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ON THE FUNCTIONAL–INTEGRAL EQUATION OF VOLTERRA TYPE WITH WEAKLY SINGULAR KERNEL

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Abstract. We give sufficient conditions for the existence of L^p-solution of a Volterra functional–integral equation in a Banach space. Our assumptions and proofs are expressed in terms of measures of noncompactness.

1. Introduction

Let E, F be Banach spaces, $D = [0, d_1] \times \cdots \times [0, d_m]$ and

$$D(t) = \{s = (s_1, \ldots, s_m) \in R^m : \ 0 \leqslant s_i \leqslant t_i, \ \ i = 1, \ldots, m\}$$

for $t = (t_1, \ldots, t_m) \in D$. Denote by $L^p(D, E)$ $(p > 1)$ the space of all strongly measurable functions $x : D \mapsto E$ with $\int_D \|x(t)\|^p dt < \infty$, provided with the norm $\|x\|_p = \left(\int_D \|x(t)\|^p dt\right)^{1/p}$.

We consider the following functional–integral equation of Volterra type

$$x(t) = \phi\left(t, \int\limits_{D(t)} K(t, s)\, g(s, x(s))\, ds\right) \tag{1}$$

with the kernel $K(t, s) = \frac{A(t,s)}{|t-s|^r}$, $0 < r < n$ $(t, s \in D,\ t \neq s)$. We give sufficient conditions for the existence of a solution $x \in L^p(D, E)$ of (1). Moreover, for $r < 1$ we present one-dimensional result involving a generalized Osgood condition. Our considerations are inspirated by a paper of Darwish [5] concerning the functional–integral equation of Hammerstein type. The existence of L^1-solution of functional–integral equation of Hammerstein type was studied in [4] and when $g(s, x) = x$ we get an equation considered in [3]. In [15] Szufla has established the existence of L^p-solution of Hammerstein integral equation with weakly singular kernel.

Throughout this paper we shall assume that:

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1° $(t,x) \mapsto \phi(t,x)$ is a function from $D \times E$ into E such that

(i) ϕ is strongly measurable in t and continuous in x;

(ii) $\|\phi(t,x)-\phi(\tau,y)\| \leqslant |a_1(t)-a_1(\tau)| + b_1\|x-y\|$ for $t,\tau \in D$ and $x,y \in E$, where $a_1 \in L^p(D,R)$ and $b_1 \geqslant 0$;

(iii) $\phi(0,0)=0$;

2° A is a bounded strongly measurable function from $D \times D$ into the space of continuous linear mappings $F \mapsto E$;

3° $(t,x) \mapsto g(t,x)$ is a function from $D \times E$ into F such that

(i) g is strongly measurable in t and continuous in x;

(ii) $\|g(t,x)\| \leqslant a_2(t) + b_2\|x\|$ for $s \in D$ and $x \in E$, where $a_2 \in L^p(D,R)$ and $b_2 \geqslant 0$.

In what follows we shall need the following lemmas:

Lemma 1. *The linear integral operator*

$$(Sx)(t) = \int_D K(t,s)\,x(s)\,ds \quad (x \in L^p(D,E),\ t \in D)$$

maps $L^p(D,E)$ into itself continuously. Moreover,

$$\|S\| \leqslant aQ, \quad \textit{where} \ \ a = \sup\{\|A(t,s)\| : t,s \in D\}$$

and

$$\frac{2\pi^{n/2}(\operatorname{diam} D)^{n-r}}{(n-r)\Gamma(n/2)} = Q \geqslant \int_D \frac{ds}{|t-s|^r} \quad \textit{for all}\ \ t \in D. \tag{2}$$

Lemma 2. *Put $G(x)(t) = g(t,x(t))$ for $x \in L^p(D,E)$ and $t \in D$. Then G is a continuous mapping of $L^p(D,E)$ into itself.*

For the proofs we refer for example to [15].

Denote by α and α_1 the Kuratowski measures of noncompactness in E and $L^1(D,E)$, respectively. For any set V of functions belonging to $L^1(D,E)$ denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in D$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \{x(t) : x \in V\}$. The next lemma clarifies the relation between α and α_1.

Lemma 3. ([7, Th.2.1]; and [16, Th.1]) *Assume that V is a countable set of strongly measurable functions $D \mapsto E$ and there exists an integrable function μ such that $\|x(t)\| \leqslant \mu(t)$ for all $x \in V$ and $t \in D$. Then the corresponding function v is integrable on D and*

$$\alpha\left(\left\{\int_D x(t)\,dt : \ x \in V\right\}\right) \leqslant 2\int_D v(t)\,dt.$$

If, in addition $\lim\limits_{h\to\infty} \sup\limits_{x\in V} \int_D \|x(t+h)-x(t)\|\,dt = 0$, *then*

$$\alpha_1(V) \leqslant 2\int_D v(t)\,dt.$$

2. The main results

Let $H : D \mapsto R_+$ be a measurable function such that the function $(t,s) \mapsto \|A(t,s)\|H(s)$ is bounded on $D \times D$.

Theorem 1. *Let* $1^\circ - 3^\circ$ *hold and* $0 < r < n$. *If*

$$\alpha(g(s,X)) \leqslant H(s)\alpha(X) \tag{3}$$

for any $s \in D$ *and for any bounded subset* X *of* E, *then the equation* (1) *has a solution* $x \in L^p(D,E)$.

In the case, when $r < 1$, we can apply the famous Mydlarczyk theorem [12, Th.3.1], and consequently we obtain a stronger theorem if we replace (3) by the condition (5) given below.

Theorem 2. *Let* $\omega : R_+ \mapsto R_+$ *be a continuous nondecreasing function such that* $\omega(0) = 0$, $\omega(t) > 0$ *for* $t > 0$ *and*

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{\omega(s)}\right]^{\frac{1}{1-r}} ds = \infty \quad (\delta > 0). \qquad (\textit{cf. [12]}) \tag{4}$$

Let 1°*–*3° *hold,* $0 < r < 1$ *and* $J = [0,d]$ *be a compact interval in* R. *If*

$$\alpha(g(s,X)) \leqslant \omega(\alpha(X)) \tag{5}$$

for any $s \in J$ *and for any bounded subset* X *of* E, *then the equation* (1) *has a solution* $x \in L^p(J,E)$.

Proof. By the theory of scalar linear Volterra integral equations it follows that there exists a nonnegative solution $u(t)$ of the equation

$$u(t) = a_1(t) + b_1 \int\limits_{D(t)} \|K(t,s)\| a_2(s)\, ds + b_1 b_2 \int\limits_{D(t)} \|K(t,s)\| u(s)\, ds.$$

More precisely, as the spectral radius $r(\mathcal{K})$ of the Volterra integral operator

$$\mathcal{K}u(t) = \int\limits_{D(t)} \|K(t,s)\| u(s)\, ds \tag{6}$$

is equal to 0, by Theorem 2.2 from [10] the sequence of successive approximations $u_n(t)$ for (6) is convergent; obviously all $u_n(t)$ are nonnegative.

Put $B = \{x \in L^p(D,E) : \ \|x(t)\| \leqslant u(t)$ for a.e. $t \in D\}$. Define $F : B \mapsto L^p(D,E)$ by

$$(Fx)(t) = \phi\left(t, \int\limits_{D(t)} K(t,s) g(s,x(s))\, ds\right) \text{ for } x \in B \text{ and } t \in D.$$

Since

$$\begin{aligned}\|(Fx)(t)\| &= \|\phi(t,SGx(t))\| \leqslant a_1(t)+b_1\|SGx(t)\| \\ &\leqslant a_1(t)+b_1\Big\|\int\limits_{D(t)} K(t,s)\,g(s,x(s))\,ds\Big\| \\ &\leqslant a_1(t)+b_1\int\limits_{D(t)} \|K(t,s)\|\big(a_2(s)+b_2\|x(s)\|\big)\,ds \\ &\leqslant a_1(t)+b_1\int\limits_{D(t)} \|K(t,s)\|\,a_2(s)\,ds+b_1b_2\int\limits_{D(t)}\|K(t,s)\|\,u(s)\,ds = u(t)\end{aligned}$$

for $x \in B$ and $t \in D$, Lemmas 1 and 2 prove that F is a continuous mapping $B \mapsto B$.

Without loss of generality we shall always assume that all functions from $L^p(D,E)$ are extended to R^n by putting $x(t)=0$ outside D. Moreover, by 1°(ii) we obtain

$$\|F(x)(t+h)-F(x)(t)\| \leqslant d(t,h) \text{ for } x\in B,\ t\in D \text{ and small } |h|,$$

where

$$d(t,h)=\begin{cases} u(t) & \text{if } t\in D \text{ and } t+h\notin D \\ \|a_1(t+h)-a_1(t)\| & \\ \quad +\, b_1\int_D \|K(t+h,s)-K(t,s)\|\big(a_2(s)+b_2u(s)\big)\,ds & \text{if } t,\ t+h\in D.\end{cases}$$

From (2) it follows that for each $z\in L^1(D,R)$ we have

$$\iint\limits_{D\times D}\frac{|z(s)|}{|t-s|^r}\,ds\,dt=\int\limits_D\bigg(\int\limits_D\frac{dt}{|t-s|^r}\bigg)|z(s)|\,ds \leqslant Q\int\limits_D |z(s)|\,ds. \tag{7}$$

In view of (7) the function $(t,s)\mapsto W(t,s)=K(t,s)(a_2(s)+b_2u(s))$ is integrable on $D\times D$. Therefore

$$\begin{aligned}\lim_{h\to0}\int\limits_D d(t,h)\,dt &= \lim_{h\to0}\int\limits_D\bigg(\int\limits_D \|K(t+h,s)-K(t,s)\|\,\big(a_2(s)+b_2u(s)\big)\,ds\bigg)\,dt \\ &= \lim_{h\to0}\int\limits_D\int\limits_D \|W(t+h,s)-W(t,s)\|\,ds\,dt=0\end{aligned}$$

for $t\in D$. Hence

$$\lim_{h\to0}\sup_{x\in B}\int\limits_{D(t)}\|(Fx)(t+h)-(Fx)(t)\|\,dt=0. \tag{8}$$

Next, let V be a countable subset of B such that

$$V\subset\overline{\mathrm{conv}}(F(V)\cup\{0\}). \tag{9}$$

Then $V(t) \subset \overline{\text{conv}}\big(F(V)(t) \cup \{0\}\big)$ for a.e. $t \in D$, so that

$$\alpha(V(t)) \leqslant \alpha(F(V)(t)) \text{ for a.e. } t \in D. \tag{10}$$

Put $v(t) = \alpha(V(t))$ for $t \in D$. From (8) and (9) we deduce that

$$\lim_{h\to 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\| \, dt = 0.$$

Moreover, $\|x(t)\| \leqslant u(t)$ for all $x \in V$ and a.e. $t \in D$. Consequently, by Lemma 3, $v \in L^p(D, R)$ and

$$\alpha_1(V) \leqslant 2 \int_D v(t)\, dt. \tag{11}$$

According to $1°$(ii), we have $\|\phi(t,x) - \phi(t,y)\| \leqslant b_1\|x-y\|$ for $t \in D$ and $x, y \in E$. Then $\alpha(\phi(t,X)) \leqslant b_1\alpha(X)$ for any bounded subset X of E.

From (7) it is clear that

$$\int_D \frac{a_2(s) + b_2 u(s)}{|t-s|^r}\, ds < \infty \text{ for a.e. } t \in D. \tag{12}$$

Fix $t \in D$ such that the integral (12) is finite. Next, we have

$$\|K(t,s)g(s,x(s))\| \leqslant a\frac{a_2(s) + b_2 u(s)}{|t-s|^r} \text{ for } x \in B \text{ and } s \in D.$$

Case 1. Suppose that the assumptions of Theorem 1 hold. Thus, by (10), (3) and Lemma 3, we get

$$\begin{aligned}
\alpha(V(t)) &\leqslant \alpha((FV)(t)) = \alpha(\phi(t, SGV(t))) \\
&\leqslant b_1\alpha\left(\left\{\int_{D(t)} K(t,s)\, g(s,x(s))\, ds : \ x \in V\right\}\right) \\
&\leqslant 2b_1 \int_{D(t)} \alpha\big(\{K(t,s)\, g(s,x(s))\, ds : \ x \in V\}\big)\, ds \\
&\leqslant 2b_1 \int_{D(t)} \|K(t,s)\|\, \alpha(g(s,V(s))\, ds \leqslant 2b_1 \int_{D(t)} \|K(t,s)\|\, H(s)\, \alpha(V(s))\, ds
\end{aligned}$$

i.e.

$$v(t) \leqslant 2b_1 \int_{D(t)} \|K(t,s)\|\, H(s)\, v(s)\, ds.$$

Putting

$$w(t) = 2b_1 c \int_{D(t)} \frac{v(s)}{|t-s|^r}\, ds,$$

where $c = \sup\big\{\|A(t,s)\|\, H(s) : \ t, s \in D\big\}$, we see that $w(t)$ is a continuous function such that $v(t) \leqslant w(t)$ for $t \in D$. Hence

$$w(t) \leqslant 2b_1 c \int\limits_{D(t)} \frac{w(s)}{|t-s|^r}\, ds. \tag{13}$$

Arguing similarly as in [8; p. 134–135] we can prove that $w(t) = 0$ for $t \in D$. Since $v(t) \leqslant w(t)$, we have $v(t) = 0$ for $t \in D$.

Case 2. Suppose that the assumptions of Theorem 2 hold. Thus, by (10), (5) and Lemma 3, we get

$$\begin{aligned}
\alpha(V(t)) &\leqslant \alpha((FV)(t)) = \alpha\Big(\phi(t, SGV(t))\Big) \\
&\leqslant b_1 \alpha\left(\left\{\int\limits_0^t K(t,s)\, g(s,x(s))\, ds :\ x \in V\right\}\right) \\
&\leqslant 2b_1 \int\limits_0^t \alpha\big(\{K(t,s)\, g(s,x(s))\, ds :\ x \in V\}\big)\, ds \\
&\leqslant 2b_1 \int\limits_0^t \|K(t,s)\|\, \alpha\big(g(s,V(s))\big)\, ds \leqslant 2b_1 \int\limits_0^t \|K(t,s)\|\, \omega\big(\alpha(V(s))\big)\, ds,
\end{aligned}$$

i.e.

$$v(t) \leqslant 2b_1 a \int\limits_0^t \frac{\omega(v(s))}{(t-s)^r}\, ds \quad \text{for } t \in J.$$

Putting

$$w(t) = 2b_1 a \int\limits_0^t \frac{\omega(v(s))}{(t-s)^r}\, ds \quad \text{for } t \in J$$

we see that w is a continuous function such that $v(t) \leqslant w(t)$ for $t \in J$. Hence

$$w(t) \leqslant 2b_1 a \int\limits_0^t \frac{\omega(w(s))}{(t-s)^r}\, ds \quad \text{for } t \in J. \tag{14}$$

By the Mydlarczyk theorem [12, Th. 3.1] and assumption (4), the integral equation

$$z(t) = 2b_1 a \int\limits_0^t \frac{\omega(z(s))}{(t-s)^r}\, ds \quad \text{for } \in J$$

has the unique continuous solution $z(t) \equiv 0$. Applying now theorem on integral inequalities [1, Th. 2], from (14) we deduce that $w(t) \equiv 0$. Thus $v(t) = 0$ for $t \in J$.

In view of (11) this shows that $\alpha_1(V) = 0$, so that V is relatively compact in $L^1(D,E)$. On the other hand, the set B has equiabsolutely continuous norms in $L^p(D,E)$ and $V \subset B$. Consequently, V is relatively compact in $L^p(D,E)$.

Applying now the following Mönch fixed point theorem [11]:

Theorem 3. *Let B be a closed, convex, and bounded subset of a Banach space such that $0 \in B$. If $F : B \mapsto B$ is a continuous mapping such that for each countable subset V of B the following implication holds*

$$V \subset \overline{\mathrm{conv}}(F(V) \cup 0) \Longrightarrow V \quad \textit{is relatively compact},$$

then F has a fixed point.

we conclude that there exists $x \in B$ such that $x = F(x)$. Obviously x is a solution of (1). □

References

[1] N. V. Azbieliev, Z. B. Caliuk, *Ob integralnych nieravienstvach*, Matem. Sbornik **56** (1962), 325–342.

[2] J. Banaś, K. Goebel, *Measure of noncompactness in Banach spaces*, Marcel Dekker, New York – Basel, 1980.

[3] J. Banaś, Z. Knap, *Integrable solutions of a functional-integral equation*, Rev. Mat. Univ. Complutense Madr. **2(1)** (1989), 31–38.

[4] G. Emmanuele, *Existence of solutions of a functional-integral equation in infinite dimensional Banach spaces*, Czech. Math. J. **44(4)** (1994), 603–609.

[5] M. A. Darwish, *Weakly singular functional-integral equation in infinite dimensional Banach spaces*, Appl. Math. Comp. **136** (2003), 123–129.

[6] G. Gripenberg, *Unique solutions of some Volterra integral equations*, Math. Scand. **48** (1981), 59–67.

[7] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983), 1351–1371.

[8] J. Januszewski, *On Volterra integral equations with weakly sigular kernel in Banach spaces*, Demonstr. Math. **26** (1993), 131–136.

[9] W. Kołodziej, *Wybrane rozdziały analizy matematycznej*, PWN, Warszawa 1982.

[10] M. A. Krasnoselskii, *Približennoe resenie operatornyh uravnenij*, Nauka, Moskva 1969.

[11] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal. **4** (1980), 985–999.

[12] W. Mydlarczyk, *The existence of nontrivial solutions of Volterra equations*, Math. Scand. **68** (1991), 83–88.

[13] D. O'Regan, M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer, Dordrecht, 1998.

[14] B. Sadovskii, *Limit-compact and condensing mappings*, Russian Math. Surveys **27** (1972), 85–155.

[15] S. Szufla, *On the Hammerstein integral equation with weakly singular kernel*, Funkcialaj Ekvacioj **34 (2)** (1991), 279–285.

[16] S. Szufla, *Appendix to the paper "An existence theorem for the Urysohn integral equation in Banach spaces"*, Comm. Math. Univ. Carolinae **25** (1984), 763–764.

[17] S. Szufla, *On the Volterra integral equation with weakly singular kernel*, Mathem. Bohem. **131(3)** (2006), 225–231.

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