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A NOTE ON SHOCK PROFILES IN DISSIPATIVE HYPERBOLIC AND PARABOLIC MODELS

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Abstract. This note presents a comparative study of shock profiles in dissipative systems. Main assumption is that both hyperbolic and parabolic model are reducible to the same underlying equilibrium system when dissipative effects are neglected. It will be shown that the highest characteristic speed of equilibrium system determines the critical value of the shock speed for which downstream equilibrium state bifurcates. It will be also shown that it obeys the same transcritical bifurcation pattern in hyperbolic, as well as in parabolic case.

1. Introduction and preliminaries

Mathematical models of dynamical processes in continuous media may have different structure and may posses different degrees of complexity. Their main ingredients are conservation laws of continuum physics adjoined with constitutive relations which describe material response. Structure of the model mainly depends on assumptions used in building up constitutive relations. Typical outcomes are hyperbolic and parabolic PDE's.

Complete models of physical phenomena could be rather complicated. Therefore, analysis may be pursued in a different direction—development of so-called model equations which have simpler form, but capture all important qualitative features of the complete model.

The purpose of this paper is to analyze common properties of shock profiles which appear both in parabolic and hyperbolic model equations. To motivate this study some preliminary assumptions will be established first. Models which will be dealt with will be confined to one space dimension, say x, without loss of generality. The basic system—we call it the equilibrium system in the remainder of the paper—will be the system of hyperbolic conservation laws

$$\partial_t u + \partial_x F(u) = 0. \tag{1.1}$$

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Here $u \in \mathbf{R}^n$ are state variables and $F(u)$ is smooth vector-function in an open set of the state space. The system (1.1) is hyperbolic in the sense that the eigenvalues $\lambda_j(u)$, $j = 1, \dots, n$ of the matrix $A(u) = \mathrm{grad} F(u)$ are real, thus forming a set of characteristic speeds. For convenience it will be assumed that the system (1.1) is strictly hyperbolic, i.e., that characteristic speeds are distinct with ordering $\lambda_1(u) < \lambda_2(u) < \dots < \lambda_{n-1}(u) < \lambda_n(u)$. It is well known that (1.1) admits piecewise smooth weak solutions with jump discontinuities—the shocks—which can evolve even from smooth initial data [**7, 12, 14**]. These jumps are localized on the lines of discontinuity—shock fronts—and satisfy Rankine–Hugoniot condition

$$s[u] = [F(u)] \tag{1.2}$$

where $[(\cdot)] = (\cdot)_+ - (\cdot)_-$ denotes the difference of variable in front of and behind the discontinuity, and s is the speed of shock. In addition to (1.2) physically relevant shocks ought to satisfy a supplementary selection rule. There are several such rules in the theory of hyperbolic systems. Here, we shall consider probably the simplest one—Lax condition [**9**]—which reads $\lambda_{j-1}(u_-) < s < \lambda_j(u_-)$, $\lambda_j(u_+) < s < \lambda_{j+1}(u_+)$ for some $1 \leqslant j \leqslant n$. From these two inequalities the following simple rule may be extracted

$$\lambda_j(u_+) < s < \lambda_j(u_-), \tag{1.3}$$

and it is said that u_- and u_+ are connected by a $j-$shock. In the sequel our attention will be focused on the simplest discontinuous solution of (1.1)

$$u(x,t) = \begin{cases} u_-, & x < st; \\ u_+, & x > st, \end{cases}$$

u_- and u_+ being constant states.

Given a state u_-, all the states u which can be connected to u_- by $j-$shock form a one-parameter family of states satisfying Rankine–Hugoniot conditions (1.2), which can be written in the form $\Psi(u) = \Psi(u_-)$ where $\Psi(u) = -su + F(u)$. This set of equations have trivial solution $u = u_-$, while non-trivial one may exist as a bifurcating solution for a certain value of shock speed. Existence of non-trivial solutions is thus related to non-uniqueness of solutions of Rankine–Hugoniot equations. Therefore, $\Psi(u)$ has to be locally non-invertible and $\det(-sI + A(u)) = 0$. This leads to a conclusion that the state u connected to u_- by a $j-$shock bifurcates from u_- in the neighborhood of the critical value of shock speed $s^* = \lambda_j(u_-)$.

Hyperbolic systems in conservative form usually do not take into account dissipative effects. Starting from (1.1) dissipation can be introduced by non-local constitutive equations, like Navier–Stokes and Fourier ones in continuum theory of fluids, arriving to a model

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x (B(u) \partial_x u), \tag{1.4}$$

where $B(u)$ is so-called viscosity matrix and ε positive parameter. Under certain conditions on $B(u)$, see Majda and Pego [**10**] for a comprehensive account, this model is parabolic and predicts infinite speed of propagation of disturbances. On the other hand, dissipation smears out jump discontinuity and transform it into a continuous travelling profile which connects two constant equilibrium states u_- and u_+. Viscous shock profile is determined by a set of ODE's

$$B(u)\dot{u} = F(u) - F(u_-) - s(u - u_-), \tag{1.5}$$

where an overdot denotes differentiation with respect to $\xi = (x-st)/\varepsilon$. It represents a heteroclinic orbit connecting two stationary points of the system (1.5), u_- and u_+, which satisfy Rankine–Hugoniot conditions (1.2) of the equilibrium system. Since right-hand side of (1.5) has the form $\Psi(u) - \Psi(u_-)$ equivalent to Rankine–Hugoniot equations, stationary point u_+ may be regarded as a bifurcating solution of the ODE system with the same critical value of the shock speed, $s^*_{\mathrm{par}} = s^*$. More involved problem of bifurcation of non-classical viscous shock profiles has recently been discussed by Azevedo et al. [**1**].

Another way of description of dissipative mechanisms emerge by taking into account relaxation effects. As a result, an extended system of balance laws is obtained

$$\partial_t U + \partial_x \hat{F}(U) = \frac{1}{\varepsilon} Q(U), \tag{1.6}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \quad \hat{F}(U) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}, \quad Q(U) = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix},$$

$v \in \mathbf{R}^k$, $n + k = N$. In physical examples v variables usually represent non-convective fluxes, like stress tensor and heat flux, whose evolution is determined by balance laws with non-zero source terms rather than by constitutive equations. It is assumed that $q(u,v) = 0$ uniquely determines "equilibrium manifold" $v_E = h(u)$ as $\varepsilon \to 0$, on which the system (1.6) reduces to the equilibrium system (1.1) with $F(u) = f(u, h(u))$. In (1.6) the first n equations are conservation laws, while k remaining ones contain source terms $q(u,v)$ which describe dissipative effects off the equilibrium manifold.

Two remarks are in order for the dissipative system (1.6). First, it will be assumed that its differential part is hyperbolic with characteristic speeds $\Lambda_i(U)$, $i = 1, \dots, N$ determined as eigenvalues of the matrix $\hat{A}(U) = \mathrm{grad}\hat{F}(U)$. This assumption is tightly related to the so-called sub-characteristic condition which impose bounds for characteristic speeds of equilibrium system

$$\min_{1 \leqslant i \leqslant N} \Lambda_i(u, h(u)) \leqslant \lambda_j(u) \leqslant \max_{1 \leqslant i \leqslant N} \Lambda_i(u, h(u)), \tag{1.7}$$

for every j, $1 \leqslant j \leqslant n$. Moreover, the spectrum $\lambda_j(u)$ of the equilibrium system does not have to be contained in the spectrum $\Lambda_i(u, h(u))$ of the hyperbolic dissipative system (1.6). Consequently, jump discontinuities may appear as bifurcating solutions for the shock speeds which do not coincide with the ones of the system (1.1). Important role of sub-characteristic condition has already been recognized by Whitham [**14**], while more detailed study of the problem has recently been done by Chen, Levermore and Liu [**5**] and Boillat and Ruggeri [**2**]. One may also consult [**11**] for a nice account on other aspects of hyperbolic systems with relaxation.

Second remark is concerned with relation between parabolic and hyperbolic systems (1.4) and (1.6). It will be assumed that the hyperbolic system can be reduced to a parabolic one by means of asymptotic expansion in the spirit of Chapman–Enskog, originally used in kinetic theory for derivation of hydrodynamic equations, see Chapman and Cowling [**4**]. In such a way parabolic system (1.4) determines the effective response of relaxation process in the neighborhood of equilibrium manifold.

This leads to the main issue of the study: in which way does the dissipation in the system (1.6) affect the jump discontinuity of the equilibrium system (1.1)? Yong and Zumbrun [**15**] gave the answer to the question of existence of relaxation shock profile following the traces of Majda and Pego [**10**]. In this paper the answer will be sketched through stability and bifurcation analysis of equilibrium state. Using Burgers' equation and a model of isothermal viscoelasticity as persuasive examples two results will be presented. First, it will be shown that hyperbolic dissipative system smears out the jump discontinuity of small strength—so-called weak shock—related to the highest characteristic speed of equilibrium sytem, i.e., $n-$shock, in the same way as parabolic model does. This result will come from comparative study of the ODE system (1.5) and the system

$$-s\dot{U} + \frac{d}{d\xi}\hat{F}(U) = Q(U)$$

obtained from hyperbolic model (1.6) for a travelling profile $U = U(\xi)$, $\xi = (x - st)/\varepsilon$. Relaxation shock profile connects two stationary points $U_- = (u_-, h(u_-))$ and $U_+ = (u_+, h(u_+))$, where U_+ may be regarded as bifurcating solution for the critical value of the shock speed $s^*_{\text{hyp}} = s^* = \lambda_n(u_-)$. Second, it will be shown that bifurcating solutions of parabolic and hyperbolic dissipative models obey the same transcritical bifurcation pattern.

2. Burgers' equation

Comparative analysis of hyperbolic and parabolic models will commence with Burgers' equation. Its non-dissipative hyperbolic version reads

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0. \tag{2.1}$$

Since $F(u) = u^2/2$, characteristic speed is $\lambda(u) = F'(u) = u$. Corresponding Rankine–Hugoniot condition yields the following solutions

$$-s[u] + [F(u)] = 0 \quad \Rightarrow \quad u_+ = u_- \quad \text{or} \quad u_+ = 2s - u_-, \tag{2.2}$$

where u_- and u_+ are respectively downstream and upstream equilibrium states. Nontrivial solution $u_+ \neq u_-$ bifurcates from trivial one for the critical value of shock speed $s^* = u_-$. The admissibility of shock can be simply determined by means of Lax condition (1.3) which in this case reads

$$\lambda(u_+) = u_+ < s < u_- = \lambda(u_-).$$

For a pair of equilibrium states u_- and u_+ determined by Rankine–Hugoniot equation and admitted by Lax condition, shock speed is $s = (u_- + u_+)/2$.

Burgers' equation can be regularized as classical parabolic equation

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = \varepsilon \partial_{xx} u. \tag{2.3}$$

where diffusive term $\varepsilon\partial_{xx}u$ takes into account dissipation in the system. Assume the solution of equation (2.3) in the form of travelling wave $u = u(\xi)$, $\xi = (x-st)/\varepsilon$, which reduces the original PDE into a second order ODE

$$-s\dot{u} + \frac{d}{d\xi}\left(\frac{1}{2}u^2\right) = \ddot{u}.$$

It can be integrated from $-\infty$ to ξ to obtain

$$\dot{u} = -s(u - u_-) + \frac{1}{2}(u^2 - u_-^2) = \varphi(u, s). \tag{2.4}$$

where boundary conditions $u(-\infty) = u_-$, $\dot{u}(-\infty) = 0$ were used, expressing the fact that u_- is equilibrium state and $\varphi(u_-, s) \equiv 0$ for any s. Moreover, upstream equilibrium u_+ determined by Rankine–Hugoniot condition is also a stationary point of dynamical equation (2.4) for a shock speed $s = (u_- + u_+)/2$. In other words, shock profile represents a heteroclinic orbit of equation (2.4) connecting two equilibrium states.

The solution of parabolic profile equation (2.4) is well known, but our intention is to perform stability and bifurcation analysis. Stability of stationary point u_- may simply be studied by means of linear stability analysis. To that end linear variational equation corresponding to (2.4) may be derived

$$\dot{y} = \Phi(u_-, s)y; \quad \Phi(u_-, s) = \varphi_u(u_-, s) = -(s - u_-),$$

where $y = u - u_-$ is perturbation. Stationary point is stable for $\Phi(u_-, s) < 0$ and unstable for $\Phi(u_-, s) > 0$. Critical value of shock speed is reached for $\Phi(u_-, s^*_{\text{par}}) = 0$ and exchange of stability occurs when $\Phi_s(u_-, s^*_{\text{par}}) \neq 0$. Both these conditions are satisfied for $s^*_{\text{par}} = s^* = u_-$, in particular $\Phi_s(u_-, s^*_{\text{par}}) = -1$ and downstream equilibrium state has the following stability properties

$$\begin{aligned} s < s^*_{\text{par}} \quad &\Rightarrow \quad u_-\text{-unstable;} \\ s > s^*_{\text{par}} \quad &\Rightarrow \quad u_-\text{-stable.} \end{aligned}$$

It is important to note that critical value s^*_{par} of shock speed s, obtained through stability analysis of equation (2.4), coincides with the value s^* for which uniqueness of solution of Rankine–Hugoniot equation (2.2) is lost. This fact motivates the analysis of bifurcation pattern in the neighborhood of bifurcation point $(u, s) = (u_-, s^*_{\text{par}})$. To achieve this goal it is sufficient to calculate all the partial derivatives up to second order (for a detailed explanation see Guckenheimer and Holmes [**8**])

$$\begin{aligned} \varphi(u_-, s^*_{\text{par}}) = 0, \quad \varphi_u(u_-, s^*_{\text{par}}) &= 0, \quad \varphi_{uu}(u_-, s^*_{\text{par}}) = 1, \\ \varphi_s(u_-, s^*_{\text{par}}) &= 0, \quad \varphi_{us}(u_-, s^*_{\text{par}}) = -1, \\ &\qquad \varphi_{ss}(u_-, s^*_{\text{par}}) = 0. \end{aligned}$$

Denoting bifurcation parameter as $\mu = s - s^*_{\text{par}}$, a bifurcation equation in normal form is obtained

$$\dot{y} \approx \frac{1}{2}\left(-\mu y + y^2\right), \tag{2.5}$$

describing the transcritical bifurcation pattern.

Second part of our analysis is devoted to dissipative hyperbolic system related to Burgers' equation (2.1). Namely, one may introduce so-called non-equilibrium variable v with intention to describe processes far from equilibrium state or ones in which large gradients of state variable u occur. Evolution of v is determined by an additional balance law, but it may also appear in basic conservation law as follows

$$\begin{aligned} \partial_t u + \partial_x\left(\frac{1}{2}u^2 + v\right) &= 0; \\ \partial_t v + \partial_x u &= -\frac{1}{\varepsilon}\, v. \end{aligned} \tag{2.6}$$

Equation $(2.6)_2$ determines an equilibrium value of v as $\varepsilon \to 0$, i.e., $v_E = 0$. In such a way $(2.6)_1$ reduces to an equilibrium system which is exactly non-dissipative equation (2.1).

On the other hand, hyperbolic system (2.6) may be analyzed *per se* and characteristic speeds determined from its differential part are

$$\Lambda_1(u) = \frac{1}{2}\left(u - \sqrt{u^2+4}\right), \quad \Lambda_2(u) = \frac{1}{2}\left(u + \sqrt{u^2+4}\right).$$

They satisfy sub-characteristic conditions for any u

$$\Lambda_1(u) < \lambda(u) < \Lambda_2(u).$$

In order to identify effective response of the relaxation process in the course of approaching equilibrium state a Chapman–Enskog-like expansion will be performed. Non-equilibrium variable may be expressed as

$$v^\varepsilon = v_E + \varepsilon S(u^\varepsilon, \partial_x u^\varepsilon, ...) + O(\varepsilon^2),$$

where function S is to be determined. Putting this expression in $(2.6)_2$ and retaining terms of the lowest order in ε one easily obtains $S = -\partial_x u^\varepsilon$, i.e., $v^\varepsilon = -\varepsilon\partial_x u^\varepsilon$. With this expansion formula equation $(2.6)_1$ reduces to a parabolic model (2.3)

$$\partial_t u^\varepsilon + \partial_x\left(\frac{1}{2}(u^\varepsilon)^2\right) = \varepsilon\partial_{xx}u^\varepsilon.$$

In this sense hyperbolic system (2.6) predicts the same response as parabolic one in the neighborhood of equilibrium manifold.

Since both assumptions about hyperbolic system are satisfied we may proceed with the analysis of shock profile determined by (2.6). Assuming the solution in the form of travelling wave, $u = u(\xi)$, $v = v(\xi)$, $\xi = (x - st)/\varepsilon$, equation $(2.6)_1$ may be integrated to obtain

$$v = s(u - u_-) - \frac{1}{2}\left(u^2 - u_-^2\right),$$

where equilibrium boundary data have been used, $u(-\infty) = u_-$, $v(-\infty) = 0$. Using this result $(2.6)_2$ reduces to a single ODE for a shock profile

$$\dot{u} = -\frac{-2su + u^2 + 2su_- - u_-^2}{2\left(-1 + s^2 - su\right)} = \theta(u, s). \tag{2.7}$$

It is easy to show that non-trivial solution $u_+ = 2s - u_-$ of Rankine–Hugoniot equation (2.2) solves $\theta(u_+, s) = 0$ meaning that u_+ is also a stationary point of (2.7).

Like in parabolic case we may perform stability and bifurcation analysis of stationary point u_-. Linear variational equation corresponding to (2.7) reads

$$\dot{z} = \Theta(u_-, s)z; \quad \Theta(u_-, s) = \theta_u(u_-, s) = \frac{s - u_-}{-1 + s^2 - su_-},$$

where $z = u - u_-$ is perturbation. Critical value of shock speed is obtained as solution of $\Theta(u_-, s^*_{\text{hyp}}) = 0$, i.e., $s^*_{\text{hyp}} = s^* = u_-$, and exchange of stability occurs since $\Theta_s(u_-, s^*_{\text{hyp}}) = -1 \neq 0$. This confirms our first conjecture that downstream equilibrium state of (2.7) changes its stability when the speed of profile coincides with the highest characteristic speed of an equilibrium system. Moreover, stability properties are changed in the same way as in the parabolic case

$$\begin{aligned} s < s^*_{\text{hyp}} \quad &\Rightarrow \quad u_-\text{-unstable;} \\ s > s^*_{\text{hyp}} \quad &\Rightarrow \quad u_-\text{-stable.} \end{aligned}$$

To determine the bifurcation pattern which occurs during the exchange of stability partial derivatives up to second order will be calculated

$$\begin{aligned} \theta(u_-, s^*_{\text{hyp}}) = 0, \quad \theta_u(u_-, s^*_{\text{hyp}}) &= 0, \quad \theta_{uu}(u_-, s^*_{\text{hyp}}) = 1, \\ \theta_s(u_-, s^*_{\text{hyp}}) &= 0, \quad \theta_{us}(u_-, s^*_{\text{hyp}}) = -1, \\ & \quad\ \ \theta_{ss}(u_-, s^*_{\text{hyp}}) = 0. \end{aligned}$$

If bifurcation parameter is denoted as $\mu = s - s^*_{\text{hyp}}$, a bifurcation equation is obtained in normal form

$$\dot{z} \approx \frac{1}{2}\left(-\mu z + z^2\right),$$

which has the same form as (2.5). This is to confirm our second conjecture: bifurcation pattern of downstream equilibrium state is the same in hyperbolic and parabolic models related to the same equilibrium system and occurs for the same same critical value of bifurcation parameter—shock speed s—coinciding with the characteristic speed of equilibrium system.

3. Isothermal viscoelasticity

In this section stability and bifurcation analysis will be performed for a model which primarily arises in isothermal elastodynamics, so-called p-system

$$\begin{aligned} \partial_t u^1 - \partial_x u^2 &= 0; \\ \partial_t u^2 - \partial_x p(u^1) &= 0, \end{aligned} \tag{3.1}$$

It is hyperbolic provided $p'(u^1) > 0$ with characteristic speeds

$$\lambda_1(u) = -\sqrt{p'(u^1)}, \quad \lambda_2(u) = \sqrt{p'(u^1)}. \tag{3.2}$$

for $u = (u^1, u^2)^T$. Furthermore, we shall assume that eigenvalues (3.2) are genuinely nonlinear, which implies $p''(u^1) \neq 0$ for all u^1. Our analysis will be focused on weak shocks which bifurcate from u_- in the neighborhood of the highest characteristic speed, i.e., when $s \to s^* = \lambda_2(u_-)$ and satisfy Lax condition $\lambda_2(u_+) < s < \lambda_2(u_-)$. They are determined as solutions of Rankine–Hugoniot equations

$$s(u^1_+ - u^1_-) + (u^2_+ - u^2_-) = 0; \quad s(u^2_+ - u^2_-) + (p(u^1_+) - p(u^1_-)) = 0.$$

In contrast to the analysis of Burgers' equation we shall first analyze a hyperbolic dissipative model

$$\begin{aligned} \partial_t u^1 - \partial_x u^2 &= 0; \\ \partial_t u^2 - \partial_x v &= 0; \\ \partial_t v - \nu \partial_x u^2 &= -\frac{1}{\varepsilon}\left(v - p(u^1)\right). \end{aligned} \tag{3.3}$$

where $\nu > 0$ and ε is small positive parameter. It was proposed by Suliciu [**13**] as a model system which describes isothermal viscoelastic response of continuum. As $\varepsilon \to 0$ an equilibrium manifold $v_E = p(u^1)$ is determined from $(3.3)_3$ and $(3.3)_{1,2}$ is reduced to the equilibrium system (3.1). Characteristic speeds of the differential part of (3.3) are

$$\Lambda_1(U) = -\sqrt{\nu}; \quad \Lambda_2(U) = 0; \quad \Lambda_3(U) = \sqrt{\nu},$$

for $U = (u^1, u^2, v)^T$. Sub-characteristic condition (1.7) is satisfied if $p'(u^1) \leqslant \nu$. In the sequel we shall assume that strict inequality holds. Otherwise, a continuous shock profile determined by (3.3) will seize to exist.

Corresponding parabolic model will be obtained by means of Chapman–Enskog expansion of non-equilibrium variable

$$v^\varepsilon = p(u^{1\varepsilon}) + \varepsilon S(u^\varepsilon, \partial_x u^\varepsilon, \dots) + O(\varepsilon^2).$$

From $(3.3)_3$ and compatibility condition $(3.3)_1$ one obtains

$$S = \left(\nu - p'(u^{1\varepsilon})\right)\partial_x u^{2\varepsilon}$$

and $(3.3)_{1,2}$ become

$$\begin{aligned} \partial_t u^1 - \partial_x u^2 &= 0; \\ \partial_t u^2 - \partial_x p(u^1) &= \varepsilon \partial_x \left(\left(\nu - p'(u^1)\right)\partial_x u^2\right), \end{aligned} \tag{3.4}$$

where superscript ε is dropped for convenience. Parabolicity of (3.4) is ensured by the same condition needed to meet the sub-characteristic condition. It is obviously reduced to (3.1) when $\varepsilon \to 0$.

Assuming solution in the form of travelling wave and integrating equations (3.4) once with the use of boundary equilibrium data, one obtains the following set of equations which determines the shock profile

$$\begin{aligned} s(u^1-u^1_-)+(u^2-u^2_-)&=0;\\ -s(u^2-u^2_-)-(p(u^1)-p(u^1_-))&=(\nu-p'(u^1))\dot u^2. \end{aligned}$$

By eliminating u^2 using an algebraic equation, a single ODE is derived which determines the structure of shock wave

$$\dot u^1=\frac{-s^2\left(u^1-u^1_-\right)+p(u^1)-g(u^1_-)}{s\left(\nu-p'(u^1)\right)}=\varphi(u^1,s) \tag{3.5}$$

Stability analysis of stationary point u^1_- of (3.5) starts with linear variational equation

$$\dot y=\Phi(u^1_-,s)y;\quad \Phi(u^1_-,s)=\varphi_{u^1}(u^1_-,s)=\frac{-s^2+p'(u^1_-)}{s(\nu-p'(u^1_-))},$$

where $y=u^1-u^1_-$ is perturbation. Critical values of shock speed are reached for $\Phi(u^1_-,s^*_{\text{par}})=0$ and coincide with characteristic speeds of equilibrium system (3.2). In the sequel critical value will be referred to as $s^*_{\text{par}}=s^*=\lambda_2(u^1_-)=\sqrt{p'(u^1_-)}$. Stability of stationary point is changed in the neighborhood of s^*_{par} since $\Phi_s(u^1_-,s^*_{\text{par}})=-2/(\nu-p'(u^1_-))\neq 0$. Bifurcation pattern is determined by Taylor expansion of the right hand side of (3.5) up to the second order terms. Corresponding partial derivatives read

$$\varphi(u^1_-,s^*_{\text{par}})=0,\quad \varphi_{u^1}(u^1_-,s^*_{\text{par}})=0,\quad \varphi_{u^1u^1}(u^1_-,s^*_{\text{par}})=\frac{p''(u^1_-)}{(\nu-p'(u^1_-))\sqrt{p'(u^1_-)}},$$
$$\varphi_s(u^1_-,s^*_{\text{par}})=0,\quad \varphi_{u^1s}(u^1_-,s^*_{\text{par}})=-\frac{2}{\nu-p'(u^1_-)},$$
$$\varphi_{ss}(u^1_-,s^*_{\text{par}})=0,$$

and bifurcation equation resembles transcritical bifurcation pattern

$$\dot y\approx\frac{1}{2(\nu-p'(u^1_-))}\left(-2\mu y+\frac{p''(u^1_-)}{\sqrt{p'(u^1_-)}}y^2\right), \tag{3.6}$$

$\mu=s-s^*_{\text{par}}$ being a bifurcation parameter.

Turning the attention now to hyperbolic dissipative system (3.3), by similar analysis the following set of equations is obtained for a shock profile

$$\begin{aligned} s(u^1-u^1_-)+(u^2-u^2_-)&=0;\\ s(u^2-u^2_-)+(v-p(u^1_-))&=0;\\ s\dot v+\nu\dot u^2&=v-p(u^1_-), \end{aligned}$$

where downstream equilibrium state $v(-\infty)=v_E(-\infty)=p(u^1_-)$ for v is used in the course of integration. By eliminating u^2 and v from this set one arrives at a single ODE which reads

$$\dot u^1=\frac{s^2\left(u^1-u^1_-\right)-p(u^1)+p(u^1_-)}{s\left(s^2-\nu\right)}=\theta(u^1,s). \tag{3.7}$$

Stability of downstream equilibrium point u^1_- of (3.7) is determined by the linear variational equation

$$\dot{z} = \Theta(u^1_-, s)z; \quad \Theta(u^1_-, s) = \theta_{u^1}(u^1_-, s) = \frac{s^2 - p'(u^1_-)}{s\,(s^2 - \nu)}.$$

where $z = u^1 - u^1_-$ is perturbation. Critical values of the shock speed s occur when $\Theta(u^1_-, s^*_{\text{hyp}}) = 0$ and again coincide with characteristic speeds of equilibrium system. Taking $s^*_{\text{hyp}} = s^* = \lambda_2(u^1_-)$ it is easy to check that stability of stationary point is changed in the neighborhood of this critical value since $\Theta_s(u^1_-, s^*_{\text{hyp}}) = -2/(\nu - p'(u^1_-)) \neq 0$. By Taylor expansion of the right hand side of (3.7) up to the second order terms a bifurcation pattern is determined. Corresponding partial derivatives read

$$\theta(u^1_-, s^*_{\text{par}}) = 0, \quad \theta_{u^1}(u^1_-, s^*_{\text{par}}) = 0, \quad \theta_{u^1u^1}(u^1_-, s^*_{\text{par}}) = \frac{p''(u^1_-)}{(\nu - p'(u^1_-))\sqrt{p'(u^1_-)}},$$
$$\theta_s(u^1_-, s^*_{\text{par}}) = 0, \qquad \theta_{u^1 s}(u^1_-, s^*_{\text{par}}) = -\frac{2}{\nu - p'(u^1_-)},$$
$$\theta_{ss}(u^1_-, s^*_{\text{par}}) = 0,$$

and bifurcation equation becomes

$$\dot{z} \approx \frac{1}{2(\nu - p'(u^1_-)} \left(-2\mu z + \frac{p''(u^1_-)}{\sqrt{p'(u^1_-)}} z^2 \right),$$

for a bifurcation parameter $\mu = s - s^*_{\text{par}}$. This equation describes the same transcritical bifurcation likewise (3.6) in the parabolic case.

Two remarks concerned with this stability analysis have to be given. First, stationary point u^1_- changes its stability in the same way as in the case of stationary point of Burgers' equation. However, complete downstream equilibrium state $U_- = (u^1_-, u^2_-, v_-)^T$ is non-hyperbolic stationary point of the ODE system derived from (3.3) for a travelling profile $U = U(\xi)$. Linearized variational equations have two zero eigenvalues, while the remaining one has the same behaviour as predicted by linearization of (3.7). Second, one may observe that right hand side of (3.7) has singularity when s approaches the characteristic speeds of (3.3). The consequence is that continuous relaxation shock profile seize to exist when shock speed exceeds the highest characteristic speed of the full system. For a discussion of this problem one may consult Boillat and Ruggeri [**3**] and Currò and Fusco [**6**].

4. Conclusions

In this paper the problem of shock structure is analyzed through the comparative study of dissipative hyperbolic and parabolic models. Two examples, Burgers' equation and a model of isothermal viscoelasticity, have been picked up to demonstrate two properties of weak shocks revealed through stability and bifurcation analysis of stationary points of an ODE system which describe the shock profile. First, it was shown that downstream equilibrium state, both in parabolic and hyperbolic case, changes its stability properties in the neighborhood of the critical value of shock speed $s^*_{\text{par}} = s^*_{\text{hyp}} = s^*$; it coincides with the critical value of shock speed s^* for the equilibrium system. Second, upstream equilibrium state can be regarded as a bifurcating solution; it obeys the same transcritical bifurcation pattern in parabolic, as well as in hyperbolic case.

Although the examples studied in this paper are rather simple, they are nevertheless convincing and trigger the interest for more involved models. In particular, hyperbolic models discussed here had only one equation in the form of balance law which contain source term. Increasing number of balance laws requires application reduction techniques in order to obtain appropriate stability and bifurcation results. This will be the subject of our prospective study.

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