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FINITE DIFFERENCE APPROXIMATION OF STRONG SOLUTIONS OF A PARABOLIC INTERFACE PROBLEM ON DISCONNECTED DOMAINS

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Abstract. We investigate an initial boundary value problem for one dimensional parabolic equation in two disconnected intervals. A finite difference scheme for its solution is proposed and investigated. Convergence rate estimate compatible with the smoothness of input data is obtained.

1. Introduction

Interface problems occur in many applications in science and engineering. From the mathematical point of view, interface problems lead to partial differential equations whose input data and solutions have discontinuities across one or several hypersurfaces, which have lower dimension than the domain where the problem is defined. Various forms of conjugation conditions satisfied by the solution and its derivatives on the interface are known. The numerical methods designed for smooth solutions do not work efficiently for interface problems. Problems of this type we considered in [**6, 7**]

There exists another similar type of problems whose solutions are defined in two (or more) disconnected domains. For example, such situation occurs when the solution in the intermediate region is known or can be determined from a simpler equation. Its effect can be modelled (see [**2**]) by means of nonlocal jump conditions across the intermediate region (layer).

In this paper we consider the following initial-boundary-value problem (IBVP): Find functions $u_1(x,t)$ and $u_2(x,t)$ that satisfy the parabolic equations

$$\frac{\partial u_1}{\partial t} - \frac{\partial}{\partial x}\left(p_1(x)\frac{\partial u_1}{\partial x}\right) = f_1(x,t), \quad x \in \Omega_1 \equiv (a_1, b_1), \quad t > 0, \tag{1}$$

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$$\frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x}\left(p_2(x)\frac{\partial u_2}{\partial x}\right) = f_2(x,t), \quad x \in \Omega_2 \equiv (a_2, b_2), \quad t > 0, \tag{2}$$

where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$, the internal conjugation conditions of Robin–Dirichlet type

$$p_1(b_1)\frac{\partial u_1(b_1,t)}{\partial x} + \alpha_1 u_1(b_1,t) = \beta_1 u_2(a_2,t) + \gamma_1(t), \tag{3}$$

$$-p_2(a_2)\frac{\partial u_2(a_2,t)}{\partial x} + \alpha_2 u_2(a_2,t) = \beta_2 u_1(b_1,t) + \gamma_2(t), \tag{4}$$

the simplest external Dirichlet boundary conditions

$$u_1(a_1,t) = 0, \quad u_2(b_2,t) = 0, \tag{5}$$

and initial conditions

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x). \tag{6}$$

Throughout the paper we assume that the data satisfy the usual regularity and ellipticity conditions

$$p_i(x) \in L_\infty(\Omega_i), \quad 0 < p_{i0} \leqslant p_i(x), \ \text{ a.e. in } \ \Omega_i, \quad i = 1, 2. \tag{7}$$

We also assume that

$$\alpha_i > 0\,, \quad \beta_i > 0, \quad i = 1, 2 \quad \text{ and } \quad \beta_1\beta_2 \leqslant \alpha_1\alpha_2. \tag{8}$$

Similar problem is investigated in [**8**] and a finite difference scheme (FDS) for its numerical solution is proposed. The convergence of FDS in the weak discrete norm ($H^{1,1/2}_{h,\tau}$) is proved. In this paper the properties of the strong solution of IBVP (1)–(6) are examined and the convergence of the corresponding FDS is proved in the strong discrete norm ($H^{2,1}_{h,\tau}$). To ensure the second order of convergence in the space step-size h a special approximation for $x = b_1$ and $x = a_2$ was needed.

By C we shall denote a positive generic constant, independent of the solution of IBVP and the mesh-sizes, which can take different values in different formulas.

The layout of the paper is as follows. Section 2 is devoted to the analysis of the existence and the uniqueness of the strong solution of IBVP (1)–(6). In Section 3 we introduce a FDS approximating IBVP (1)–(6) and investigate its convergence. A convergence rate estimate compatible with the smoothness of input data is obtained.

2. Existence and uniqueness of the strong solution

Let the conditions (8) hold. We introduce the product space

$$L = L_2(\Omega_1) \times L_2(\Omega_2) = \{v = (v_1, v_2) \,|\, v_i \in L_2(\Omega_i)\},$$

endowed with the inner product and associated norm

$$(u, v)_L = \beta_2(u_1, v_1)_{L_2(\Omega_1)} + \beta_1(u_2, v_2)_{L_2(\Omega_2)}, \quad \|v\|_L = (v, v)_L^{1/2},$$

where

$$(u_i, v_i)_{L_2(\Omega_i)} = \int\limits_{\Omega_i} u_i v_i \, dx, \quad i = 1, 2.$$

We also define the spaces

$$H^k = \{v = (v_1, v_2) \mid v_i \in H^k(\Omega_i)\}, \quad k = 1, 2, \dots,$$

endowed with the inner products and norms

$$(u, v)_{H^k} = \beta_2 (u_1, v_1)_{H^k(\Omega_1)} + \beta_1 (u_2, v_2)_{H^k(\Omega_2)}, \quad \|v\|_{H^k} = (v, v)^{1/2}_{H^k},$$

where

$$(u_i, v_i)_{H^k(\Omega_i)} = \sum_{j=0}^{k} \Big(\frac{d^j u_i}{dx^j}, \frac{d^j v_i}{dx^j} \Big)_{L_2(\Omega_i)}, \quad i = 1, 2, \quad k = 1, 2, \dots.$$

In particular, we set

$$H_0^1 = \{v = (v_1, v_2) \in H^1 \mid v_1(a_1) = 0,\ v_2(b_2) = 0\}.$$

Finally, we define the bilinear form

$$A(u, v) = \beta_2 \int\limits_{\Omega_1} p_1 \frac{du_1}{dx} \frac{dv_1}{dx}\, dx + \beta_1 \int\limits_{\Omega_2} p_2 \frac{du_2}{dx} \frac{dv_2}{dx}\, dx$$
$$+ \beta_2 \alpha_1 v_1(b_1) w_1(b_1) + \beta_1 \alpha_2 v_2(a_2) w_2(a_2) - \beta_1 \beta_2 \big[v_1(b_1) w_2(a_2) + v_2(a_2) w_1(b_1) \big]. \tag{9}$$

LEMMA 2.1. [**8**] *Under the conditions* (7) *and* (8) *the bilinear form A, defined by* (9)*, is symmetric and bounded on $H^1 \times H^1$. Moreover, this form is also coercive on H_0^1, i.e., there exists a constant $c_0 > 0$ such that $A(v, v) \geqslant c_0 \|v\|^2_{H^1}$, $\forall\, v \in H_0^1$.*

Let Ω be a domain in $\mathbb{R}^n$ and $u(t)$ a function mapping Ω into a Hilbert space H. In a standard manner (see [**9**]) we define the Sobolev space of vector valued functions $H^k(\Omega, H)$, endowed with the inner product

$$(u, v)_{H^k(\Omega, H)} = \int\limits_{\Omega} \sum_{|\alpha| \leqslant k} (D^\alpha u(t),\, D^\alpha v(t))_H \, dt\,, \quad k = 0, 1, 2, \dots$$

and with the usual modification for non-integer k. For $k = 0$ we set $L_2(\Omega, H) = H^0(\Omega, H)$.

We define the spaces $H^{k, k/2} = L_2((0, T), H^k) \cap H^{k/2}((0, T), L)$. We also denote $Q_i = \Omega_i \times (0, T)$, $i = 1, 2$.

THEOREM 2.1. *Let the assumptions* (7) *and* (8) *hold and $u_0 = (u_{10}, u_{20}) \in H_0^1$, $f = (f_1, f_2) \in L_2((0, T), L)$, $\gamma_i \in H^{1/4}(0, T)$, $p_i' \in L_\infty(\Omega_i)$, $i = 1, 2$. Then the IBVP* (1)–(6) *has a unique strong solution $u = (u_1, u_2) \in H^{2,1}$ and the following a priori estimate holds true*

$$\|u\|^2_{H^{2,1}} \leqslant C \sum_{i=1}^{2} \beta_{3-i} \Big(\|u_{i0}\|^2_{H^1(\Omega_i)} + \|f_i\|^2_{L_2(Q_i)} + \|\gamma_i\|^2_{H^{1/4}(0,T)} \Big). \tag{10}$$

PROOF. Existence and uniqueness of the weak solution of IBVP (1)–(6) is proved in [**?**, Theorems 2.1 and 3.1]. In such a way, it remains to prove that under our assumptions this solution is strong, i.e., satisfies the a priori estimate (10).

Multiplying the equation (1) by $\partial u_1/\partial t$ and integrating by parts, we obtain

$$\begin{aligned}\Big\|\frac{\partial u_1}{\partial t}(\cdot,t)\Big\|^2_{L_2(\Omega_1)} &+ \int\limits_{\Omega_1} p_1\frac{\partial u_1}{\partial x}\,\frac{\partial^2 u_1}{\partial t\partial x}\,dx \\ &+ \alpha_1 u_1(b_1,t)\,\frac{\partial u_1}{\partial t}(b_1,t) - \beta_1 u_2(a_2,t)\,\frac{\partial u_1}{\partial t}(b_1,t) \\ &= \Big(f_1(\cdot,t),\frac{\partial u_1}{\partial t}(\cdot,t)\Big)_{L_2(\Omega_1)} + \gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t) \\ &\leqslant \frac{1}{2}\,\|f_1(\cdot,t)\|^2_{L_2(\Omega_1)} + \frac{1}{2}\,\Big\|\frac{\partial u_1}{\partial t}(\cdot,t)\Big\|^2_{L_2(\Omega_1)} + \gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t)\end{aligned}$$

and analogously

$$\begin{aligned}\frac{1}{2}\,\Big\|\frac{\partial u_2}{\partial t}(\cdot,t)\Big\|^2_{L_2(\Omega_2)} &+ \int\limits_{\Omega_2} p_2\frac{\partial u_2}{\partial x}\,\frac{\partial^2 u_2}{\partial t\partial x}\,dx \\ &+ \alpha_2 u_2(a_2,t)\,\frac{\partial u_2}{\partial t}(a_2,t) - \beta_2 u_1(b_1,t)\,\frac{\partial u_2}{\partial t}(a_2,t) \\ &\leqslant \frac{1}{2}\,\|f_2(\cdot,t)\|^2_{L_2(\Omega_2)} + \gamma_2(t)\,\frac{\partial u_2}{\partial t}(a_2,t).\end{aligned}$$

Multiplying the first of these inequalities by $2\beta_2$, the second by $2\beta_1$ and summing up we get

$$\begin{aligned}\Big\|\frac{\partial u}{\partial t}(\cdot,t)\Big\|^2_{L} &+ \frac{d}{dt}\Big(A(u(\cdot,t),\,u(\cdot,t))\Big) \\ &\leqslant \|f(\cdot,t)\|^2_L + 2\Big(\beta_2\gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t) + \beta_1\gamma_2(t)\,\frac{\partial u_2}{\partial t}(a_2,t)\Big).\end{aligned}$$

Integrating this inequality on $t\in(0,T)$ and using Lemma 2.1 one obtains

$$\begin{aligned}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L_2((0,T),L)} &\leqslant C\|u_0\|^2_{H^1} + \|f\|^2_{L_2((0,T),L)} \\ &+ 2\int\limits_0^T\Big(\beta_2\gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t) + \beta_1\gamma_2(t)\,\frac{\partial u_2}{\partial t}(a_2,t)\Big)dt.\end{aligned}\tag{11}$$

Analogously

$$\begin{aligned}\|Au\|^2_{L_2((0,T),L)} &\leqslant C\|u_0\|^2_{H^1} + \|f\|^2_{L_2((0,T),L)} \\ &+ 2\int\limits_0^T\Big(\beta_2\gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t) + \beta_1\gamma_2(t)\,\frac{\partial u_2}{\partial t}(a_2,t)\Big)dt,\end{aligned}\tag{12}$$

where we denoted $Au=(A_1u_1,A_2u_2)$ and

$$A_iu_i = -\frac{\partial}{\partial x}\Big(p_i(x)\frac{\partial u_i}{\partial x}\Big),\quad i=1,2.$$

Let us estimate $\|Au\|_L = \|Au(\cdot,t)\|_L$. From the inequalities

$$\Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_L \leqslant C\,\Big\|p\,\frac{\partial^2 u}{\partial x^2}\Big\|_L = C\,\Big\|Au + \frac{dp}{dx}\,\frac{\partial u}{\partial x}\Big\|_L \leqslant C\Big(\|Au\|_L + \|u\|_{H^1}\Big)$$

and

$$\begin{aligned} c_0\,\|u\|^2_{H^1} &\leqslant A(u,u) = (Au,u)_L + \beta_2\gamma_1(t)\,u_1(b_1,t) + \beta_1\gamma_2(t)u_2(a_2,t) \\ &\leqslant \varepsilon\Big(\|u\|^2_L + \beta_2 u_1^2(b_1,t) + \beta_1 u_2^2(a_2,t)\Big) + \frac{1}{4\varepsilon}\Big(\|Au\|^2_L + \beta_2\gamma_1^2(t) + \beta_1\gamma_2^2(t)\Big) \\ &\leqslant C\varepsilon\,\|u\|^2_{H^1} + \frac{1}{4\varepsilon}\Big(\|Au\|^2_L + \beta_2\gamma_1^2(t) + \beta_1\gamma_2^2(t)\Big) \end{aligned}$$

after taking $\varepsilon = c_0/(2C)$ one obtains

$$\|u\|^2_{H^2} = \Big\|\frac{\partial^2 u}{\partial x^2}\Big\|^2_L + \|u\|^2_{H^1} \leqslant C\Big(\|Au\|^2_L + \beta_2\gamma_1^2(t) + \beta_1\gamma_2^2(t)\Big). \tag{13}$$

Let us now estimate $\int_0^T \gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t)\,dt$. Using Fourier sine and cosine expansions

$$\gamma_1(t) = \sum_{j=1}^{\infty} b_j[\gamma_1]\,\sin\frac{j\pi t}{T}, \quad u_1(b_1,t) = \frac{a_0[u_1(b_1,\cdot)]}{2} + \sum_{j=1}^{\infty} a_j[u_1(b_1,\cdot)]\,\cos\frac{j\pi t}{T}$$

(where $b_j[\gamma_1]$ and $a_j[u_1(b_1,\cdot)]$ are the corresponding Fourier coefficients) and the orthogonality of sines, we obtain

$$\begin{aligned} \int_0^T \gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t)\,dt &= -\frac{\pi}{2}\sum_{k=1}^{\infty} k\,b_k[\gamma_1]a_k[u_1(b_1,\cdot)] \\ &\leqslant \frac{\pi}{2}\left(\sum_{k=1}^{\infty} k^{1/2}\,b_k^2[\gamma_1]\right)^{1/2}\left(\sum_{k=1}^{\infty} k^{3/2}\,a_k^2[u_1(b_1,\cdot)]\right)^{1/2}. \end{aligned}$$

From this inequality, using Lemma 3.1 from [**8**] and the trace theorem for anisotropic Sobolev spaces [**9**], one obtains

$$\begin{aligned} \left|\int_0^T \gamma_1(t)\,\frac{\partial u_1}{\partial t}(b_1,t)\,dt\right| &\leqslant C\|\gamma_1\|_{H^{1/4}(0,T)}\|u_1(b_1,\cdot)\|_{H^{3/4}(0,T)} \\ &\leqslant C\|\gamma_1\|_{H^{1/4}(0,T)}\|u_1\|_{H^{2,1}(Q_1)} \leqslant \varepsilon\,\|u_1\|^2_{H^{2,1}(Q_1)} + \frac{C}{\varepsilon}\,\|\gamma_1\|^2_{H^{1/4}(0,T)} \end{aligned} \tag{14}$$

and analogously

$$\left|\int_0^T \gamma_2(t)\,\frac{\partial u_2}{\partial t}(a_2,t)\,dt\right| \leqslant \varepsilon\,\|u_2\|^2_{H^{2,1}(Q_2)} + \frac{C}{\varepsilon}\,\|\gamma_2\|^2_{H^{1/4}(0,T)}. \tag{15}$$

Finally, for sufficiently small $\varepsilon > 0$, from (11)–(15) we get the a priori estimate (10). □

3. Finite difference approximation

3.1. Meshes, finite differences and discrete norms. Let $\bar{\omega}_{1,h_1}$ be an uniform mesh in $\bar{\Omega}_1$ with the step-size $h_1 = (b_1 - a_1)/n_1$, $\omega_{1,h_1} = \bar{\omega}_{1,h_1} \cap \Omega_1$, $\omega^-_{1,h_1} = \omega_{1,h_1} \cup \{a_1\}$ and $\omega^+_{1,h_1} = \omega_{1,h_1} \cup \{b_1\}$. Analogously, in $\bar{\Omega}_2$ we define uniform mesh $\bar{\omega}_{2,h_2}$ with the step-size $h_2 = (b_2 - a_2)/n_2$ and its submeshes $\omega_{2,h_2} = \bar{\omega}_{2,h_2} \cap \Omega_2$, $\omega^-_{2,h_2} = \omega_{2,h_2} \cup \{a_2\}$ and $\omega^+_{2,h_2} = \omega_{2,h_2} \cup \{b_2\}$. Finally, in $[0, T]$ we introduce uniform mesh $\bar{\omega}_\tau$ with the step-size $\tau = T/n$ and set $\omega_\tau = \bar{\omega}_\tau \cap (0, T)$, $\omega^-_\tau = \omega_\tau \cup \{0\}$ and $\omega^+_\tau = \omega_\tau \cup \{T\}$. We will consider vector-functions of the form $v = (v_1, v_2)$ where v_i is mesh function defined on $\bar{\omega}_{i,h_i} \times \bar{\omega}_\tau$, $i = 1, 2$. We define finite differences in the usual way [**10**]

$$v_{i,x}(x,t) = \frac{v_i(x + h_i,\, t) - v_i(x,\, t)}{h_i} = v_{i,\bar{x}}(x + h_i,\, t),$$
$$v_{i,t}(x,t) = \frac{v_i(x,\, t + \tau) - v_i(x,\, t)}{\tau} = v_{i,\bar{t}}(x,\, t + \tau),$$
$$v_x = (v_{1,x}, v_{2,x}), \quad v_{\bar{x}} = (v_{1,\bar{x}}, v_{2,\bar{x}}), \quad v_t = (v_{1,t}, v_{2,t}), \quad v_{\bar{t}} = (v_{1,\bar{t}}, v_{2,\bar{t}}).$$

We also introduce the discrete inner products and norms

$$(v,w)_{L_h} = \beta_2 \sum_{x \in \omega^+_{1,h_1}} v_1 w_1 \hbar_1 + \beta_1 \sum_{x \in \omega^-_{2,h_2}} v_2 w_2 \hbar_2,$$
$$(v,w)_{L_{h'}} = \beta_2\, h_1 \sum_{x \in \omega^+_{1,h_1}} v_1 w_1 + \beta_1\, h_2 \sum_{x \in \omega^+_{2,h_2}} v_2 w_2,$$
$$\|v\|^2_{L_h} = (v,v)_{L_h}, \quad \|v\|^2_{L_{h'}} = (v,v)_{L_{h'}}, \quad \|v_i\|^2_{L_2(\omega^+_\tau)} = \tau \sum_{t \in \omega^+_\tau} v_i^2,$$
$$\|v\|^2_{L_2(\omega^+_\tau, L_h)} = \tau \sum_{t \in \omega^+_\tau} \|v(\cdot,t)\|^2_{L_h}, \quad \|v\|^2_{L_2(\omega^+_\tau, L_{h'})} = \tau \sum_{t \in \omega^+_\tau} \|v(\cdot,t)\|^2_{L_{h'}},$$
$$|v|^2_{H^1(\omega^+_\tau, L_h)} = \tau \sum_{t \in \omega^+_\tau} \|v_{\bar{t}}(\cdot,t)\|^2_{L_h},$$
$$\|v\|^2_{H^{2,1}_{h,\tau}} = \|v_{\bar{x}x}\|^2_{L_2(\omega^+_\tau, L_h)} + \|v_{\bar{x}}\|^2_{L_2(\omega^+_\tau, L_{h'})} + \|v\|^2_{L_2(\omega^+_\tau, L_h)} + |v|^2_{H^1(\omega^+_\tau, L_h)},$$

where

$$\hbar_i = h_i, \quad x \in \omega_{i,h_i}, \ \ i = 1, 2, \quad \hbar_1(b_1) = h_1/2, \quad \hbar_2(a_2) = h_2/2,$$
$$v_{1,\bar{x}x}(b_1,t) = \frac{-2}{h_1}\Big[v_{1,\bar{x}}(b_1,t) + \alpha_1^0 v_1(b_1,t) - \beta_1^0 v_2(a_2,t)\Big],$$
$$v_{2,\bar{x}x}(a_2,t) = \frac{2}{h_2}\Big[v_{2,x}(a_2,t) - \alpha_2^0 v_2(a_2,t) + \beta_2^0 v_1(b_1,t)\Big],$$

and

$$\alpha_1^0 = \frac{\alpha_1}{p_1(b_1)}, \quad \beta_1^0 = \frac{\beta_1}{p_1(b_1)}, \quad \alpha_2^0 = \frac{\alpha_2}{p_2(a_2)}, \quad \beta_2^0 = \frac{\beta_2}{p_2(a_2)}.$$

3.2. Finite difference scheme. In this and subsequent sections we shall assume that u_i belongs to $H^{4,2}(Q_i)$, while $p_i \in H^3(\Omega_i)$. Consequently, $f_i \in H^{2,1}(Q_i)$ is continuous function. We approximate the equations (1) and (2) in the following manner:

$$v_{1,\bar{t}} - (\bar{p}_1 v_{1,\bar{x}})_x = f_1, \qquad x \in \omega_{1,h_1}, \quad t \in \omega_\tau^+, \tag{16}$$

$$v_{2,\bar{t}} - (\bar{p}_2 v_{2,\bar{x}})_x = f_2, \qquad x \in \omega_{2,h_2}, \quad t \in \omega_\tau^+, \tag{17}$$

where $\bar{p}_i(x) = \frac{1}{2}\left[p_i(x) + p_i(x - h_i)\right]$, $i = 1, 2$. To ensure the same order of approximation for $x = b_1$ and $x = a_2$ we set:

$$\begin{aligned} v_{1,\bar{t}}(b_1,t) - \frac{h_1}{3}\, v_{1,\bar{t}\bar{x}}(b_1,t) + \frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\, v_{1,\bar{t}}(b_1,t) + \frac{2}{h_1}\Bigg\{ \bar{p}_1(b_1) v_{1,\bar{x}}(b_1,t) \\ + \left[1 + \frac{h_1^2}{12}\left(\frac{p_{1,\bar{x}}'(b_1)}{p_1(b_1)} + \frac{(p_1'(b_1))^2}{p_1^2(b_1)}\right)\right](\alpha_1 v_1(b_1,t) - \beta_1 v_2(a_2,t))\Bigg\} \\ = f_1(b_1,t) - \frac{h_1}{3}\, f_{1,\bar{x}}(b_1,t) + \frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\, f_1(b_1,t) \\ + \frac{2}{h_1}\left[1 + \frac{h_1^2}{12}\left(\frac{p_{1,\bar{x}}'(b_1)}{p_1(b_1)} + \frac{(p_1'(b_1))^2}{p_1^2(b_1)}\right)\right]\gamma_1(t), \quad t \in \omega_\tau^+, \end{aligned} \tag{18}$$

$$\begin{aligned} v_{2,\bar{t}}(a_2,t) + \frac{h_2}{3}\, v_{2,\bar{t}x}(a_2,t) - \frac{h_2}{6}\,\frac{p_2'(a_2)}{p_2(a_2)} v_{2,\bar{t}}(a_2,t) - \frac{2}{h_2}\Bigg\{ \bar{p}_2(a_2 + h_2) v_{2,x}(a_2,t) \\ - \left[1 + \frac{h_2^2}{12}\left(\frac{p_{2,x}'(a_2)}{p_2(a_2)} + \frac{(p_2'(a_2))^2}{p_2^2(a_2)}\right)\right](\alpha_2 v_2(a_2,t) - \beta_2 v_1(b_1,t))\Bigg\} \\ = f_2(a_2,t) + \frac{h_2}{3}\, f_{2,x}(a_2,t) - \frac{h_2}{6}\,\frac{p_2'(a_2)}{p_2(a_2)}\, f_2(a_2,t) \\ + \frac{2}{h_2}\left[1 + \frac{h_2^2}{12}\left(\frac{p_{2,x}'(a_2)}{p_2(a_2)} + \frac{(p_2'(a_2))^2}{p_2^2(a_2)}\right)\right]\gamma_2(t), \quad t \in \omega_\tau^+. \end{aligned} \tag{19}$$

The Dirichlet boundary conditions (5) and initial conditions (6) can be satisfied exactly:

$$v_1(a_1,t) = 0, \qquad v_2(b_2,t) = 0, \qquad t \in \omega_\tau^+, \tag{20}$$

$$v_i(x,0) = u_{i0}(x), \qquad x \in \bar{\omega}_{i,h_i}, \qquad i = 1, 2. \tag{21}$$

In each time level $t = j\tau$ FDS (16)–(21) reduces to a tridiagonal linear system with $n_1 + n_2$ unknowns. In such a way, FDS (16)–(21) is computationally efficient. From the general theory of difference schemes [**10**] it follows that FDS (16)–(21) is unconditionally stable.

3.3. Convergence of the finite difference scheme. Let $u = (u_1, u_2)$ be the solution of the IBVP (1)–(6) and $v = (v_1, v_2)$ the solution of the FDS (16)–(21). Then the error $z = u - v$ satisfies the following FDS:

$$z_{1,\bar{t}} - (\bar{p}_1 z_{1,\bar{x}})_x = \varphi_1, \qquad x \in \omega_{1,h_1}, \quad t \in \omega_\tau^+, \tag{22}$$

$$z_{2,\bar{t}} - (\bar{p}_2 z_{2,\bar{x}})_x = \varphi_2, \qquad x \in \omega_{2,h_2}, \quad t \in \omega_\tau^+, \tag{23}$$

$$(24)\quad z_{1,\bar t}(b_1,t)-\frac{h_1}{3}\,z_{1,\bar t\bar x}(b_1,t)+\frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\,z_{1,\bar t}(b_1,t)+\frac{2}{h_1}\Big\{\bar p_1(b_1)z_{1,\bar x}(b_1,t)$$
$$+\Big[1+\frac{h_1^2}{12}\Big(\frac{p_{1,\bar x}'(b_1)}{p_1(b_1)}+\frac{(p_1'(b_1))^2}{p_1^2(b_1)}\Big)\Big](\alpha_1 z_1(b_1,t)-\beta_1 z_2(a_2,t))\Big\}=\varphi_1(b_1,t),\quad t\in\omega_\tau^+,$$

$$(25)\quad z_{2,\bar t}(a_2,t)+\frac{h_2}{3}\,z_{2,\bar t x}(a_2,t)-\frac{h_2}{6}\frac{p_2'(a_2)}{p_2(a_2)}z_{2,\bar t}(a_2,t)-\frac{2}{h_2}\Big\{\bar p_2(a_2+h_2)z_{2,x}(a_2,t)$$
$$-\Big[1+\frac{h_2^2}{12}\Big(\frac{p_{2,x}'(a_2)}{p_2(a_2)}+\frac{(p_2'(a_2))^2}{p_2^2(a_2)}\Big)\Big](\alpha_2 z_2(a_2,t)-\beta_2 z_1(b_1,t))\Big\}=\varphi_2(a_2,t),\quad t\in\omega_\tau^+,$$

$$(26)\qquad z_1(a_1,t)=0,\qquad z_2(b_2,t)=0,\qquad t\in\omega_\tau^+,$$

$$(27)\qquad z_i(x,0)=0,\qquad x\in\bar\omega_{i,h_i},\qquad i=1,2,$$

where

$$\varphi_i=\psi_i+\chi_i\,,\quad x\in\omega_{i,h_i},\quad t\in\omega_\tau^+,\quad i=1,2,$$
$$\varphi_1(b_1,t)=\psi_1(b_1,t)-\frac{h_1}{3}\,\psi_{1,\bar x}(b_1,t)+\frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\,\psi_1(b_1,t)+\widetilde\chi_1(b_1,t)\,,$$
$$\varphi_2(a_2,t)=\psi_2(a_2,t)+\frac{h_2}{3}\,\psi_{2,x}(a_2,t)-\frac{h_2}{6}\,\frac{p_2'(a_2)}{p_2(a_2)}\,\psi_2(a_2,t)+\widetilde\chi_2(a_2,t)\,,$$
$$\psi_i=u_{i,\bar t}-\frac{\partial u_i}{\partial t}\,,\qquad \chi_i=\frac{\partial}{\partial x}\Big(p_i\,\frac{\partial u_i}{\partial x}\Big)-(\bar p_i\,u_{i,\bar x})_x,$$
$$\widetilde\chi_1=\frac{\partial}{\partial x}\Big(p_1\,\frac{\partial u_1}{\partial x}\Big)+\frac{2}{h_1}\Big(\bar p_1 u_{1,\bar x}-p_1\,\frac{\partial u_1}{\partial x}\Big)$$
$$+\frac{h_1}{6}\Big(p_1'\,\frac{\partial^2 u_1}{\partial x^2}-p_{1,\bar x}'\,\frac{\partial u_1}{\partial x}\Big)-\frac{h_1}{3}\Big[\frac{\partial}{\partial x}\Big(p_1\,\frac{\partial u_1}{\partial x}\Big)\Big]_{\bar x},$$
$$\widetilde\chi_2=\frac{\partial}{\partial x}\Big(p_2\,\frac{\partial u_2}{\partial x}\Big)-\frac{2}{h_2}\Big(\bar p_2(x+h_2)u_{2,x}-p_2\,\frac{\partial u_2}{\partial x}\Big)$$
$$-\frac{h_2}{6}\Big(p_2'\,\frac{\partial^2 u_2}{\partial x^2}-p_{2,x}'\,\frac{\partial u_2}{\partial x}\Big)+\frac{h_2}{3}\Big[\frac{\partial}{\partial x}\Big(p_2\,\frac{\partial u_2}{\partial x}\Big)\Big]_x.$$

Analogously as in the continuous case one obtains the next assertion.

THEOREM 3.1. *Let the assumptions* (7) *and* (8) *hold and let* $p_i\in C^1(\bar\Omega_i)$, *for* $i=1,2$. *Then, for sufficiently small* h_1 *and* h_2, *the solution* z *of FDS* (22)–(27) *satisfies the a priori estimate*

$$(28)\qquad \|z\|_{H^{2,1}_{h,\tau}}\leqslant C\|\varphi\|_{L_2(\omega_\tau^+,L_h)}.$$

Therefore, in order to determine the convergence rate of the FDS (16)–(21), it is enough to estimate the right hand side term in the inequality (28).

From the integral representation

$$\psi_1(x,t)=\psi_{10}(x,t)+\psi_{11}(x,t)=-\frac{1}{\tau h_1}\int\limits_{t-\tau}^{t}\int\limits_{t'}^{t}\int\limits_{x-h_1}^{x}\frac{\partial^2 u_1}{\partial t^2}(x',t'')\,dx'\,dt''\,dt'$$

$$+\frac{1}{\tau h_1}\int\limits_{t-\tau}^{t}\int\limits_{x-h_1}^{x}\left[\frac{\partial u_1}{\partial t}(x,t')-\frac{\partial u_1}{\partial t}(x,t)-\frac{\partial u_1}{\partial t}(x',t')+\frac{\partial u_1}{\partial t}(x',t)\right]dx'\,dt'$$

we immediately obtain

$$|\psi_{10}(x,t)|\leqslant\frac{\tau}{\tau h_1}\int\limits_{t-\tau}^{t}\int\limits_{x-h_1}^{x}\left|\frac{\partial^2 u_1}{\partial t^2}(x',t')\right|dx'\,dt'\leqslant\frac{\tau}{\sqrt{\tau h_1}}\left\|\frac{\partial^2 u_1}{\partial t^2}\right\|_{L_2(e(x,t))},$$

where $e(x,t)=(x-h_1,x)\times(t-\tau,t)$. Summation over the mesh yields

$$\tau\sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}^+}\hbar_1|\psi_{10}|^2\leqslant C\tau^2\left\|\frac{\partial^2 u_1}{\partial t^2}\right\|^2_{L_2(Q_1)}\leqslant C\tau^2\|u_1\|^2_{H^{4,2}(Q_1)}. \tag{29}$$

The term of the same form as ψ_{11} is estimated in [**5**] wherefrom follows

$$\tau\sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}^+}\hbar_1|\psi_{11}|^2\leqslant C(h_1^4+\tau^2)\|u_1\|^2_{H^{4,2}(Q_1)}. \tag{30}$$

For $x\in\omega_{1,h_1}$, $t\in\omega_\tau^+$ term χ_1 can be represented in the following way

$$\begin{aligned}
\chi_1(x,t)&=\chi_{10}(x,t)+\chi_{11}(x,t)+\chi_{12}(x,t)+\chi_{13}(x,t)+\chi_{14}(x,t)\\
&=-\frac{p_1(x)}{\tau h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{x'}^{x}\int\limits_{x''}^{x}\int\limits_{t-\tau}^{t}\Big(1-\frac{|x'-x|}{h_1}\Big)\frac{\partial^4 u_1}{\partial x^4}(x''',t')\,dt'\,dx'''\,dx''\,dx'\\
&\quad+\frac{p_1(x)}{\tau h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{t-\tau}^{t}\Big(1-\frac{|x'-x|}{h_1}\Big)\left[\frac{\partial^2 u_1}{\partial x^2}(x,t)-\frac{\partial^2 u_1}{\partial x^2}(x',t)\right.\\
&\qquad\qquad\left.-\frac{\partial^2 u_1}{\partial x^2}(x,t')+\frac{\partial^2 u_1}{\partial x^2}(x',t')\right]dt'\,dx'\\
&\quad-\frac{p_1'(x)}{2h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{x'}^{x}\int\limits_{x''}^{x}\frac{\partial^3 u_1}{\partial x^3}(x''',t)\,dx'''\,dx''\,dx'\\
&\quad-\frac{h_1}{4}\,p_1''(x)\int\limits_{x-h_1}^{x+h_1}\Big(1-\frac{|x'-x|}{h_1}\Big)\frac{\partial^2 u_1}{\partial x^2}(x',t)\,dx'\\
&\quad-\frac{1}{2h_1^2}\left[\left(\int\limits_{x}^{x+h_1}\int\limits_{x}^{x'}\int\limits_{x}^{x''}p_1'''(x''')\,dx'''\,dx''\,dx'\right)\left(\int\limits_{x}^{x+h_1}\frac{\partial u_1}{\partial x}(x',t)\,dx'\right)\right.\\
&\qquad\left.+\left(\int\limits_{x-h_1}^{x}\int\limits_{x'}^{x}\int\limits_{x''}^{x}p_1'''(x''')\,dx'''\,dx''\,dx'\right)\left(\int\limits_{x-h_1}^{x}\frac{\partial u_1}{\partial x}(x',t)\,dx'\right)\right].
\end{aligned}$$

Summands χ_{10} and χ_{11} can be estimated in the same manner as ψ_{10} and ψ_{11}:

$$h_1\tau \sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}} |\chi_{10}|^2 \leqslant Ch_1^4 \|p_1\|^2_{C(\bar\Omega_1)} \left\| \frac{\partial^4 u_1}{\partial x^4}\right\|^2_{L_2(Q_1)} \leqslant Ch_1^4 \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{H^{4,2}(Q_1)}, \tag{31}$$

$$h_1\tau \sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}} |\chi_{11}|^2 \leqslant C(h_1^4+\tau^2) \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{H^{4,2}(Q_1)}. \tag{32}$$

Remaining terms can be estimated directly using Cauchy–Schwartz inequality and Sobolev imbedding theorem:

$$h_1\tau \sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}} |\chi_{12}|^2 \leqslant Ch_1^4 \|p_1\|^2_{C^1(\bar\Omega_1)} \max_{t\in[0,T]} \|u_1(\cdot,t)\|^2_{H^3(\Omega_1)} \leqslant Ch_1^4 \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{H^{4,2}(Q_1)}, \tag{33}$$

$$h_1\tau \sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}} |\chi_{13}|^2 \leqslant Ch_1^4 \|p_1\|^2_{C^2(\bar\Omega_1)} \|u_1\|^2_{C^2(\bar Q_1)} \leqslant Ch_1^4 \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{H^{4,2}(Q_1)}, \tag{34}$$

$$h_1\tau \sum_{t\in\omega_\tau^+}\sum_{x\in\omega_{1,h_1}} |\chi_{14}|^2 \leqslant Ch_1^4 \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{C^1(\bar Q_1)} \leqslant Ch_1^4 \|p_1\|^2_{H^3(\Omega_1)} \|u_1\|^2_{H^{4,2}(Q_1)}. \tag{35}$$

Similarly, term $\widetilde{\chi}_1$ can be represented in the following manner

$$\begin{aligned}
&\widetilde{\chi}_1(b_1,t) = \widetilde{\chi}_{10}(b_1,t) + \widetilde{\chi}_{11}(b_1,t) + \widetilde{\chi}_{12}(b_1,t) + \widetilde{\chi}_{13}(b_1,t) + \chi_{14}(b_1,t)\\
&= \frac{2p_1(b_1)}{\tau h_1^3} \int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x''''}^{x'''}\int\limits_{t-\tau}^{t} \frac{\partial^4 u_1}{\partial x^4}(x'''',t')\,dt'\,dx''''dx''''dx'''\,dx''\,dx'\\
&+ \frac{2p_1(b_1)}{\tau h_1^2}\Bigg\{ \frac{2}{3}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{t-\tau}^{t}\Bigg[\frac{\partial^2 u_1}{\partial x^2}(b_1,t) - \frac{\partial^2 u_1}{\partial x^2}(x'',t)\\
&\qquad - \frac{\partial^2 u_1}{\partial x^2}(b_1,t') + \frac{\partial^2 u_1}{\partial x^2}(x'',t')\Bigg]dt'dx''dx'\\
&\qquad + \frac{1}{3}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{t-\tau}^{t}\Bigg[\frac{\partial^2 u_1}{\partial x^2}(b_1-h_1,t) - \frac{\partial^2 u_1}{\partial x^2}(x'',t)\\
&\qquad - \frac{\partial^2 u_1}{\partial x^2}(b_1-h_1,t') + \frac{\partial^2 u_1}{\partial x^2}(x'',t')\Bigg]dt'dx''dx'\Bigg\}\\
&+ \Bigg\{\frac{1}{3}\left(\int\limits_{b_1-h_1}^{b_1} p_1'(x')\,dx'\right)\left(\int\limits_{b_1-h_1}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x',t)\,dx'\right)\\
&\quad + \frac{p_1'(b_1-h_1)}{3}\left(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x'',t)\,dx''dx'\right)
\end{aligned}$$

$$- \frac{p_1'(b_1)}{h_1}\left(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x''',t)\,dx'''\,dx''\,dx'\right)\Bigg\}$$

$$+\left\{\left(\frac{h_1}{3}\int\limits_{b_1-h_1}^{b_1}p_1''(x')\,dx' + \frac{1}{3}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}p_1''(x'')\,dx''dx'\right)\frac{\partial^2 u_1}{\partial x^2}(b_1,t)\right.$$

$$\left.- \frac{1}{h_1^2}\left(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}p_1''(x'')\,dx''dx'\right)\left(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\frac{\partial^2 u_1}{\partial x^2}(x'',t)\,dx''\,dx'\right)\right\}$$

$$+\frac{1}{h_1^2}\left(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'''}^{x''}p_1'''(x'''')\,dx''''\,dx'''\,dx''\,dx'\right)\frac{\partial u_1}{\partial x}(b_1,t).$$

Analogously as in the previous case one obtains:

$$h_1\tau\sum_{t\in\omega_\tau^+}|\widetilde{\chi}_{11}(b_1,t)|^2 \leqslant C(h_1^4+\tau^2)\,\|p_1\|^2_{H^3(\Omega_1)}\,\|u_1\|^2_{H^{4,2}(Q_1)}, \tag{36}$$

$$h_1\tau\sum_{t\in\omega_\tau^+}|\widetilde{\chi}_{1j}(b_1,t)|^2 \leqslant Ch_1^4\,\|p_1\|^2_{H^3(\Omega_1)}\,\|u_1\|^2_{H^{4,2}(Q_1)}, \quad j=0,2,3,4. \tag{37}$$

From (29)–(37), analogous inequalities for ψ_2, χ_2 and $\widetilde{\chi}_2$ and the a priori estimate (28) one obtains the next result.

THEOREM 3.2. *Let $p_i \in H^3(\Omega_i)$, $i=1,2$, and let assumptions* (7) *and* (8) *hold. Let the functions γ_i and f_i be continuous and sufficiently smooth to ensure that the solution of IBVP* (1)–(6) *belongs to the space $H^{4,2}$. Let also the step sizes h_1 and h_2 be sufficiently small so that the a priori estimate* (28) *holds. Then the solution v of FDS* (16)–(21) *converges to the solution u of IBVP* (1)–(6) *in $H^{2,1}_{h,\tau}$ and the following convergence rate estimate holds true:*

$$\|u-v\|_{H^{2,1}_{h,\tau}} \leqslant C(h^2+\tau)\Big(1+\max_i\|p_i\|_{H^3(\Omega_i)}\Big)\|u\|_{H^{4,2}}, \quad h=\max\{h_1,h_2\}.$$

REMARK 3.1. For $u \in H^{s,s/2}$, $3.5 < s < 4$, using Bramble–Hilbert lemma [**1, 3**] and methodology proposed in [**4**], one can obtain convergence rate $O(h^{s-2})$ assuming $\tau \asymp h^2$. The same result holds for $2 < s < 3.5$, but in this case a FDS with averaged data must be used, because the right-hand sides f_i may be discontinuous functions.

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