T-\lambda) = \alpha(T^{-1} - \frac{1}{\lambda})$.

Proof. We only have to show that $N\big((T-\lambda)^k\big) \subseteq N\big((T^{-1} - \frac{1}{\lambda})^k\big)$. Take $x \in N\big((T-\lambda)^k\big)$. Then $0 = (T-\lambda)^k x$, thus $0 = (T^{-1})^k (T-\lambda)^k x = (1 - \lambda T^{-1})^k x$, hence $x \in N\big((T^{-1} - \frac{1}{\lambda})^k\big)$. □

2.7. Proposition. *Suppose that $T \in \mathcal{L}(X)$, $0 \in \sigma(T)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and k is a nonnegative integer. Furthermore suppose that T is Drazin invertible and that C is the Drazin inverse of T. Then:*

(1) $N((T-\lambda)^k) = N\big((C - \frac{1}{\lambda})^k\big)$;
(2) $\alpha(T-\lambda) = \alpha(C - \frac{1}{\lambda})$;

Proof. (2) follows from (1).

(2) Let $\nu = i(T)$. We use induction. First we show that $N(T-\lambda) = N(C - \frac{1}{\lambda})$. Let $x \in N(T-\lambda)$, then $Tx = \lambda x$ and $T^\nu x = \lambda^\nu x$. We have

$$\lambda C^2 x = C^2 T x = CTCx = Cx,$$

hence $C(1-\lambda C)x = 0$, thus $(1-\lambda C)x \subseteq N(C)$. By Proposition 1.2, $N(C) = N(T^\nu)$, therefore

$$0 = T^\nu (1-\lambda C)x = (1-\lambda C)T^\nu x = (1-\lambda C)\lambda^\nu x,$$

therefore $x \in N(C - \frac{1}{\lambda})$. Now let $x \in N(C - \frac{1}{\lambda})$. From $Cx = \frac{1}{\lambda}x$ we see that $x \in C(X) = N(P)$, where P is as in Proposition 1.2. From $P = I - TC$ we get $x = TCx = T(\frac{1}{\lambda}x)$, thus $Tx = \lambda x$, hence $x \in N(T-\lambda)$. Now suppose that n is a positive integer and that

$$N\big((T-\lambda)^n\big) = N\big((C - \tfrac{1}{\lambda})^n\big).$$

Take $x \in N\big((T-\lambda)^{n+1}\big)$. Then $(T-\lambda)x \in N\big((T-\lambda)^n\big) = N\big((C - \frac{1}{\lambda})^n\big)$, thus

$$0 = (C - \tfrac{1}{\lambda})^n (T-\lambda)x = (T-\lambda)(C - \tfrac{1}{\lambda})^n x.$$

This gives

$$(C - \tfrac{1}{\lambda})^n x \in N(T-\lambda) = N(C - \tfrac{1}{\lambda}),$$

therefore $x \in N\big((C - \frac{1}{\lambda})^{n+1}\big)$. Similar arguments show that $N\big((C - \frac{1}{\lambda})^{n+1}\big) \subseteq N\big((T-\lambda)^{n+1}\big)$. □

In what follows we use the notation of the beginning of this section. Recall that we have $\sigma(A) = \{\lambda_1, \dots, \lambda_k\}$. If $0 \in \sigma(A)$, then we always assume that $\lambda_1 = 0$, hence $\sigma(A) \setminus \{0\} = \{\lambda_2, \dots, \lambda_k\}$.

2.8. PROPOSITION.

(1) *If* $0 \in \rho(A)$, *then* $\sigma(A^{-1}) = \{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}\}$.

(2) *If* $0 \in \sigma(A)$ *and if* C *is the Drazin inverse of* A, *then* $0 \in \sigma(C)$ *and* $\sigma(C) \setminus \{0\} = \{\frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_k}\}$.

PROOF. (1) follows from the spectral mapping theorem.
(2) is a consequence of Proposition 1.2. □

For our next result recall from Corollary 1.4 that if $0 \in \rho(A)$, then $A^{-1} \in \mathcal{F}(X)$.

2.9. THEOREM. *Suppose that* $0 \in \rho(A)$. *Then*

(1) *If the minimal polynomial* m_A *has the representation* (2.1), *then the minimal polynomial* $m_{A^{-1}}$ *of* A^{-1} *is given by*

$$m_{A^{-1}}(\lambda) = \left(\lambda - \tfrac{1}{\lambda_1}\right)^{m_1} \cdots \left(\lambda - \tfrac{1}{\lambda_k}\right)^{m_k}.$$

(2) *If the minimal polynomial* m_A *has the representation* (2.2), *then* $m_{A^{-1}}$ *is given by*

$$m_{A^{-1}}(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0}\lambda + \cdots + \frac{a_1}{a_0}\lambda^{n-1} + \lambda^n.$$

PROOF. Proposition 2.6 shows that

$$\alpha(A^{-1} - \tfrac{1}{\lambda_j}) = \alpha(A - \lambda_j) = m_j \quad (j = 1, \dots, k),$$

thus (1) is shown. Furthermore $m_{A^{-1}}$ has degree $m_1 + \cdots + m_k = n$. Now define the polynomial q by

$$q(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0}\lambda + \cdots + \frac{a_1}{a_0}\lambda^{n-1} + \lambda^n.$$

Then

$$\begin{aligned} a_0 A^n q(A^{-1}) &= A^n(a_0(A^{-1})^n + a_1(A^{-1})^{n-1} + \cdots + a_{n-1}A^{-1} + 1) \\ &= m_A(A) = 0. \end{aligned}$$

Since $a_0 \neq 0$ and $0 \in \rho(A)$, it results that $q(A^{-1}) = 0$. Because of degree of $q = n =$ degree of $m_{A^{-1}}$, we get $q = m_{A^{-1}}$. □

REMARK. The proof just given shows that there is a polynomial q such that $q(A^{-1}) = 0$. Therefore we have a simple proof for the fact that $A^{-1} \in \mathcal{F}(X)$.

2.10. THEOREM. *Suppose that* $0 \in \sigma(A)$ *and that* 0 *is a pole of* $R_\lambda(A)$ *of order* $\nu \geqslant 1$. *Let* C *denote the Drazin inverse of* A (recall from Corollary 2.2 that $C \in \mathcal{F}(X)$).

(1) *If* m_A *has the representation* (2.1), *then*

$$m_C(\lambda) = \lambda(\lambda - \tfrac{1}{\lambda_2})^{m_2} \cdots (\lambda - \tfrac{1}{\lambda_k})^{m_k}.$$

(2) *If* m_A *has the representation* (2.2), *then*

$$m_C(\lambda) = \frac{1}{a_\nu}\lambda + \frac{a_{n-1}}{a_\nu}\lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu}\lambda^{n+1-(\nu+1)} + \lambda^{n+1-\nu}.$$

Proof. Proposition 2.7 gives

$$\alpha(C - \tfrac{1}{\lambda_j}) = \alpha(A - \lambda_j) = m_j \quad (j = 2, \dots, k).$$

By Proposition 1.1 and Proposition 1.2, $\alpha(C) = 1$. Thus (1) is valid. We have

$$m_A(\lambda) = a_\nu \lambda^\nu + a_{\nu+1}\lambda^{\nu+1} + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n,$$

hence

$$0 = m_A(A) = a_\nu A^\nu + a_{\nu+1}A^{\nu+1} + \cdots + a_{n-1}A^{n-1} + A^n. \tag{2.3}$$

If $\nu \leqslant l \leqslant n$, then

$$\begin{aligned} C^{n+1}A^l &= C^{n+1}C^lA^l = C^{n+1-l}(CA)^l \\ &= C^{n+1-l}CA = C^{n-l}CAC = C^{n+1-l}. \end{aligned}$$

Then multiplying (2.3) by C^{n+1}, it follows that

$$0 = a_\nu C^{n+1-\nu} + a_{\nu+1}C^{n+1-(\nu+1)} + \cdots + a_{n-1}C^2 + C.$$

Now define the polynomial q by

$$q(\lambda) = \frac{1}{a_\nu}\lambda + \frac{a_{n-1}}{a_\nu}\lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu}\lambda^{n+1-(\nu+1)} + \lambda^{n+1-\nu}.$$

Then $q(C) = 0$. Since degree of $q = n+1-\nu = 1 + m_2 + \cdots + m_k =$ degree of m_C, we get $q = m_C$. □

2.11. Corollary. *With the notation in Theorem 2.10 we have*

$$C(A-\lambda_2)^{m_2} \cdots (A-\lambda_k)^{m_k} = 0.$$

Proof. Let $D = (A-\lambda_2)^{m_k} \cdots (A-\lambda_k)^{m_k}$. From $A^\nu D = m_A(A) = 0$ we see that $D(X) \subseteq N(A^\nu)$. Since $N(A^\nu) = N(C)$ (Proposition 1.2), $CD = 0$. □

Notation. X^* denotes the dual space of X and we write T^* for the adjoint of an operator $T \in \mathcal{L}(X)$. Recall from [**4**, Theorem IV. 8.4] that

$$\overline{T(X)} = N(T^*)^\perp \quad (T \in \mathcal{L}(X)). \tag{2.4}$$

2.12. Proposition. *Suppose that $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and that j is a nonnegative integer. Then*

1. *If $0 \in \rho(T)$, then $(T-\lambda)^j(X) = (T^{-1} - \frac{1}{\lambda})^j(X)$.*
2. *If $0 \in \sigma(T)$, if T is Drazin invertible and if C denotes the Drazin inverse of T, then $\overline{(T-\lambda)^j(X)} = \overline{(C - \frac{1}{\lambda})^j(X)}$.*

Proof. (1) Let $y = (T-\lambda)^j x \in (T-\lambda)^j(x) \quad (x \in X)$. Then

$$\begin{aligned} (T^{-1} - \tfrac{1}{\lambda})^j T^j x &= \big((T^{-1} - \tfrac{1}{\lambda})T\big)^j x = (1 - \tfrac{T}{\lambda})^j x \\ &= \frac{(-1)^j}{\lambda^j}(T-\lambda)^j x = \frac{(-1)^j}{\lambda^j} y, \end{aligned}$$

therefore $y \in (T^{-1} - \frac{1}{\lambda})^j(X)$.

(2) Let $\nu = i(T)$. Then $T^{\nu+1}C = T^\nu$, $TC = CT$ and $CTC = C$. Hence

$$(T^*)^{\nu+1}C^* = (T^*)^\nu, \quad T^*C^* = C^*T^* \quad \text{and} \quad C^*T^*C^* = C^*.$$

Thus T^* is Drazin invertible and C^* is the Drazin inverse of T^*. By Proposition 2.7,

$$N\big((T^*-\lambda)^j\big) = N\big((C^*-\tfrac{1}{\lambda})^j\big),$$

therefore the result follows in view of (2.4). □

2.13. Corollary.

(1) *If* $0 \in \rho(A)$, *then* $(A-\lambda_j)^{m_j}(X) = (A^{-1}-\frac{1}{\lambda_j})^{m_j}(X) \quad (j = 1,\dots,k)$.

(2) *If* $0 \in \sigma(A)$ *is a pole of order* $\nu \geqslant 1$ *of* $R_\lambda(A)$ *and if* C *is the Drazin inverse of* A, *then* $A^\nu(X) = C(X)$ *and*

$$(A-\lambda_J)^{m_j}(X) = (C-\tfrac{1}{\lambda_j})^{m_j}(X) \quad (j = 2,\dots,k).$$

Proof. (1) is a consequence of Proposition 2.12.

(2) That $A^\nu(X) = C(X)$ is a consequence of Proposition 1.2. Now let $j \leqslant \{2,\dots,k\}$. Because of Proposition 1.1 and Theorem 2.10 we see that

$$\alpha(C-\tfrac{1}{\lambda_j}) = \delta(C-\tfrac{1}{\lambda_j}) = m_j = \alpha(A-\lambda_j) = \delta(A-\lambda_j).$$

By [**3**, Satz 101.2], the subspaces $(A-\lambda_j)^{m_j}(X)$ and $(C-\frac{1}{\lambda_j})^{m_j}(X)$ are closed. Now apply Proposition 2.12. □

For $j = 1,\dots,k$ let P_j denote the spectral projection of A associated with the spectral set $\{\lambda_j\}$. Observe that

$$P_iP_j = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad P_1 + \dots + P_k = 1.$$

If $0 \in \rho(A)$, then denote by Q_j the spectral projection of A^{-1} associated with the spectral set $\{\frac{1}{\lambda_j}\}$ $(j = 1,\dots,k)$. If $0 \in \sigma(A)$ and if C is the Drazin inverse, then denote by Q_1 the spectral projection of C associated with the spectral set $\{0\}$ and by Q_j the spectral projection of C associated with the spectral set $\{\frac{1}{\lambda_j}\}$ $(j = 2,\dots,k)$.

2.14. Corollary. $P_j = Q_j$ $(j = 1,\dots,k)$.

Proof. By [**3**, Satz 101.2], we have

$$P_j(X) = N((A-\lambda_j)^{m_j}) \quad \text{and} \quad N(P_j) = (A-\lambda_j)^{m_j}(X)$$

$(j = 1,\dots,k)$. If $0 \in \rho(A)$, then

$$Q_j(X) = N\big((A^{-1}-\tfrac{1}{\lambda_j})^{m_j}\big) \quad \text{and} \quad N(Q_j) = (A^{-1}-\tfrac{1}{\lambda_j})^{m_j}(X)$$

$(j = 1,\dots,k)$. Now apply Proposition 2.6 and Corollary 2.13 (1) to get

$$P_j(X) = Q_j(X) \quad \text{and} \quad N(P_j) = N(Q_j),$$

hence $P_j = Q_j$ $(j = 1,\dots,k)$.

Now let $0 \in \sigma(A)$. By Proposition 1.2, Proposition 2.7, Corollary 2.13 (2) and [**3**, Satz 101.2], we derive

$$\begin{aligned}
&P_1(X) = N(C) = Q_1(X),\ N(P_1) = C(X) = N(Q_1),\\
&P_j(X) = N\big((C-\tfrac{1}{\lambda_j})^{m_j}\big) = Q_j(X)\\
&N(P_j) = \big(C-\tfrac{1}{\lambda_j}\big)^{m_j}(X) = N(Q_j)
\end{aligned}$$

$(j = 2, \dots, k)$. Hence $P_j = Q_j$ $(j = 1, \dots, k)$. □

For A we have the representation $A = \sum_{j=1}^{k} \lambda_j P_j + N$, where $N \in \mathcal{L}(X)$ is nilpotent and $N = \sum_{j=1}^{k} (A - \lambda_j) P_j$ (see [**4**, Chapter V. 11]). If $p = \max\{m_1, \dots, m_k\}$, then it is easily seen that $N^p = 0$. If A has only simple poles, then $N = 0$.

2.15. Corollary.

(1) *If* $0 \in \rho(A)$, *then there is a nilpotent operator* $N_1 \in \mathcal{L}(X)$ *with*

$$A^{-1} = \sum_{j=1}^{k} \frac{1}{\lambda_j} P_j + N_1$$

(2) *If* $0 \in \sigma(A)$ *and if* C *is the Drazin inverse of* A, *then*

$$C = \sum_{j=2}^{k} \frac{1}{\lambda_j} P_j + N_1, \qquad \text{where } N_1 \in \mathcal{L}(X) \text{ is nilpotent.}$$

Proof. Corollary 2.14. □

With the notation of Corollary 2.15 (2) we have $AC = 1 - P_1$, $CP_1 = 0$ (see Proposition 1.2) and

$$ACA = (1 - P_1)\Big(\sum_{j=2} k\lambda_j P_j + N\Big) = A - P_1\Big(\sum_{j=2}^{k} \lambda_j P_k + N\Big) = A - P_1 N;$$

hence $A = ACA + P_1 N$, $P_1 N$ is nilpotent and

$$(ACA)P_1 N = ACP_1 AN = 0 = NACP_1 A = P_1 N(ACA).$$

Recall that ACA is the Drazin inverse of C and that $i(ACA) = 1$. The following more general result holds:

2.16. Theorem. *Suppose that* $T \in \mathcal{L}(X)$ *is Drazin invertible,* $i(T) = \nu \geqslant 1$ *and that* C *is the Drazin inverse of* T. *Then there is a nilpotent* $N \in \mathcal{L}(X)$ *such that* $T = TCT + N$, $N(TCT) = (TCT)N = 0$ *and* $N^\nu = 0$.

This decomposition is unique in the following sense: if $S, N_1 \in \mathcal{L}(X)$, S *is Drazin invertible,* $i(S) = 1$, N_1 *is nilpotent,* $N_1 S = S N_1 = 0$ *and if* $T = S + N_1$, *then* $S = TCT$ *and* $N = N_1$.

Proof. Let $N = T - TCT$; then

$$\begin{aligned} N^\nu &= (T(1 - CT))^\nu = T^\nu (1 - CT)^\nu = T^\nu (1 - CT) \\ &= T^\nu - T^\nu CT = T^\nu - T^{\nu+1} C = T^\nu - T^\nu = 0. \end{aligned}$$

For the uniqueness of the decomposition we only have to show that $S = TCT$. There is $R \in \mathcal{L}(X)$ such that $SRS = S$, $RSR = R$ and $SR = RS$. Consequently,

$$N_1 R = N_1 RSR = N_1 SR^2 = 0 = R^S S N_1 = RN_1,$$

hence

$$TR = (S + N_1)R = SR = RS = R(S + N_1) = RT.$$

Now let n be a nonnegative integer such that $N_1^n = 0$. Since $SN_1 = 0 = N_1S$, it follows that

$$T^n = (S + N_1)^n = S^n + N_1^n = S^n.$$

We can assume that $n \geqslant \nu$. Thus

$$T^{n+1}R = S^{n+1}R = S^{n-1}SRS = S^n = T^n.$$

Furthermore we have $TR = RT$ and

$$RTR = R(S + N_1)R = RSR = R,$$

hence $R = C$. With $S_1 = TCT$ we get

$$\begin{aligned} S_1RS_1 &= TCTCTCT = TCT = S_1, \\ RS_1R &= CTCTC = CTC = RTR = R \\ S_1R &= TCTC = CTCT = RS_1. \end{aligned}$$

This shows that $S = S_1 = TCT$. □

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