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ON THE MEAN SQUARE OF THE RIEMANN ZETA FUNCTION AND THE DIVISOR PROBLEM

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Abstract. Let $\Delta(T)$ and $E(T)$ be the error terms in the classical Dirichlet divisor problem and in the asymptotic formula for the mean square of the Riemann zeta function in the critical strip, respectively. We show that $\Delta(T)$ and $E(T)$ are asymptotic integral transforms of each other. We then use this integral representation of $\Delta(T)$ to give a new proof of a result of M. Jutila.

1. Introduction and statement of results

Let $\zeta(s)$ be the Riemann zeta function, and let $d(n)$ denote the number of positive divisors of n. The error terms $\Delta(T)$ and $E(T)$ in the classical Dirichlet divisor problem and in the asymptotic formula for the mean square of $\zeta(s)$ on the critical line $\operatorname{Re} s = 1/2$ are defined by

$$\Delta(T) = \sum_{k<T} d(k) + \frac{1}{2}d(T) - T\log T - (2\gamma-1)T - \frac{1}{4}$$

with the convention that $d(T)=0$ if T is not an integer and

$$E(T) = \int_0^T |\zeta(1/2+it)|^2\,dt - T\log\frac{T}{2\pi} - (2\gamma-1)T,$$

respectively. The properties of $\Delta(T)$ and $E(T)$ have been the subject of numerous papers. (For example, [**2**], [**3**], [**4**], [**5**], [**7**], [**8**], [**9**], [**10**], [**13**], [**14**], [**16**], [**17**], [**18**] and [**19**]. For a general overview of the subject see the book [**6**] or the survey article [**15**].)

Two of the most frequently used tools in the study of $\Delta(T)$ and $E(T)$ are the following two remarkable formulas due to Voronoi [**20**] and Atkinson [**1**], respectively.

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Lemma 1.1 (Voronoi). *We have*

$$\Delta(T)=\frac{T^{1/4}}{\pi\sqrt{2}}\sum_{k\leqslant K}\frac{d(k)}{k^{3/4}}\cos\{4\pi(kT)^{1/2}-\pi/4\}+O\Big(\frac{T^{1/2+\epsilon}}{K^{1/2}}\Big)+O(T^{\epsilon})$$

for any $K>0$.

Lemma 1.2 (Atkinson). *Let $0<A<A'$ be constants, and suppose that $AT\leqslant K\leqslant A'T$. Put*

$$K'=K'(T)=\frac{T}{2\pi}+\frac{K}{2}-\sqrt{\frac{KT}{2\pi}+\frac{K^2}{4}}. \tag{1.1}$$

Then $E(T)=\Sigma_1+\Sigma_2+O((\log T)^2)$, where

$$\Sigma_1=\frac{1}{\sqrt{2}}\sum_{k\leqslant K}(-1)^k d(k)\Big(\frac{kT}{2\pi}+\frac{k^2}{4}\Big)^{-1/4}\Big\{\sinh^{-1}\Big(\frac{\pi k}{2T}\Big)^{1/2}\Big\}^{-1}\cos 2\pi\theta_k(T/(2\pi))$$

with

$$\theta_k(T)=2T\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{T}}+\sqrt{kT+\frac{k^2}{4}}-\frac{1}{8},$$

and

$$\Sigma_2=2\sum_{k\leqslant K'}\frac{d(k)}{\sqrt{k}}\Big(\log\frac{T}{2\pi k}\Big)^{-1}\sin\rho_k(T)\quad \textit{with}\quad \rho_k(T)=T\log\frac{T}{2\pi k}-T-\frac{\pi}{4}.$$

We note that the contribution to $E(T)$ is mainly from the first sum Σ_1. For instance, on the Lindelöf Hypothesis, the second sum Σ_2 can be shown to be bounded by T^{ϵ}. In applications we can usually employ averaging techniques to show that the contribution from Σ_2 is less significant than that from Σ_1. Thus, to study the properties of $E(T)$, one would generally focus on Σ_1.

Using the Taylor expansions

$$2\pi\theta_k(T/(2\pi))=4\pi\Big(\frac{kT}{2\pi}\Big)^{1/2}-\frac{\pi}{4}+O(k^{3/2}T^{-1/2})$$

$$\frac{1}{\sqrt{2}}\Big(\frac{kT}{2\pi}+\frac{k^2}{4}\Big)^{-1/4}\Big\{\sinh^{-1}\Big(\frac{\pi k}{2T}\Big)^{1/2}\Big\}^{-1}=\sqrt{2}\Big(\frac{T}{2\pi}\Big)^{1/4}k^{-3/4}+O(T^{-3/4}k^{1/4})$$

it can be seen that, aside from the alternating factor $(-1)^k$, the first $o(T^{1/3})$ terms in Σ_1 are asymptotically equal to the corresponding terms in Voronoi's formula for $2\pi\Delta(T/(2\pi))$. This analogy between $\Delta(T)$ and $E(T)$ motivates the work of Jutila [**8**], [**9**] and [**13**]. Jutila introduced a new function

$$\Delta^*(T)=-\Delta(T)+2\Delta(2T)-\frac{1}{2}\Delta(4T).$$

This function can be interpreted as the error term in the approximation of the summatory function of a certain arithmetic function, and there is a formula analogous to Voronoi's formula for $\Delta^*(T)$, namely,

$$\Delta^*(T)=\frac{T^{1/4}}{\pi\sqrt{2}}\sum_{k\leqslant K}(-1)^k\frac{d(k)}{k^{3/4}}\cos\big(4\pi\sqrt{kT}-\pi/4\big)+O(T^{1/2+\epsilon}K^{-1/2}+T^{\epsilon}).$$

Since this formula also contains the alternating factor $(-1)^k$, the magnitude of the function $\Delta^*(T)$ is more comparable to that of $E(T)$ than that of $\Delta(T)$. In fact, Jutila [**9**] showed that

$$\int_T^{T+H} \bigl(E(u) - 2\pi\Delta^*(u/(2\pi))\bigr)^2\, du \ll HT^{1/3+\epsilon} + T^{1+\epsilon}$$

for $2 \leqslant H \leqslant T$, while the corresponding integrals for $E(T)^2$ and $\Delta(u)^2$ (and hence $\Delta^*(u)^2$) are known to be bounded by $HT^{1/2+\epsilon}+T^{1+\epsilon}$. Using this similarity between $2\pi\Delta^*(T/(2\pi))$ and $E(T)$, Jutila [**8**] further proved that the truth of the conjecture $\Delta(T) \ll T^{1/4+\epsilon}$ implies the bound $E(T) \ll T^{5/16+\epsilon}$, and later [**13**] improved this conditional bound to $T^{3/10+\epsilon}$.

The main purpose of the present paper is to provide a different perspective on the connection between $\Delta(T)$ and $E(T)$. We will show that these two functions are in fact asymptotic integral transforms of each other.

Theorem 1.1. *Define two functions $f(u)$ and $g(u)$ by*

$$f(u) = f_T(u) = \frac{\log(u/T)}{\sqrt{T}(u/T-1)} \exp\Bigl\{-2\pi i\Bigl(u\log\frac{u}{T} - u + T - \frac{1}{8}\Bigr)\Bigr\}$$
$$g(u) = g_T(u) = \frac{T/u-1}{\sqrt{u}\log(T/u)} \exp\Bigl\{2\pi i\Bigl(T\log\frac{T}{u} - T + u - \frac{1}{8}\Bigr)\Bigr\}.$$

Let $\epsilon > 0$, $0 < A < 1$ and $B > 0$ be constants, and put $B' = 1+\sqrt{B+B^2/4}+B/2$. For $T - AT \leqslant u \leqslant T + AT$ let $E_1(2\pi u)$ denote the main sum in Atkinson's formula

$$E_1(2\pi u) = \sum_{k\leqslant BT} \frac{d(k)\cos\{2\pi\theta_k(u)\}}{a_k(u)},$$

where

$$\theta_k(u) = 2u\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{u}} + \sqrt{ku+\frac{k^2}{4}} - \frac{k}{2} - \frac{1}{8},$$
$$a_k(u) = (4ku+k^2)^{1/4}\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{u}}.$$

Set $T_1 = T - AT$, $T_2 = T + AT$, $T_3 = T/B'$, $T_4 = B'T$. Then we have

$$\Delta(T) = \frac{1}{2\pi}\int_{T_1}^{T_2} E_1(2\pi u) f(u)\, du + O(T^\epsilon),$$
$$E_1(2\pi T) = 2\pi\int_{T_3}^{T_4} \Delta(u) g(u)\, du + O(T^\epsilon),$$

where the O-constants depend only on ϵ, A and B.

The underlying idea of our approach evolves from the fact that the function $\chi(1-s)$ is in fact the Mellin transform of $2\cos(2\pi x)$, where

$$\chi(s) = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)} \Bigl(= 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\Bigr)$$

is the function in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ for $\zeta(s)$. Thus, by the Mellin inversion formula, we have

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \chi(1-s)x^{-s}\,ds = 2\cos(2\pi x)$$

whenever the integral converges absolutely. Moreover, for the truncated integral

$$\frac{1}{2\pi i}\int_{\sigma}^{\sigma+2\pi iT} \chi(1-s)x^{-s}\,ds = \int_0^T \chi(1-\sigma-2\pi it)x^{-\sigma-2\pi it}\,dt,$$

using the asymptotic expansion

$$\chi(1-\sigma-2\pi it) = t^{\sigma-1/2}\exp\{2\pi i(t\log t - t - 1/8)\}\left(1 + O\left((1+|t|)^{-1}\right)\right)$$

and the stationary phase method (see Lemma 2.1 below), we see that

$$\int_0^T \chi(1-\sigma-2\pi it)x^{-\sigma-2\pi it}\,dt = \exp\{-2\pi ix\} \times \begin{cases} 1 + A_{\sigma,T}(x), & \text{if } x \leqslant T, \\ A_{\sigma,T}(x), & \text{if } x > T, \end{cases}$$

where $A_{\sigma,T}(x)$ is differentiable for $x \neq T$ and satisfies

$$\lim_{x\to T-} A_{\sigma,T}(x) = -\frac{1}{2} + O_\sigma\big(1/\sqrt{T}\big),$$

$$\lim_{x\to T+} A_{\sigma,T}(x) = \frac{1}{2} + O_\sigma\big(1/\sqrt{T}\big),$$

$$A_{\sigma,T}(x) \ll_\sigma \left(\frac{T}{x}\right)^{\sigma} \min\left(1, \frac{1}{\sqrt{T}|\log(T/x)|}\right).$$

In particular, if we write $\zeta(s)^2$ as

$$\zeta(s)^2 = \sum_{k\leqslant 2T} d(k)k^{-s} + B(s,T),$$

then we have

$$\begin{aligned}\int_0^T |\zeta(1/2+2\pi it)|^2\,dt &= \int_0^T \zeta(1/2+2\pi it)^2\chi(1/2-2\pi it)\,dt \\ &= \sum_{k\leqslant T} d(k) + \sum_{k\leqslant 2T} d(k)A_{1/2,T}(k) + C_1(T),\end{aligned}$$

where $C_1(T)$ may be thought of as a secondary error term. Thus, integrating by parts on the second sum yields

$$\begin{aligned}\int_0^T |\zeta(1/2+2\pi it)|^2\,dt = T(\log T + 2\gamma - 1) &+ \int_0^{T-} \Delta(u)A'_{1/2,T}(u)\,du \\ &+ \int_{T+}^{2T} \Delta(u)A'_{1/2,T}(u)\,du + C_2(T).\end{aligned}$$

This shows that $E(2\pi T)$ is representable asymptotically as an integral transform of $\Delta(u)$. Conversely, we can express $\Delta(T)$ asymptotically in terms of the "inverse" integral transform of $E(2\pi u)$, and hence to study the properties of $\Delta(T)$ we may employ this integral representation of $\Delta(T)$, instead of the usual Voronoi's formula. As an illustration we will give a new proof of a result of Jutila [**10**].

THEOREM 1.2 (Jutila). *Suppose that $HU \gg T^{1+\epsilon}$ and $U \leqslant \sqrt{T}/2$. We have*

$$\int_T^{T+H} (\Delta(u+U) - \Delta(u))^2 \, du \ll HU \log^3 \frac{\sqrt{T}}{U}.$$

The problem of estimating integrals of $(\Delta(u+U) - \Delta(u))^2$ over an interval is closely related to that of sign changes of $\Delta(u)$ (see [**4**]).

There are other possible applications of our main result. For instance, we may use our integral representation of $\Delta(T)$ to show that $\int_T^{T+H} \Delta(u)^4 \, du \ll T^\epsilon(HT + H^{1/5}T^{8/5})$ holds for all $H \leqslant T$. However, this result is inferior to that obtainable by the method of Ivić [**5**]. It seems to us that in order to achieve a stronger result, properties that are specifically related to $d(n)$, or equivalently, to the Riemann zeta function, must be utilized. Another natural question to ask is whether our result will yield a good bound for $|E(2\pi T) - 2\pi\Delta(T)|$, or a result that connects a bound for $\Delta(T)$ with that for $E(T)$. We are unable to give an affirmative answer at present either.

As usual, the notations $f(x) \ll g(x)$ and $f(x) = O(g(x))$ mean that there is a positive constant c such that $|f(x)| \leqslant c|g(x)|$ for x in the range under consideration. When $\lim_{x\to a} f(x)/g(x) = 0$, we use the notation $f(x) = o(g(x))$. The letter ϵ will always denote a small, but fixed positive number, though the number may not be the same at each occurrence. For example, we may write $T^\epsilon \log T \ll T^\epsilon$.

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2. Proof of Theorem 1.1

We first quote an analytic lemma regarding exponential integrals. The first part of the lemma is due to Atkinson, and the second part is due to Jutila [**12**].

LEMMA 2.1. *Let $\mu(x)$ be a positive differentiable function in the interval $[a, b]$. Suppose that $f(z)$ and $g(z)$ are functions satisfying the following conditions:*

(1) *the function $f(x)$ is real and $f''(x) > 0$ for $x \in [a, b]$;*
(2) *$f(z)$ and $g(z)$ are analytic for all z in the domain*

$$\bigcup_{x\in[a,b]} \{z : |z - x| \leqslant \mu(x)\};$$

(3) *there exist positive functions $F(x)$ and $G(x)$ such that for $x \in [a, b]$ and $|z - x| \leqslant \mu(x)$ we have*

$$F(x) \gg 1, \qquad |g(z)| \ll G(x),$$
$$|f'(z)| \ll F(x)\mu(x)^{-1}, \qquad f''(x) \gg F(x)\mu(x)^{-2};$$

(4) $\mu'(x) \ll 1$.

Let $H_J(x)$ *denote the function* $H_J(x) = G(x)\left(|f'(x)| + |f''(x)|^{1/2}\right)^{-J-1}$. *Assume that* $f'(x)$ *has a zero* c *in the interval* $[a,b]$. *We then have*

$$\begin{aligned}\int_a^b g(x)\exp\{2\pi i f(x)\}\,dx &= \frac{e^{\pi i/4}g(c)}{\sqrt{|f''(c)|}}\exp\{2\pi i f(c)\}\\ &+ O\Big(\int_a^b G(x)\exp\{-CF(x)\}\,dx\Big) + O\big(G(c)\mu(c)F(c)^{-3/2}\big)\\ &+ O(H_0(a)) + O(H_0(b)),\end{aligned}$$

where C *is a positive number determined by the* O*-constants in condition (3).*

Furthermore, if U *is a positive number and* J *is a positive integer such that* $JU < (b-a)/2$, $a + JU < c < b - JU$ *and* $U \gg \mu(c)F(c)^{-1/2}$, *we have*

$$\begin{aligned}U^{-J}&\int_0^U du_1 \cdots \int_0^U du_J \int_{a+u_1+\cdots+u_J}^{b-u_1-\cdots-u_J} g(x)\exp\{2\pi i f(x)\}\,dx\\ &= \frac{e^{\pi i/4}g(c)}{\sqrt{|f''(c)|}}\exp\{2\pi i f(c)\} + O\bigg(\int_a^b \Big(1 + \Big(\frac{\mu(x)}{U}\Big)^J\Big)G(x)\exp\{-CF(x)\}\,dx\bigg)\\ &+ O\big(G(c)\mu(c)F(c)^{-3/2}\big) + O(H_J(a)) + O(H_J(b)).\end{aligned}$$

In the case when $f'(x)$ *does not vanish in* $[a,b]$, *the above estimates hold without the terms involved with* c. *Moreover, if the condition* $f''(x) > 0$ *is replaced by* $f''(x) < 0$, *then the factor* $e^{\pi i/4}$ *in the main terms is replaced by* $e^{-\pi i/4}$.

The next lemma constitutes the essential part of the proof of Theorem 1.1.

Lemma 2.2. (i) *Let* $A < 1$ *be a positive constant, and* T *and* K *be positive numbers with* $K \leqslant AT$. *Let* T_1 *denote* $T - \sqrt{KT}$, *and* T_2 *denote* $T + \sqrt{KT}$. *Set*

$$\theta(u) = \theta_k(u) = 2u\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{u}} + \sqrt{ku + k^2/4} - \frac{k}{2} - \frac{1}{8},$$

$$f(u) = f_T(u) = \frac{\log(u/T)}{\sqrt{T}(u/T-1)}\exp\Big\{-2\pi i\Big(u\log\frac{u}{T} - u + T - \frac{1}{8}\Big)\Big\},$$

and $a(u) = a_k(u) = \big(4ku + k^2\big)^{1/4}\sinh^{-1}\dfrac{\sqrt{k}}{2\sqrt{u}}$. *Then we have, for* $0 < k \leqslant K$,

$$\int_{T_1}^{T_2}\frac{\cos\{2\pi\theta(u)\}}{a(u)}f(u)\,du = \frac{\sqrt{2}T^{1/4}}{k^{3/4}}\cos\Big\{4\pi(kT)^{1/2} - \pi/4\Big\} + O(\delta(k,T)),$$

where

$$\delta(k,T) = \frac{1}{T^{1/4}k^{3/4}}\left(1 + \min\left(\sqrt{T}, \frac{T}{|\sqrt{KT} - \sqrt{kT}|}\right)\right).$$

If $k > K$, *then the estimate holds without the leading term.*

(ii) *Conversely, let* K *be a positive number, and set*

$$T_3 = T - \sqrt{KT + K^2/4} + K/2, \qquad T_4 = T + \sqrt{KT + K^2/4} + K/2,$$

and

$$g(u)=g_T(u)=\frac{T/u-1}{\sqrt{u}\log(T/u)}\exp\Big\{2\pi i\Big(T\log\frac{T}{u}-T+u-\frac{1}{8}\Big)\Big\}.$$

Then we have, for $k\leqslant K$*,*

$$\sqrt{2}\int_{T_3}^{T_4}\frac{u^{1/4}}{k^{3/4}}\cos\Big\{4\pi(ku)^{1/2}-\pi/4\Big\}g(u)\,du=\frac{\cos\{2\pi\theta(T)\}}{a(T)}+O(\eta(k,T)),$$

where

$$\eta(k,T)=\frac{1}{T^{1/4}k^{3/4}}\Bigg(1+\min\Big(\sqrt{T},\frac{T}{|T_3-T+\sqrt{kT+k^2/4}-k/2|}\Big)\\ +\min\Big(\sqrt{T},\frac{T}{|T_4-T-\sqrt{kT+k^2/4}-k/2|}\Big)\Bigg).$$

When $k>K$*, the estimate holds without the main term.*

Proof. To prove the first part of the lemma, we first write the cosine function as a sum of two exponentials, and then evaluate two branches separately. Let I_k denote the integral

$$I_k=\frac{1}{2}\int_{T_1}^{T_2}\frac{\exp\{2\pi i\theta(u)\}}{a(u)}f(u)\,du,$$

and set $h(u)=\theta(u)-\big(u\log(u/T)-u+T-1/8\big)$. Let u_k be the solution of the equation $h'(u)=0$. Since

$$h'(u)=2\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{u}}-\log\frac{u}{T}, \tag{2.1}$$

we have

$$\sqrt{\frac{T}{u_k}}=\sqrt{1+\frac{k}{4u_k}}-\frac{\sqrt{k}}{2\sqrt{u_k}},$$

and thus

$$u_k=T+\sqrt{kT}. \tag{2.2}$$

Note that when $k\leqslant K$, the stationary point u_k lies in the interval $[T_1,T_2]$. We now apply the first part of Lemma 2.1 with

$$f(u)=h(u),\qquad g(u)=\frac{\log(u/T)}{a(u)\sqrt{T}(u/T-1)},$$
$$\mu(u)=T\big(1-\sqrt{A}\big)/2,\qquad F(u)=T,\qquad G(u)=1/T^{1/4}k^{3/4},$$
$$a=T_1,\qquad b=T_2,\qquad c=u_k.$$

Since $f''(u)$ is of constant sign and $|f''(u)|\gg 1/T$, we have

$$|f'(b)|=|f'(b)-f'(c)|\gg|b-c|/T=\big|\sqrt{KT}-\sqrt{kT}\big|/T,$$

and the same lower bound holds for $|f'(a)|$. Thus, Lemma 2.1 yields

$$I_k = \begin{cases} \dfrac{e^{-\pi i/4}}{2\sqrt{T|h''(u_k)|}} \dfrac{\log(u_k/T)}{u_k/T-1} \dfrac{\exp\{2\pi i h(u_k)\}}{a(u_k)} & \text{if } 0 < k \leqslant K, \\ \qquad\qquad + O(\delta(k,T)), & \\ O(\delta(k,T)), & \text{if } k > K, \end{cases} \tag{2.3}$$

since

$$h''(u_k) = \frac{2}{\sqrt{1+k/(4u_k)}}\left(-\frac{\sqrt{k}}{4u_k^{3/2}}\right) - \frac{1}{u_k} < 0. \tag{2.4}$$

We now show that the main term in (2.3) is actually equal to

$$\frac{T^{1/4}}{\sqrt{2}k^{3/4}} \exp\bigl\{4\pi i(kT)^{1/2} - \pi i/4\bigr\}.$$

By (2.1) and the definition of u_k, we have

$$\sinh^{-1} \frac{\sqrt{k}}{2\sqrt{u_k}} = \log\sqrt{u_k/T}. \tag{2.5}$$

It follows that, by (2.2),

$$\begin{aligned} h(u_k) &= \sqrt{ku_k + k^2/4} + (u_k - T) - k/2 \\ &= \sqrt{kT + k^{3/2}T^{1/2} + k^2/4} + \sqrt{kT} - k/2 = 2\sqrt{kT}. \end{aligned}$$

Moreover, from (2.2) we have

$$\frac{u_k}{T} - 1 = \frac{1}{T}\sqrt{kT} = \sqrt{k/T},$$

$$\bigl(4ku_k + k^2\bigr)^{1/4} = \bigl(4kT + 4k^{3/2}T^{1/2} + k^2\bigr)^{1/4} = \bigl(2\sqrt{kT} + k\bigr)^{1/2},$$

and hence, by (2.5), $a(u_k) = \frac{1}{2}\log(u_k/T)\bigl(2\sqrt{kT}+k\bigr)^{1/2}$. By (2.4), we have

$$\begin{aligned} h''(u_k) &= -\frac{1}{u_k}\left(\frac{\sqrt{k}}{\sqrt{4u_k+k}} + 1\right) \\ &= -\frac{1}{T+\sqrt{kT}}\left(\frac{\sqrt{k}}{\sqrt{4T + 4(kT)^{1/2} + k}} + 1\right) \\ &= -\frac{1}{T+\sqrt{kT}}\left(\frac{2\sqrt{k}+2\sqrt{T}}{\sqrt{k}+2\sqrt{T}}\right) = -\frac{2}{\sqrt{kT}+2T}. \end{aligned}$$

Inserting these expressions into (2.3), we obtain, for $k \leqslant K$,

$$\begin{aligned} I_k &= \frac{\bigl(\sqrt{kT}+2T\bigr)^{1/2}}{2^{3/2}\sqrt{T}} \frac{\sqrt{T}}{\sqrt{k}} \frac{2}{\bigl(2\sqrt{kT}+k\bigr)^{1/2}} \exp\bigl\{4\pi i(kT)^{1/2} - \pi i/4\bigr\} + O(\delta(k,T)) \\ &= \frac{T^{1/4}}{\sqrt{2}k^{3/4}} \exp\bigl\{4\pi i(kT)^{1/2} - \pi i/4\bigr\} + O(\delta(k,T)). \end{aligned}$$

For the other integral

$$\frac{1}{2}\int_{T_1}^{T_2} \frac{\exp\{-2\pi i\theta(u)\}}{a(u)} f(u)\, du,$$

we can show that the function $-\theta(u) - (u\log(u/T) - u + T - 1/8)$ has a root at $u = T - \sqrt{kT}$, and Lemma 2.1 yields

$$\frac{1}{2}\int_{T_1}^{T_2} \frac{\exp\{-2\pi i\theta(u)\}}{a(u)} f(u)\, du = \frac{T^{1/4}}{\sqrt{2}k^{3/4}} \exp\{-4\pi i(kT)^{1/2} + \pi i/4\} + O(\delta(k,T))$$

for $0 < k \leqslant K$ and

$$\int_{T_1}^{T_2} \frac{\exp\{-2\pi i\theta(u)\}}{a(u)} f(u)\, du \ll \delta(k,T)$$

for $k > K$. The first part of the lemma follows by combining these estimates with (2.3).

The proof of the second part is analogous, and the calculation is essentially the same as that in the proof of Theorem 7.2 of [**6**] and that in the proof of Theorem 1 of [**11**]. For completeness we sketch the proof as follows. We consider the integral

$$J_k = \frac{1}{\sqrt{2}}\int_{T_1}^{T_2} \frac{u^{1/4}}{k^{3/4}} \exp\left\{4\pi i(ku)^{1/2} - \pi i/4\right\} g(u)\, du.$$

Let $h(u)$ denote $h(u) = 2(ku)^{1/2} + T\log(T/u) - T + u - 1/4$. We have

$$h'(u) = \frac{(ku)^{1/2}}{u} - \frac{T}{u} + 1$$

$$h''(u) = -\frac{\sqrt{k}}{2u^{3/2}} + \frac{T}{u^2}. \tag{2.6}$$

Thus, if u_k is the real root of the equation $h'(u) = 0$, then we have

$$\frac{\sqrt{k}}{\sqrt{T}} = \sqrt{\frac{T}{u_k}} - \sqrt{\frac{u_k}{T}}. \tag{2.7}$$

It follows that

$$\log\sqrt{\frac{T}{u_k}} = \sinh^{-1}\frac{\sqrt{k}}{2\sqrt{T}}, \tag{2.8}$$

$$u_k = T - \sqrt{kT + k^2/4} + \frac{k}{2}, \tag{2.9}$$

and $h''(u_k) > 0$. By Lemma 2.1, if $k \leqslant K$, then we have

$$J_k = \frac{1}{\sqrt{2u_k}} \frac{e^{\pi i/4}}{\sqrt{|h''(u_k)|}} \frac{u_k^{1/4}}{k^{3/4}} \frac{T/u_k - 1}{\log(T/u_k)} \exp\{2\pi i h(u_k)\} + O(\eta(k,T)).$$

In light of (2.7), (2.8) and (2.9) we see that

$$\frac{T/u_k - 1}{u_k^{1/4}k^{3/4}\log(T/u_k)} = \frac{1}{2u_k^{3/4}k^{1/4}\sinh^{-1}(\sqrt{k}/(2\sqrt{T}))}$$

and

$$\begin{aligned}
h(u_k) &= 2(ku_k)^{1/2} + \big(T\log(T/u_k) - T + u_k\big) - 1/4 \\
&= (ku_k)^{1/2} + 2T\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{T}} \\
&= \sqrt{k}\big(-\sqrt{k}/2 + \sqrt{T+k/4}\big) + 2T\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{T}} - 1/4 \\
&= 2T\sinh^{-1}\frac{\sqrt{k}}{2\sqrt{T}} + \sqrt{kT + k^2/4} - \frac{k}{2} - 1/4 = \theta(T) - 1/8
\end{aligned}$$

Furthermore, by (2.6) and (2.7), we have

$$\begin{aligned}
h''(u_k) &= -\frac{\sqrt{k}}{2u_k^{3/2}} + \frac{T}{u_k^2} = \left(-\frac{1}{2}\sqrt{\frac{k}{T}} + \sqrt{\frac{T}{u_k}}\right)\frac{T^{1/2}}{u_k^{3/2}} \\
&= \frac{1}{2}\left(\sqrt{\frac{T}{u_k}} + \sqrt{\frac{u_k}{T}}\right)\frac{T^{1/2}}{u_k^{3/2}} = \frac{T^{1/2}}{u_k^{3/2}}\sqrt{\frac{1}{4}\left(\sqrt{\frac{T}{u_k}} - \sqrt{\frac{u_k}{T}}\right)^2 + 1} \\
&= u_k^{-3/2}\sqrt{T+k/4}.
\end{aligned}$$

Hence, for $k \leqslant K$, the integral J_k can be estimated as

$$\begin{aligned}
J_k &= \frac{1}{\sqrt{2u_k}}\frac{e^{\pi i/4}}{\sqrt{|h''(u_k)|}}\frac{u_k^{1/4}}{k^{3/4}}\frac{T/u_k - 1}{\log(T/u_k)}\exp\{2\pi i h(u_k)\} + O(\eta(k,T)) \\
&= \frac{u_k^{3/4}}{\sqrt{2}(T+k/4)^{1/4}}\frac{\exp\{2\pi i\theta(T)\}}{2u_k^{3/4}k^{1/4}\sinh^{-1}\big(\sqrt{k}/(2\sqrt{T})\big)} + O(\eta(k,T)) \\
&= \frac{\exp\{2\pi i\theta(T)\}}{2a(T)} + O(\eta(k,T)).
\end{aligned}$$

For the case where $k \geqslant K$, the same lemma implies that $J_k \ll \eta(k,T)$. Similarly, we can show that

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\int_{T_1}^{T_2}\frac{u^{1/4}}{k^{3/4}}\exp\big\{-4\pi i(ku)^{1/2} + \pi i/4\big\}g(u)\,du \\
&\qquad = \begin{cases} \dfrac{\exp\{-2\pi i\theta(T)\}}{2a(T)} + O(\eta(k,T)), & \text{if } k \leqslant K, \\ O(\eta(k,T)), & \text{if } k > K. \end{cases}
\end{aligned}$$

combining this with estimates for J_k the second part of the lemma follows, and the proof of the lemma is complete. □

Proof of Theorem 1.1. The proof of Theorem 1.1 is a straightforward application of Lemma 2.2. By Lemma 2.2, we have

$$\frac{1}{2\pi}\int_{T_1}^{T_2} E_1(2\pi u)f(u)\,du$$
$$= \frac{T^{1/4}}{\pi\sqrt{2}} \sum_{k\leqslant \min(A^2T,BT)} \frac{d(k)}{k^{3/4}} \cos\bigl\{4\pi(kT)^{1/2}-\pi/4\bigr\} + O\biggl(\sum_{k\leqslant BT}\delta(k,T)\biggr)$$

In light of Voronoi's formula (Lemma 1.1) the main term in the last expression is $\Delta(T)+O(T^\epsilon)$, while the O-term is bounded by

$$\ll \frac{1}{T^{1/4}}\sum_{k\leqslant BT}\frac{d(k)}{k^{3/4}} + \frac{1}{T^{1/4}}\sum_{k\leqslant A^2T/2}\frac{d(k)}{k^{3/4}} + T^{1/4}\sum_{|k-A^2T|\leqslant AT^{1/2}}\frac{d(k)}{k^{3/4}}$$
$$+ T^{3/4}\sum_{AT^{1/2}\leqslant|k-A^2T|\leqslant A^2T/2}\frac{d(k)}{k^{3/4}\bigl|\sqrt{KT}-\sqrt{kT}\bigr|} + T^{1/4}\sum_{k\geqslant 3A^2T/2}\frac{d(k)}{k^{5/4}}.$$

Using the bound $d(k)\ll k^\epsilon$ for any fixed $\epsilon>0$ we see that the last expression is bounded by T^ϵ, and thus

$$\frac{1}{2\pi}\int_{T_1}^{T_2} E_1(2\pi u)f(u)\,du = \Delta(T)+O(T^\epsilon).$$

This proves the first part of the theorem.

The proof of the other part of the theorem is very similar, and the details are omitted. However, we note that we need the following form of Voronoi's formula

$$\Delta(T) = \frac{T^{1/4}}{\pi\sqrt{2}}\sum_{k\leqslant K}\frac{d(k)}{k^{3/4}}\cos\bigl\{4\pi(kT)^{1/2}-\pi/4\bigr\}$$
$$-\frac{3}{32\sqrt{2}\pi^2}T^{-1/4}\sum_{k\leqslant K}\frac{d(k)}{k^{5/4}}\sin\bigl\{4\pi(kT)^{1/2}-\pi/4\bigr\}+O(T^{-3/4})$$

in order to show that the error term is of order T^ϵ. □

3. A new proof of Theorem 1.2

In this section we will give a new proof of Theorem 1.2 using the integral representation of $\Delta(T)$ obtained in the previous section. We shall provide details only when our arguments differ from the usual methods, and sketch the proof when the arguments are identical or similar to that in literature. We first prove a lemma that generalizes the Halász–Montgomery inequality.

Lemma 3.1. *Let the inner product of two complex-valued functions $\xi(u)$ and $\phi(u)$ be defined by $(\xi,\phi)=\int\xi\overline{\phi}\,du$, and let $\|\xi\|$ denote $\|\xi\|=\bigl(\int|\xi|^2\,du\bigr)^{1/2}$. Suppose that $\xi_\lambda(u)=\xi(u,\lambda)$ and $\phi_{\lambda,r}(u)=\phi_r(u,\lambda)$, $r=1,2,\dots,R$, are integrable with respect to λ for $0\leqslant\lambda\leqslant L$. We have*

$$\sum_{r\leqslant R}\biggl|\frac{1}{L}\int_0^L(\xi_\lambda,\phi_{\lambda,r})\,d\lambda\biggr|^2 \leqslant \frac{1}{L}\int_0^L\|\xi_\lambda\|^2\,d\lambda\times\max_{r\leqslant R}\sum_{s\leqslant R}\biggl|\frac{1}{L}\int_0^L(\phi_{\lambda,r},\phi_{\lambda,s})\,d\lambda\biggr|$$

PROOF. For any complex scalars c_r we have

$$\begin{aligned}\sum_{r\leqslant R}\frac{c_r}{L}\int_0^L(\xi_\lambda,\phi_{\lambda,r})\,d\lambda &= \frac{1}{L}\int_0^L\Big(\xi_\lambda,\sum\overline{c}_r\phi_{\lambda,r}\Big)\,d\lambda\\ &\leqslant \frac{1}{L}\int_0^L\|\xi_\lambda\|\,\Big\|\sum\overline{c}_r\phi_{\lambda,r}\Big\|\,d\lambda\\ &\leqslant \bigg(\frac{1}{L}\int_0^L\|\xi_\lambda\|^2d\lambda\bigg)^{1/2}\bigg(\frac{1}{L}\int_0^L\Big\|\sum\overline{c}_r\phi_{\lambda,r}\Big\|^2\,d\lambda\bigg)^{1/2}\end{aligned}$$

Expanding $\|\sum\overline{c}_r\phi_{\lambda,r}\|^2$ and noting that $|\overline{c}_rc_s|\leqslant(|c_r|^2+|c_s|^2)/2$ we obtain

$$\begin{aligned}\int_0^L\Big\|\sum\overline{c}_r\phi_{\lambda,r}\Big\|^2\,d\lambda &\leqslant \sum_{r,s\leqslant R}|\overline{c}_rc_s|\bigg|\int_0^L(\phi_{\lambda,r},\phi_{\lambda,s})\,d\lambda\bigg|\\ &\leqslant \frac{1}{2}\sum_{r,s\leqslant R}\big(|c_r|^2+|c_s|^2\big)\bigg|\int_0^L(\phi_{\lambda,r},\phi_{\lambda,s})\,d\lambda\bigg|\\ &\leqslant \max_r\sum_s\bigg|\int_0^L(\phi_{\lambda,r},\phi_{\lambda,s})\,d\lambda\bigg|\times\sum_{r\leqslant R}|c_r|^2.\end{aligned}$$

Choosing

$$c_r=\overline{\frac{1}{L}\int_0^L(\xi_\lambda,\phi_{\lambda,r})\,d\lambda},$$

the claimed inequality follows immediately. □

PROOF OF THEOREM 1.2. Let

$$S(u,K_1,K_2)=u^{1/4}\sum_{K_1<k\leqslant K_2}\frac{d(k)}{k^{3/4}}\cos\{4\pi(ku)^{1/2}-\pi/4\}$$

denote the partial sum in Voronoi's formula, and set

$$S_U(u,K_1,K_2)=S(u+U,K_1,K_2)-S(u,K_1,K_2).$$

In view of Voronoi's formula, to prove the theorem it suffices to consider the integral

$$\int_T^{T+H}S_U(u,0,T)^2\,du.$$

Assume that $U\leqslant\sqrt{T}/2$. Let m be the integer such that $2^m<T^{1/3}U^{-2/3}\leqslant 2^{m+1}$, and let M denote 2^m. We have trivially

$$\cos\big\{4\pi\sqrt{k(u+U)}-\pi/4\big\}-\cos\big\{4\pi\sqrt{ku}-\pi/4\big\}\ll\frac{\sqrt{k}U}{\sqrt{T}},$$

and thus

$$\int_T^{T+H}S_U(u,0,M)^2\,du\ll H\bigg(T^{1/4}\sum_{k\leqslant M}\frac{d(k)}{k^{3/4}}\frac{\sqrt{k}U}{\sqrt{T}}\bigg)^2\ll HU\log^2\big(\sqrt{T}/U\big),$$

which is contained in the claimed bound.

We next consider the case when $M^3 < k \leqslant T$. Using standard arguments (see, for example, [**6**, p.363]) we see that

$$\int_T^{T+H} S_U(u, M^3, T)^2\,du \ll HT^{1/2} \sum_{k \geqslant M^3} \frac{d(k)^2}{k^{3/2}} + T^{1+\epsilon} \ll HU \log^3 \frac{\sqrt{T}}{U} + T^{1+\epsilon}.$$

Thus it remains to deal with the cases when $M < k \leqslant M^3$. We write $S_U(u, M, M^3)$ as $\sum_K S_U(u, K, 2K)$, where K runs over integers of the form $2^m M$. When $K_1 \leqslant K_2/4$, we have

$$\int_T^{T+H} S_U(u, K_1, 2K_1) S_U(u, K_2, 2K_2)\,du \ll T^{1+\epsilon} K_1^{1/4} K_2^{-1/4}.$$

When $K_1 = K_2/2$, we use the inequality $2|ab| \leqslant |a|^2 + |b|^2$, and obtain

$$\int_T^{T+H} S_U(u, M, M^3)^2\,du \ll \sum_K \int_T^{T+H} S_U(u, K, 2K)^2\,du + T^{1+\epsilon}.$$

Thus, the proof of the result will be complete if we can show that

$$\int_T^{T+H} S_U(u, K, 2K)^2\,du \ll \frac{HU^2\sqrt{K}}{\sqrt{T}} \log^3 K \tag{3.1}$$

for $K \leqslant \sqrt{T}/U$.

Let $T_h \in [T+h, T+h+1]$ denote a point with

$$|S_U(T_h, K, 2K)| = \max_{T+h \leqslant u \leqslant T+h+1} |S_U(u, K, 2K)|.$$

We then have

$$\int_T^{T+H} S_U(u, K, 2K)^2\,du \ll \sum_{h=0}^{H} |S_U(T_h, K, 2K)|^2.$$

For $h \leqslant H$ we denote by T_h' and T_h'' the points $T_h - 4\sqrt{KT}$ and $T_h + 4\sqrt{KT}$, respectively. Set $L = \sqrt{KT}$. Applying the second part of Lemma 2.1 and following the calculation in Lemma 2.2 we obtain

$$S_U(T_h, K, 2K) = \frac{1}{L} \int_0^L \int_{T_h'+\lambda}^{T_h''-\lambda} \Sigma(u, K) \left(f(u, T_h + U) - f(u, T_h) \right)\,du\,d\lambda$$
$$+ O\bigg(\frac{T^{1/4}}{K^{9/4}} \sum_{K < k \leqslant 2K} d(k) \bigg),$$

where

$$\Sigma(u, K) = \sum_{K < k \leqslant 2K} \frac{d(k) \cos\{2\pi\theta_k(u)\}}{\sqrt{2} a_k(u)},$$

$$f(u, v) = \frac{\log(u/v)}{\sqrt{v}(u/v - 1)} \exp\big\{-2\pi i (u \log(u/v) - u + v - 1/8)\big\}$$

with

$$\theta_k(u) = 2u \sinh^{-1} \frac{\sqrt{k}}{2\sqrt{u}} + \sqrt{ku + k^2/4} - \frac{k}{2} - \frac{1}{8},$$

$$a_k(u) = (4ku + k^2)^{1/4} \sinh^{-1} \frac{\sqrt{k}}{2\sqrt{u}}.$$

Setting $f_U(u, v) = f(u, v + U) - f(u, v)$, we obtain

$$\sum_{h=0}^{H} S_U(T_h, K, 2K)^2 \ll \sum_{h=0}^{H} \left| \frac{1}{L} \int_0^L \int_{T'_h+\lambda}^{T''_h-\lambda} \Sigma(u, K) f_U(u, T_h)\, du\, d\lambda \right|^2 + HT^{1/2}K^{-5/2} \log^2 K$$

Since $K \geqslant T^{1/3}U^{-2/3}$, the term $HT^{1/2}K^{-5/2}\log^2 K$ is bounded by $HU \log^2\big(\sqrt{T}/U\big)$. We now apply Lemma 3.1 with

$$\xi_\lambda(u) = \Sigma(u, K),$$

$$\phi_{\lambda,h}(u) = \begin{cases} \overline{f_U(u, T_h)}, & \text{when } T'_h + \lambda \leqslant u \leqslant T''_h - \lambda, \\ 0, & \text{else}, \end{cases}$$

and the inner product (ξ, ϕ) given by

$$(\xi, \phi) = \int_{T-5L}^{T+H+5L} \xi(u)\overline{\phi(u)}\, du.$$

It follows from Lemma 3.1 that

$$\sum_{h=0}^{H} S(T_h, K, 2K)^2 \ll \frac{1}{L} \int_0^L \|\xi_\lambda\|^2\, d\lambda \times \max_{r \leqslant R} \sum_{s \leqslant R} \left| \frac{1}{L} \int_0^L (\phi_{\lambda,r}, \phi_{\lambda,s})\, d\lambda \right| \tag{3.2}$$

Using standard arguments ([**6**, p. 363]) again we see that

$$\|\xi_\lambda\|^2 \ll HT^{1/2}K^{-1/2}\log^3 K + T^{1+\epsilon}. \tag{3.3}$$

Moreover, we have

$$f_U(u, T_h) \ll U \max_{T'_h \leqslant u \leqslant T''_h + L + U} \frac{\partial}{\partial u} f(u, T_h) \ll \frac{U\sqrt{K}}{T}$$

for all $T1_h \leqslant u \leqslant T''_h + L$, and thus

$$\frac{1}{L} \int_0^L (\phi_{\lambda,h_1}, \phi_{\lambda,h_2})\, d\lambda \ll \sqrt{TK}\left(\frac{U\sqrt{K}}{T}\right)^2 = \frac{U^2K^{3/2}}{T^{3/2}}$$

for all $h_1, h_2 \leqslant H$. On the other hand, integrating by parts, we see that the main contribution to the integral $L^{-1}\int_0^L (\phi_{\lambda,h_1}, \phi_{\lambda,h_2})\, d\lambda$ can be written as a sum of quantities of the form

$$c_1 \Big\{ g(T_{h_1} + U, T_{h_2} + U, t + c_2 L) - g(T_{h_1}, T_{h_2} + U, t + c_2 L) - g(T_{h_1} + U, T_{h_2}, t + c_2 L) + g(T_{h_1}, T_{h_2}, t + c_2 L) \Big\}$$

where t is

$$T''_{h_1,h_2} = \min(T''_{h_1}, T''_{h_2}) \quad \text{or} \quad T'_{h_1,h_2} = \max(T'_{h_1}, T'_{h_2}),$$

c_2 is 0 or 1 for $t = T'_{h_1,h_2}$, 0 or -1 for $t = T''_{h_1,h_2}$, and c_1 is 1 or -1, depending on t and c_2, and

$$g(u,v,t) = -\frac{\log(t/u)\log(t/v)}{4\pi^2\sqrt{uv}\log^2(u/v)(t/u-1)(t/v-1)}.$$

It follows that, for $|h_1 - h_2| \geqslant \sqrt{T/K}$,

$$\frac{1}{L}\int_0^L (\phi_{\lambda,h_1}, \phi_{\lambda,h_2})\, d\lambda \ll \frac{1}{LT\log^2(T_{h_1}/T_{h_2})}\left(\frac{U\sqrt{K}}{\sqrt{T}}\right)^2 \ll \frac{U^2\sqrt{K}}{\sqrt{T}|h_1-h_2|^2},$$

and thus

$$\sum_{h_2 \leqslant H} \frac{1}{L}\int_0^L (\phi_{\lambda,h_1}, \phi_{\lambda,h_2})\, d\lambda \ll \frac{U^2 K}{T}$$

for all $h_1 \leqslant H$. Inserting this estimate and (3.3) into (3.2), we hence obtain (3.1). This completes the proof of Theorem 1.2. □

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