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A NEW APPROACH TO TEMPERATE GENERALIZED COLOMBEAU FUNCTIONS

Antoine Delcroix

Abstract. A new approach to the algebra $\mathcal{G}_\tau$ of temperate nonlinear generalized functions is proposed, in which $\mathcal{G}_\tau$ is based on the space $\mathcal{O}_M$ endowed with is natural topology in contrary to previous constructions. Thus, this construction fits perfectly in the general scheme of construction of Colombeau type algebras and reveals better properties of $\mathcal{G}_\tau$. This is illustrated by the natural introduction of a regularity theory in $\mathcal{G}_\tau$, of the Fourier transform, with the definition of $\mathcal{G}_{\mathcal{O}'_C}$, the space of rapidly generalized distributions which is the Fourier image of $\mathcal{G}_\tau$.

1. Introduction

The theory of generalized functions is nowadays well established. Many applications have been carried out in various fields of mathematics such as partial differential equations, Lie analysis, local and microlocal analysis, probability theory, differential geometry. (See for examples the monographies [**1, 2, 7, 15, 16**] and the references therein.)

This paper develops some remarks about a new approach to temperate generalized functions. In order to justify the introduction of this new construction, we first recall the main types of special (or simplified) algebras of generalized based on spaces of smooth functions considered in the literature.

The original simplified *Colombeau algebra of generalized functions* $\mathcal{G}$ is based on the space $\mathcal{E} = \mathrm{C}^\infty$ of smooth functions and contains the space of Schwartz distributions as a subvector space [**1, 7, 15, 16, 18**]. The duality in the background of this construction is, of course, $(\mathcal{D}, \mathcal{D}')$. As all spaces considered in the sequel, $\mathcal{G}$ is a factor space of moderate nets modulo negligible ones, the moderateness and the negligibility being given by the asymptotic behavior of the nets with respect to an asymptotic scale. When an algebra containing the space of tempered distributions is needed, the so-called *algebra of temperate generalized functions* $\mathcal{G}_\tau$ [**1, 7, 17, 18**], based on the space $\mathcal{O}_M$ of slowly increasing smooth functions, is considered. The duality is in this case $(\mathcal{S}, \mathcal{S}')$. Note that this construction is not, at first sight,

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related to the topology of $\mathcal{O}_M$. Finally, an algebra based on the space $\mathcal{S}$ of rapidly decreasing functions has also been considered [**3, 17, 19**], with applications (for example) in the field of pseudo differential operators [**5, 6**] or of microlocal analysis of generalized functions [**3, 8, 9, 19**]. This algebra $\mathcal{G}_{\mathcal{S}}$ of *rapidly decreasing generalized functions* contains as a linear subspace $\mathcal{O}'_C$, the space of rapidly decreasing distributions.

The first and the last constructions are based on the natural topology of the underlying space, which can be described by (countable) families of semi-norms. We propose here a new version of the construction of temperate generalized functions based on the usual topology of $\mathcal{O}_M$, which therefore fits in the general scheme of construction of Colombeau type algebras. The price to be paid is the non countability of the family of semi-norms defining the topology of $\mathcal{O}_M$.

The paper is organized as follows. Section 2 is devoted to a short presentation of the construction of the spaces of Colombeau type generalized functions and of the examples quoted above. In Section 3, we briefly recall the construction of the classical space of temperate generalized functions $\mathcal{G}_\tau$, develop the new construction and show that it leads to the same space. In Section 4, we turn to the definition of the Fourier transform of elements of $\mathcal{G}_\tau$. Using the classical theorem asserting that the Fourier image of $\mathcal{O}_M$ is $\mathcal{O}'_C$, we introduce the (new) space $\mathcal{G}_{\mathcal{O}'_C}$ of rapidly generalized distributions which is the Fourier image of $\mathcal{G}_\tau$. Of course, this Fourier transform will share the classical expected properties. Finally, in Section 5, we introduce the subspace $\mathcal{G}_\tau^\infty$ of regular elements of $\mathcal{G}_\tau$ and show that the result $\mathcal{G}_\tau^\infty \cap \mathcal{S}' = \mathcal{O}_M$ holds in the spirit of the more classical one $\mathcal{G}^\infty \cap \mathcal{D}' = \mathrm{C}^\infty$ [**16**]. (More generally, we could have introduced the notion of $\mathcal{R}$-regularity [**3**].) We also show, in the spirit of [**14**], that some subspaces of $\mathcal{G}$ of regular temperate elements can be considered leading to the corresponding local analysis of elements of $\mathcal{G}$.

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2. Simplified or special algebras of generalized functions

2.1. Colombeau type algebras based on locally convex algebras. Let d be an integer and denote by $\mathbb{K}$ the field of real or complex numbers. Let $E(\cdot)$ be a presheaf (resp. sheaf) of $\mathbb{K}$-topological algebras of $\mathbb{K}$ valued functions over $\mathbb{R}^d$. (Thus, the presheaf restriction operator is the usual restriction of $\mathbb{K}$ valued functions.)

Suppose that, for any open set Ω in $\mathbb{R}^d$, the topology of $E(\Omega)$ can be described by a family $\mathcal{P}(\Omega) = (p_i)_{i\in I(\Omega)}$ of semi-norms verifying:

$$\forall i\in I(\Omega)\ \exists (j,k,C)\in I(\Omega)\times I(\Omega)\times \mathbb{R}_+^*\ \forall f,g\in E(\Omega)\ \ p_i(fg) \leqslant C p_j(f) p_k(g).$$

Set

$$\mathcal{M}_{(E,\mathcal{P})}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \mid \forall i\in I\ \exists m\in\mathbb{N}\ \ p_i(f_\varepsilon) = o(\varepsilon^{-m}) \text{ as } \varepsilon\to 0 \right\},$$
$$\mathcal{N}_{(E,\mathcal{P})}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \mid \forall i\in I\ \forall m\in\mathbb{N}\ \ p_i(f_\varepsilon) = o(\varepsilon^{m}) \text{ as } \varepsilon\to 0 \right\}.$$

(The letter $\mathcal{M}$ (resp. $\mathcal{N}$) stands for moderate (resp. negligible). In the sequel, we shall omit the precision "as $\varepsilon \to 0$.")

From [**12**], it follows that:

Proposition 2.1. (a) *Suppose that the following assertion holds:*
(1) *For any Ω_1 and Ω_2, open subsets of $\mathbb{R}^d$ with $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$. Moreover, if ρ_1^2 is the restriction operator $E(\Omega_2) \to E(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the semi-norm $\widetilde{p}_i = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$.*

Then $\mathcal{M}_{(E,\mathcal{P})}(\cdot)$ is a presheaf of $\mathbb{K}$-algebras and $\mathcal{N}_{(E,\mathcal{P})}(\cdot)$ a presheaf of ideals of $\mathcal{M}_{(E,\mathcal{P})}(\cdot)$.

(b) *Suppose that $E(\cdot)$ is a sheaf of $\mathbb{K}$-topological algebras and that assumption* (1) *and the following hold:*
(2) *For any family $(\Omega_h)_{h\in H}$ of open sets in $\mathbb{R}^d$ with $\Omega = \bigcup_{h\in H}\Omega_h$ and for any $p \in \mathcal{P}(\Omega)$, there exist a finite subfamily $(\Omega_j)_{1\leqslant j\leqslant n}$ and corresponding semi-norms $p_j \in \mathcal{P}(\Omega_j)$ such that, for any $u \in E(\Omega)$,*

$$p(u) \leqslant C \max_{1\leqslant j\leqslant n} p_j(u|_{\Omega_j}), \quad C > 0.$$

Then $\mathcal{M}_{(E,\mathcal{P})}(\cdot)$ is a sheaf of $\mathbb{K}$-algebras and $\mathcal{N}_{(E,\mathcal{P})}(\cdot)$ a presheaf of ideals of $\mathcal{M}_{(E,\mathcal{P})}(\cdot)$.

Definition 2.1. For any Ω open subset of $\mathbb{R}^d$, the *Colombeau type algebra associated* to $E(\Omega)$ is the factor algebra $\mathcal{G}(\Omega) = \mathcal{M}_{(E,\mathcal{P})}(\Omega)/\mathcal{N}_{(E,\mathcal{P})}(\Omega)$.

Proposition 2.2. [**1, 12**] (a) *Under assumption* (1), *$\mathcal{G}(\cdot)$ is a presheaf of algebras.*

(b) *In addition, suppose that assumption* (2) *is fulfilled. Then, the localization principle (F_1) holds for $\mathcal{G}(\cdot)$:*
(F_1) *Let $(\Omega_h)_{h\in H}$ be a family of open sets in $\mathbb{R}^d$ with $\Omega = \bigcup_{h\in H}\Omega_h$. Consider $u, v \in \mathcal{G}(\Omega)$ such that all restrictions $u|_{\Omega_h}$ and $v|_{\Omega_h}$ $(h \in H)$ coincide. Then $u = v$.*

(c) *Moreover, if $E(\cdot)$ is a fine sheaf of algebras, $\mathcal{G}(\cdot)$ is also a fine sheaf of algebras.*

There is a natural presheaf (resp. sheaf) embedding of $E(\cdot)$ into $\mathcal{G}(\cdot)$ defined by

$$\sigma_{E,\mathcal{G}}(\Omega) : E(\Omega) \to \mathcal{G}(\Omega), \quad f \mapsto (f)_\varepsilon + \mathcal{N}_{(E,\mathcal{P})}(\Omega). \tag{2.1}$$

The presheaf (rest. sheaf) $\mathcal{G}(\cdot)$ turns out to be a presheaf (rest. sheaf) of modules on the factor ring $\widetilde{\mathbb{C}} = \mathcal{M}(\mathbb{C})/\mathcal{N}(\mathbb{C})$ with

$$\mathcal{M}(\mathbb{K}) = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \exists m \in \mathbb{N} \ |r_\varepsilon| = o(\varepsilon^{-m})\},$$
$$\mathcal{N}(\mathbb{K}) = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \forall m \in \mathbb{N} \ |r_\varepsilon| = o(\varepsilon^{m})\},$$

with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}, \mathbb{R}_+$. Moreover, for the cases under consideration in this paper, $E(\cdot)$ is a subpresheaf (rest. subsheaf) of the sheaf $\mathrm{C}^\infty(\cdot)$ of smooth functions.

Then, one easily checks that, for $\alpha \in \mathbb{N}^d$, a presheaf family of differential operators $\partial^\alpha f$ is defined component-wise on $\mathcal{G}(\Omega)$ by

$$\partial^\alpha f = \left(\partial^\alpha f_\varepsilon\right)_\varepsilon + \mathcal{N}_{(E,\mathcal{P})}(\Omega) \text{ with } \left(f_\varepsilon\right)_\varepsilon \in f.$$

The family of differential operators $\left(\partial^\alpha\right)_{\alpha \in \mathbb{N}^d}$ satisfies the usual rules (such as the Leibniz rule) and $\mathcal{G}(\cdot)$ turns out to be a presheaf (rest. sheaf) of differential algebras. The embedding defined by (2.1) turns out to be an embedding of differential algebras.

2.2. Examples. Example 2.1. Take $E(\cdot) = \mathrm{C}^\infty(\cdot)$. For any Ω open subset of $\mathbb{R}^d$, $\mathrm{C}^\infty(\Omega)$ is endowed with the family of semi-norms $\mathcal{P}(\Omega) = \left(p_{K,l}\right)_{K \Subset \Omega, l \in \mathbb{N}}$ defined by

$$p_{K,l}(f) = \sup_{|\alpha| \leqslant l,\, x \in K} |\partial^\alpha f(x)|,$$

where the notation $K \Subset \Omega$ means that the set K is a compact set included in Ω. We set

$$\mathcal{M}_{\mathrm{C}^\infty}(\cdot) = \mathcal{M}_{(\mathrm{C}^\infty,\mathcal{P})}(\cdot), \ \ \mathcal{N}_{\mathrm{C}^\infty}(\cdot) = \mathcal{N}_{(\mathrm{C}^\infty,\mathcal{P})}(\cdot).$$

The sheaf $\mathcal{G}(\cdot) = \mathcal{M}_{\mathrm{C}^\infty}(\cdot)/\mathcal{N}_{\mathrm{C}^\infty}(\cdot)$ is the *sheaf of special or simplified Colombeau algebras of generalized functions* [**1, 7, 16, 18**].

Example 2.2. Take for $E(\cdot)$ the presheaf $H^\infty(\cdot) = \mathcal{D}_{\mathrm{L}_2}(\cdot)$, with

$$H^\infty(\Omega) = \bigcap_{m \in \mathbb{N}} H^m(\Omega), \ \ H^m(\Omega) = W^{m,2}(\Omega).$$

From Sobolev inequalities, it follows that $H^\infty(\Omega)$ is continuously embedded into $C^\infty(\Omega)$. We may suppose a priori that elements of $H^\infty(\Omega)$ are C^∞. $H^\infty(\Omega)$ is endowed with the family of norms $\mathcal{P}_{L_2}(\Omega) = \left(\|\cdot\|_{m,\Omega}\right)_{m \in \mathbb{N}}$ defined by

$$\|f\|_{m,\Omega} = \sup_{|\alpha| \leqslant m} \|\partial^\alpha f\|_{L^2(\Omega)}.$$

We set

$$\mathcal{M}_H(\cdot) = \mathcal{M}_{(H^\infty,\mathcal{P}_{L_2})}(\cdot), \ \ \mathcal{N}_H(\cdot) = \mathcal{N}_{(H^\infty,\mathcal{P}_{L_2})}(\cdot).$$

The presheaf $\mathcal{G}_H(\cdot) = \mathcal{M}_H(\cdot)/\mathcal{N}_H(\cdot)$ is a *presheaf of Sobolev generalized functions.*

For the following example, we set for $f \in \mathrm{C}^\infty(\Omega)$, $r \in \mathbb{Z}$ and $l \in \mathbb{N}$,

$$p_{r,l}(f) = \sup_{x \in \Omega,\, |\alpha| \leqslant l} \langle x \rangle^r |\partial^\alpha f(x)| \text{ with } \langle x \rangle = (1 + |x|^2)^{1/2}.$$

Example 2.3. Take $E(\cdot) = \mathcal{S}(\cdot)$, the presheaf of rapidly decreasing smooth functions. For any Ω open subset of $\mathbb{R}^d$, the topology of $\mathcal{S}(\Omega)$ is described by the family of semi-norms $\mathcal{P}_{\mathcal{S}}(\Omega) = \left(p_{q,l}\right)_{(q,l) \in \mathbb{N}^2}$. We set

$$\mathcal{M}_{\mathcal{S}}(\cdot) = \mathcal{M}_{(\mathcal{S},\mathcal{P}_{\mathcal{S}})}(\cdot), \ \ \mathcal{N}_{\mathcal{S}}(\cdot) = \mathcal{N}_{(\mathcal{S},\mathcal{P}_{\mathcal{S}})}(\cdot).$$

The presheaf $\mathcal{G}_{\mathcal{S}}(\cdot) = \mathcal{M}_{\mathcal{S}}(\cdot)/\mathcal{N}_{\mathcal{S}}(\cdot)$ is the *presheaf of algebras of rapidly decreasing generalized functions* [**3, 5, 6, 17, 19**].

Remark 2.1. More general constructions can be given, for example if $E(\Omega)$ is a projective or inductive limit of topological algebras. We refer the reader to [**4**] for these cases.

2.3. Topology on $\mathcal{G}(\cdot)$. We follow [**4**] and use the notations of Subsection 2.1. Set, for $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]}$ and $i \in I(\Omega)$,

$$\|f_\varepsilon\|_i = \limsup_{\varepsilon\to 0} p_i(f_\varepsilon)^{|\ln\varepsilon|^{-1}}$$

and $d_i(f_\varepsilon, g_\varepsilon) = \|f_\varepsilon - g_\varepsilon\|_i$. We get [**4**, Proposition-definition 2]

$$\mathcal{M}_{(E,\mathcal{P})}(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \mid \forall i \in I \;\; \|f_\varepsilon\|_i < +\infty\big\},$$
$$\mathcal{N}_{(E,\mathcal{P})}(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \mid \forall i \in I \;\; \|f_\varepsilon\|_i = 0\big\}.$$

The family $(d_i)_{i\in I(\Omega)}$ defines a family of ultrapseudometrics on $\mathcal{M}_{(E,\mathcal{P})}(\Omega)$, inducing on $\mathcal{M}_{(E,\mathcal{P})}(\Omega)$ the structure of a topological ring such that the intersection of neighborhoods of 0 is equal to $\mathcal{N}_{(E,\mathcal{P})}(\Omega)$. Thus, this topology transfers to the factor space $\mathcal{G}(\Omega)$ which turns out to be a topological ring, and a topological algebra over the factor ring $\widetilde{\mathbb{C}} = \mathcal{M}(\mathbb{C})/\mathcal{N}(\mathbb{C})$. (By setting $\|r_\varepsilon\|' = \limsup_{\varepsilon\to 0} |r_\varepsilon|^{|\ln\varepsilon|^{-1}}$, one easily get that $\mathcal{M}(\mathbb{C})$ (resp. $\mathcal{N}(\mathbb{C})$) $= \big\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \|r_\varepsilon\|' < +\infty$ (resp. $= 0)\big\}$. This topological structure turns $\widetilde{\mathbb{C}}$ into a topological ring. This topology coincides with the sharp topology, usually defined in terms of valuations [**17, 18**].

3. Temperate generalized functions

3.1. Classical construction [7, 15, 17]. We recall that

$$\mathcal{O}_M(\Omega) = \big\{f \in \mathrm{C}^\infty(\Omega) \mid \forall l \in \mathbb{N}\; \exists q \in \mathbb{N}\;\; p_{-q,l}(f) < +\infty\big\}.$$

Define

$$\mathcal{M}_\tau(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\Omega)^{(0,1]} \mid \forall l \in \mathbb{N}\; \exists q \in \mathbb{N}\; \exists m \in \mathbb{N}\;\; p_{-q,l}(f_\varepsilon) = o(\varepsilon^{-m})\big\},$$
$$\mathcal{N}_\tau(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\Omega)^{(0,1]} \mid \forall l \in \mathbb{N}\; \exists q \in \mathbb{N}\; \forall m \in \mathbb{N}\;\; p_{-q,l}(f_\varepsilon) = o(\varepsilon^{m})\big\}.$$

One can show that $\mathcal{M}_\tau(\Omega)$ is a subalgebra of $\mathcal{O}_M(\Omega)^{(0,1]}$ and $\mathcal{N}_\tau(\Omega)$ an ideal of $\mathcal{M}_\tau(\Omega)$. The algebra $\mathcal{G}_\tau(\Omega) = \mathcal{M}_\tau(\Omega)/\mathcal{N}_\tau(\Omega)$ is called the *algebra of tempered generalized functions*.

3.2. New construction. The topology of $\mathcal{O}_M(\Omega)$ may be described by the non-countable family of semi-norms $\mathcal{P}_{\mathcal{O}_M}(\Omega) = (\nu_{\varphi,l})_{(\varphi,l)\in\mathcal{S}(\Omega)\times\mathbb{N}}$ defined by

$$\nu_{\varphi,l}(f) = \sup_{x\in\Omega, |\alpha|\leqslant l} |\varphi(x)\partial^\alpha f(x)|.$$

Proposition 3.1. *$\mathcal{O}_M(\Omega)$ endowed with the family $\mathcal{P}_{\mathcal{O}_M}(\Omega)$ is a topological algebra.*

This result is classical. For the continuity of the product, one establishes the property

$$\forall(\varphi,l) \in \mathcal{S}(\Omega)\times\mathbb{N}\; \exists\psi \in \mathcal{S}(\Omega)\; \exists C > 0\; \forall(f,g) \in \mathcal{O}_M(\Omega)^2\;\; \nu_{\varphi,l}(fg) \leqslant C\nu_{\psi,l}(f)\nu_{\psi,l}(g),$$

which is a consequence of the following:

Lemma 3.1. *For any $\psi \in C^0(\Omega)$ with positive values such that, for any $q > 0$, $p_{q,0}(\psi) < +\infty$ there exists $\varphi \in \mathcal{S}(\Omega)$ such that $\psi \leqslant \varphi$.*

With the previous notations, we set

$$\begin{aligned}\mathcal{M}_{\mathcal{O}_M}(\Omega) &= \mathcal{M}_{(\mathcal{O}_M,\mathcal{P}_{\mathcal{O}_M})}(\Omega)\\ &= \left\{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\Omega)^{(0,1]} \mid \forall\varphi\in\mathcal{S}(\Omega)\ \forall l\in\mathbb{N}\ \exists m\in\mathbb{N}\ \ \nu_{\varphi,l}(f_\varepsilon) = o(\varepsilon^{-m})\right\},\\ \mathcal{N}_{\mathcal{O}_M}(\Omega) &= \mathcal{N}_{(\mathcal{O}_M,\mathcal{P}_{\mathcal{O}_M})}(\Omega)\\ &= \left\{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\Omega)^{(0,1]} \mid \forall\varphi\in\mathcal{S}(\Omega)\ \forall l\in\mathbb{N}\ \forall m\in\mathbb{N}\ \ \nu_{\varphi,l}(f_\varepsilon) = o(\varepsilon^{m})\right\}.\end{aligned}$$

Proposition 3.2. *We have $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) = \mathcal{M}_\tau(\mathbb{R}^d)$ and $\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \mathcal{N}_\tau(\mathbb{R}^d)$.*

Proof. From the definitions, we immediately get that $\mathcal{M}_\tau(\mathbb{R}^d) \subset \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$ (resp. $\mathcal{N}_\tau(\mathbb{R}^d) \subset \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$.) For the inverse inclusions, we begin by proving that, for $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$, $(f_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d)$ if, and only if, $(f_\varepsilon)_\varepsilon$ satisfies the following characteristic property

$$\forall\alpha\in\mathbb{N}^d\ \exists q\in\mathbb{N}\ \exists m\in\mathbb{N}\ \exists\varepsilon_0\in(0,1]\ \exists r>0\ \forall\varepsilon\in(0,\varepsilon_0]\ \forall x\notin B(0,r)\quad \langle x\rangle^{-q}|\partial^\alpha f_\varepsilon(x)| \leqslant \varepsilon^{-m}. \tag{3.1}$$

Indeed, we can easily see that if $(f_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d)$, the property (3.1) holds even if $(f_\varepsilon)_\varepsilon \notin \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$. Conversely suppose that $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$ and that (3.1) holds. Fix $\alpha \in \mathbb{N}^d$. There exist $q \in \mathbb{N}$, $m \in \mathbb{N}$, $\varepsilon_0 \in (0,1]$, $r > 0$ such that (3.1) holds. Let us show that $\langle x\rangle^{-q}\,|\partial^\alpha f_\varepsilon(x)| \leqslant \varepsilon^{-m'}$ for some $m' \in \mathbb{N}$, ε small enough and all $x \in B(0,r)$. Consider $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B(0,r)$. According to the definition of $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$, used with $l = |\alpha|$, there exists $\varepsilon_0' \in (0,1]$ such that

$$\forall\varepsilon\in(0,\varepsilon_0']\ \forall x\in B(0,r)\ \ \langle x\rangle^{-q}\,|\partial^\alpha f_\varepsilon(x)| \leqslant |\partial^\alpha f_\varepsilon(x)| \leqslant \nu_{\varphi,l}(f_\varepsilon) \leqslant \varepsilon^{-m'}.$$

Taking $\varepsilon_1 = \min(\varepsilon_0,\varepsilon_0')$, $m_1 = \max(m,m')$, we obtain that

$$\forall\varepsilon\in(0,\varepsilon_1]\ \forall x\in\mathbb{R}^d,\ \ \langle x\rangle^{-q}\,|\partial^\alpha f_\varepsilon(x)| \leqslant \varepsilon^{-m_1}.$$

From this last property, a classical argument shows that

$$p_{-q,l}(f_\varepsilon) = \sup_{x\in\mathbb{R}^d,\,|\alpha|\leqslant l} \langle x\rangle^{-q}\,|\partial^\alpha f_\varepsilon(x)| = o\left(\varepsilon^{-M}\right)$$

provided M is chosen big enough. Thus $(f_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d)$.

Let us return to the proof of the inclusion $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) \subset \mathcal{M}_\tau(\mathbb{R}^d)$. Take $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\Omega)$ and suppose that (3.1) does not hold. There exist $\alpha \in \mathbb{N}^d$ for which we can built by induction a sequence $(\varepsilon_q)_{q\geqslant 0}$ with $\varepsilon_q \overset{q\to+\infty}{\longrightarrow} 0$ and a sequence $(x_q)_{q\geqslant 0}$ with $|x_{q+1}| \geqslant |x_q| + 2$ such that

$$\langle x_q\rangle^{-q}\left|\partial^\alpha f_{\varepsilon_q}(x_q)\right| > \varepsilon_q^{-q}.$$

Consider $\theta \in \mathcal{D}\left(\mathbb{R}^d\right)$ with $\operatorname{supp}\theta \subset B(0,1)$, $0 \leqslant \theta \leqslant 1$ and, say, $\theta(0) = 1$. Set

$$\varphi(x) = \sum_{q=0}^{+\infty} \langle x_q\rangle^{-q}\theta(x - x_q).$$

Following [**11**], it can be verified that φ belongs to $\mathcal{S}(\mathbb{R}^d)$. (Note that

$$\operatorname{supp}(x \mapsto \theta(x - x_q)) \cap \operatorname{supp}(x \mapsto \theta(x - x_{q'})) = \emptyset \text{ for } q \neq q',$$

justifying the choice of $(x_q)_{q \geqslant 0}$.) We have

$$\varphi(x_q) \left| \partial^\alpha f_{\varepsilon_q}(x_q) \right| > \varepsilon_q^{-q} \theta(0) = \varepsilon_q^{-q}.$$

Thus, for all $q \in \mathbb{N}$, $\nu_{\varphi,|\alpha|}(f_{\varepsilon_q}) > \varepsilon_q^{-q}$, with $\varepsilon_q \overset{q \to +\infty}{\longrightarrow} 0$ in contradiction with the definition of $\mathcal{M}_{\mathcal{O}_M}(\Omega)$. Finally $(f_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d)$. The proof of the inclusion $\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) \subset \mathcal{N}_\tau(\mathbb{R}^d)$ is quite similar. □

Corollary 3.1. *We have* $\mathcal{G}_\tau(\mathbb{R}^d) = \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)/\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$.

Remark 3.1. (i) Following subsection 2.3, $\mathcal{G}_\tau(\mathbb{R}^d)$ is naturally equipped with a topological structure, given by the non countable family of ultrapseudometrics $(d_{\varphi,l})_{(\varphi,l) \in \mathcal{S}(\Omega) \times \mathbb{N}}$ defined by

$$d_{\varphi,l}(f,g) = \limsup_{\varepsilon \to 0} \nu_{\varphi,l}(f_\varepsilon - g_\varepsilon)^{|\ln \varepsilon|^{-1}}, \text{ where } (f_\varepsilon)_\varepsilon \in f,\ (g_\varepsilon)_\varepsilon \in g.$$

(ii) According to Proposition 2.1, $\mathcal{G}_\tau(\cdot)$ is a presheaf of algebras. However, the localization principle (F_1) does not hold for $\mathcal{G}_\tau(\cdot)$ as shown by the following example.

Example 3.1. We adapt a classical example, which was first used to show that $\mathcal{G}_\tau(\cdot)$ is not a subpresheaf of $\mathcal{G}(\cdot)$ [**17**]. Consider $\Psi \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \Psi \leqslant 1$ and, say, $\Psi(0) = 1$. Set $f_\varepsilon(\cdot) = \Psi(\cdot - |\ln \varepsilon|^{1/2})$. Obviously $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$, defining $f \in \mathcal{G}_\tau(\mathbb{R})$. Consider $\Omega_h = \left]-h, h\right[$ for $h \in \mathbb{N}^*$. As $|\ln \varepsilon|^{1/2} \overset{\varepsilon \to 0}{\longrightarrow} +\infty$, we have $f|_{\Omega_h} = 0$. However, $f_\varepsilon(|\ln \varepsilon|^{1/2}) = \Psi(0) = 1$. Take $\varphi \in \mathcal{S}(\mathbb{R})$ defined by $\varphi(x) = \exp(-x^2)$. We have $\varphi(|\ln \varepsilon|^{1/2}) f_\varepsilon(|\ln \varepsilon|^{1/2}) = \varepsilon$. Thus $\nu_{\varphi,0}(f_\varepsilon) \geqslant \varepsilon$ and $(f_\varepsilon)_\varepsilon \notin \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. Therefore f is non equal to 0 on $\mathbb{R} = \bigcup_{h \in \mathbb{N}} \Omega_h$.

We set

$$\mathcal{N}_{\mathcal{O}_M,0} = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\Omega) \mid \forall \varphi \in \mathcal{S}(\Omega)\ \forall m \in \mathbb{N} : \nu_{\varphi,0}(f_\varepsilon) = o(\varepsilon^m) \right\}.$$

We have the same result as theorem 1.2.27 in [**7**] concerning $\mathcal{N}_\tau(\cdot)$ (the proof is similar):

Proposition 3.3. *If the open set* Ω *is a product of* d *intervals, then* $\mathcal{N}_{\mathcal{O}_M}(\Omega)$ *is equal to* $\mathcal{N}_{\mathcal{O}_M,0}(\Omega) \cap \mathcal{M}_{\mathcal{O}_M}(\Omega)$.

This result renders easier the proof of:

Proposition 3.4. [**7, 18**] *Consider* $\rho \in \mathcal{S}(\mathbb{R}^d)$ *such that*

$$\int \rho(x)\, dx = 1, \quad \forall \alpha \in \mathbb{N}^d \smallsetminus \{0\}, \int x^\alpha \rho(x)\, dx = 0. \tag{3.2}$$

Set

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(x/\varepsilon\right). \tag{3.3}$$

(i) *The map*

$$\sigma_\tau : \mathcal{O}_M(\mathbb{R}^d) \to \mathcal{G}_\tau(\mathbb{R}^d),\ u \mapsto (u)_\varepsilon + \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$$

is an embedding of differential algebras.

(ii) *The map*

$$\iota_\tau : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{G}_\tau(\mathbb{R}^d),\ T \mapsto (T * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$$

is an embedding of differential vector spaces.

(iii) *Moreover,* $\iota_\tau|_{\mathcal{O}_M(\mathbb{R}^d)} = \sigma_\tau$, *which means that the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{O}_M(\mathbb{R}^d) \arrow[rr, "\sigma_\tau"] \arrow[dr] & & \mathcal{G}_\tau(\mathbb{R}^d). \\ & \mathcal{S}'(\mathbb{R}^d) \arrow[ur, "\iota_\tau"] & \end{tikzcd} \tag{3.4}$$

(The arrow without label is the classical embedding of $\mathcal{O}_M(\mathbb{R}^d)$ *into* $\mathcal{S}'(\mathbb{R}^d)$*.)*

The assertion (iii) is an improvement of the classical one which only gives $\iota_\tau|_{\mathcal{O}_M(\mathbb{R}^d)} = \sigma_\tau|_{\mathcal{O}_C(\mathbb{R}^d)}$.

Proof. The assertion (i) is an application of the general principle recalled in subsection 2.1 to the case of $\mathcal{O}_M(\cdot)$. We refer the reader to [**7, 18**] for the proof of the assertion (ii) which uses mainly the structure of elements of $\mathcal{S}'(\mathbb{R}^d)$. We shall prove the assertion (iii) in the case $d = 1$; the general case only differs by more complicate algebraic expressions. Let f be in $\mathcal{O}_M(\mathbb{R})$ and set $\Delta = \iota_\tau(f) - \sigma_\tau(f)$. One representative of Δ is given by $(\Delta_\varepsilon : \mathbb{R} \to \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}))_\varepsilon$ with

$$\begin{aligned} \Delta_\varepsilon(y) &= (f * \theta_\varepsilon)(y) - f(y) = \int f(y-x)\rho_\varepsilon(x)\,dx - f(y) \\ &= \int (f(y-x) - f(y))\,\rho_\varepsilon(x)\,dx = \int (f(y-\varepsilon u) - f(y))\,\rho(u)\,du \end{aligned}$$

since $\int \rho_\varepsilon(x)\,dx = 1$. Let k be a positive integer. Taylor's formula gives

$$f(y-\varepsilon u) - f(y) = \sum_{i=1}^{k} \frac{(-\varepsilon u)^i}{i!} f^{(i)}(y) + \frac{(-\varepsilon u)^k}{k!} \int_0^1 f^{(k+1)}(y-\varepsilon uv)(1-v)^k dv.$$

Using $\int x^i \rho_\varepsilon(x)\,dx = 0$, for $i \in \{1,\ldots,k\}$, we get

$$\Delta_\varepsilon(y) = \int \frac{(-\varepsilon u)^k}{k!} \int_0^1 f^{(k+1)}(y-\varepsilon uv)(1-v)^k dv\,\rho(u)\,du.$$

As $f \in \mathcal{O}_M(\mathbb{R})$, there exists $p \in \mathbb{N}$ and $C_3 > 0$ such that $|f^{(k+1)}(\xi)| \leqslant C_3(1+|\xi|)^p$. Thus

$$\forall (u,y) \in \mathbb{R}^2\ \forall v \in [0,1]\ \forall \varepsilon \in (0,1]\ \left| f^{(k+1)}(y-\varepsilon uv) \right| \leqslant C_3(1+|y|)^p\,(1+|u|)^p.$$

As ρ is rapidly decreasing, the integral $\int |u|^k (1+|u|)^p \rho(u)\,du$ converges and

$$|\Delta_\varepsilon(y)| \leqslant \frac{\varepsilon^k}{k!} C_3 (1+|y|)^p \int |u|^k (1+|u|)^p \rho(u)\,du \leqslant \varepsilon^k C_4 (1+|y|)^p.$$

Consider $\varphi \in \mathcal{S}(\mathbb{R})$. The function $(1+|\cdot|)^p\,|\varphi(\cdot)|$ is bounded. Thus

$$\sup_{y\in\mathbb{R}} |\varphi(y)\Delta_\varepsilon(y)| = o(\varepsilon^k) \ \text{ as } \varepsilon \to 0.$$

As $(\Delta_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ and $\sup_{y\in\mathbb{R}} |\varphi(y)\Delta_\varepsilon(y)| = o(\varepsilon^k)$, we can conclude without estimating the derivatives that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ by using Proposition 3.3. $\square$

4. Fourier transform and space of rapidly decreasing generalized distributions

There is no need to recall the importance of spectral analysis, based on the Fourier transform in the theories of distributions [**10**] and Colombeau generalized functions (See, for example, [**3, 8, 9, 19**]). In this section, we first define in a new way the Fourier transform of elements of $\mathcal{G}_\tau(\mathbb{R}^d)$ in relationship with a (new) space of generalized distributions.

The Fourier transform of elements of $\mathcal{G}_\tau(\Omega)$ has already been defined with the help of ad hoc cutoff functions [**17, 18**]. More precisely, one sets

$$\forall u \in \mathcal{G}_\tau(\Omega),\ \mathcal{F}(u) = \int e^{-\imath xy} u_\varepsilon(y)\widehat{\rho}(\varepsilon y)\,dy + \mathcal{N}_\tau(\mathbb{R}^d) \ \text{ with } \ (u_\varepsilon)_\varepsilon \in u,$$

where $\rho \in \mathcal{S}(\mathbb{R}^d)$ satisfies (3.2) so that $\widehat{\rho}(\varepsilon y) \overset{\varepsilon\to 0}{\longrightarrow} 1$. One shows that this definition makes sense for $\mathcal{F}(u)$ does not depend on the chosen representative $(u_\varepsilon)_\varepsilon \in u$. Analogously, one defines $\mathcal{F}^{-1}$. However, this Fourier transform lacks some expected properties such as $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{id}$. (The reader will find a complete discussion on this subject in [**18**].)

Recalling that $\mathcal{O}_M(\mathbb{R}^d)$ is the Fourier image of $\mathcal{O}'_C(\mathbb{R}^d)$ (and reciprocally), we prefer here to construct the Fourier transform starting from this fact since $\mathcal{G}_\tau(\mathbb{R}^d)$ is directly built on $\mathcal{O}_M(\mathbb{R}^d)$. In other words, we consider $\mathcal{G}_\tau(\mathbb{R}^d)$ as a space of multiplicators and we introduce a space of convolutors, both of them being linked as usual by the Fourier Transform and its inverse.

Set

$$\mathcal{M}_{\mathcal{O}'_C}(\mathbb{R}^d) = \left\{ (T_\varepsilon)_\varepsilon \in \mathcal{O}'_C(\mathbb{R}^d)^{(0,1]} \mid (\mathcal{F}^{-1}(T_\varepsilon))_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) \right\},$$
$$\mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d) = \left\{ (T_\varepsilon)_\varepsilon \in \mathcal{O}'_C(\mathbb{R}^d)^{(0,1]} \mid (\mathcal{F}^{-1}(T_\varepsilon))_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) \right\}.$$

From the linearity of $\mathcal{F}^{-1}$ and the linear properties of the spaces $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$ and $\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$, we immediately get that $\mathcal{M}_{\mathcal{O}'_C}(\mathbb{R}^d)$ is a $\widetilde{\mathbb{C}}$-submodule (resp. $\mathbb{C}$-subvector space) of $\mathcal{O}'_C(\mathbb{R}^d)^{(0,1]}$ and $\mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d)$ a $\widetilde{\mathbb{C}}$-submodule (resp. $\mathbb{C}$-subvector space) of $\mathcal{M}_{\mathcal{O}'_C}(\mathbb{R}^d)$.

Definition 4.1. The factor space $\mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d) = \mathcal{M}_{\mathcal{O}'_C}(\mathbb{R}^d)/\mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d)$ is called the space of *rapidly decreasing generalized distributions*.

With this previous material, the Fourier transform of elements of $\mathcal{G}_\tau(\mathbb{R}^d)$ (we keep the notation $\mathcal{F}$) $\mathcal{F} : \mathcal{G}_\tau(\mathbb{R}^d) \to \mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d)$ is well defined by

$$\forall u \in \mathcal{G}_\tau(\mathbb{R}^d),\ \mathcal{F}(u) = \mathcal{F}(u_\varepsilon) + \mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d) \ \text{ with } \ (u_\varepsilon)_\varepsilon \in u.$$

The inverse Fourier transform from $\mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d)$ into $\mathcal{G}_\tau(\mathbb{R}^d)$ is defined analogously. This Fourier transform has the expected classical properties as they only have to be verified component-wise.

Proposition 4.1. (i) *The map*

$$\sigma_{\mathcal{O}'_C} : \mathcal{O}'_C(\mathbb{R}^d) \to \mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d),\ u \mapsto (u)_\varepsilon + \mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d)$$

is an embedding of $\mathbb{C}$*-vector spaces.*

(ii) *Take, as in Proposition* 3.4*,* $\rho \in \mathcal{S}(\mathbb{R}^d)$ *satisfying* (3.2) *and* $(\rho_\varepsilon)_\varepsilon$ *defined by* (3.3)*. The map*

$$\iota_{\mathcal{O}'_C} : \mathcal{S}'_C(\mathbb{R}^d) \to \mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d),\ T \mapsto (T\widehat{\rho}(\varepsilon\cdot))_\varepsilon + \mathcal{N}_{\mathcal{O}'_C}(\mathbb{R}^d)$$

is an embedding of $\mathbb{C}$*-vector spaces.*

The proof of (i) is immediate, whereas (ii) is obtained by "taking the Fourier transform image of the diagram (3.4)" in Proposition 3.4. In fact, the following diagram is commutative

$$\begin{tikzcd}
\mathcal{O}_M(\mathbb{R}^d) \arrow[rr, "\sigma_\tau"] \arrow[dr] \arrow[ddd, shift right, "\mathcal{F}"'] & & \mathcal{G}_\tau(\mathbb{R}^d) \arrow[ddd, shift right, "\mathcal{F}"'] \\
& \mathcal{S}'(\mathbb{R}^d) \arrow[ur, "\iota_\tau"] \arrow[d, shift right, "\mathcal{F}"'] & \\
& \mathcal{S}'(\mathbb{R}^d) \arrow[u, shift right, "\mathcal{F}^{-1}"'] \arrow[dr, "\iota_{\mathcal{O}'_C}"] & \\
\mathcal{O}'_C(\mathbb{R}^d) \arrow[rr, "\sigma_{\mathcal{O}'_C}"'] \arrow[ur] \arrow[uuu, shift right, "\mathcal{F}^{-1}"'] & & \mathcal{G}_{\mathcal{O}'_C}(\mathbb{R}^d) \arrow[uuu, shift right, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

(The arrows without labels are the usual embeddings of $\mathcal{O}_M(\mathbb{R}^d)$ and $\mathcal{O}'_C(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.)

Remark 4.1. Following ideas of Jean-André Marti (private communication), the Fourier transform in $\mathcal{G}_\tau(\mathbb{R}^d)$ can be used to define Sobolev type subspaces of $\mathcal{G}_\tau(\mathbb{R}^d)$. More precisely, we say that $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d)$ is of H^s *type* if, for all $\varepsilon \in (0,1]$, $\langle\cdot\rangle^s(\cdot)\widehat{u}_\varepsilon(\cdot) \in L^2(\mathbb{R}^d)$ and $(\|\langle\cdot\rangle^s\widehat{u}_\varepsilon(\cdot)\|_{L^2})_\varepsilon \in \mathcal{M}(\mathbb{R})$. One shows that the space $H^s(\mathbb{R}^d)$ is embedded into $\mathcal{G}_\tau^{(s)}(\mathbb{R}^d)$ through ι_τ defined in Proposition 3.4. This will be used in a forthcoming paper to introduce a H^s local and microlocal analysis in spaces of generalized functions.

5. Introduction to regularity theory

5.1. The spaces $\mathcal{M}_\tau^\infty(\Omega)$ and $\mathcal{G}_\tau^\infty(\Omega)$. In analogy to the definition of $\mathcal{G}^\infty$ [**7, 16**], we set

$$\mathcal{M}_{\mathcal{O}_M}^\infty(\Omega) = \left\{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\Omega)^{(0,1]} \mid \forall\varphi\in\mathcal{S}(\Omega)\ \exists m\in\mathbb{N}\ \forall l\in\mathbb{N}\ \ \nu_{\varphi,l}(f_\varepsilon) = o(\varepsilon^{-m})\right\}.$$

It is easy to check that $\mathcal{M}_{\mathcal{O}_M}^\infty(\cdot)$ is a subpreasheaf of algebras of $\mathcal{M}_{\mathcal{O}_M}(\cdot)$. From this, we get:

Proposition 5.1. *$\mathcal{G}_\tau^\infty(\cdot) = \mathcal{M}_{\mathcal{O}_M}^\infty(\cdot)/\mathcal{N}_{\mathcal{O}_M}^\infty(\cdot)$ is a subpresheaf of differential algebras of $\mathcal{G}_\tau(\cdot)$.*

Going further with the above mentioned analogy, we recall that $\mathcal{G}^\infty(\mathbb{R}^d) \cap \mathcal{D}'(\mathbb{R}^d) = \mathrm{C}^\infty(\mathbb{R}^d)$ [**16**]. This result can be interpreted as follows: The subsheaf $\mathcal{G}^\infty$ of regular sections of $\mathcal{G}$ is such that the sheaf embedding $\mathcal{G}^\infty \to \mathcal{G}$ is the natural extension of the classical one $\mathrm{C}^\infty \to \mathcal{D}'$. We have here the same situation (*modulo* the fact $\mathcal{G}_\tau(\cdot)$ is only a presheaf) that given by:

Proposition 5.2. $\mathcal{G}_\tau^\infty(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d) = \mathcal{O}_M(\mathbb{R}^d)$.

The result should be understood as follows. For $u \in \mathcal{S}'(\mathbb{R}^d)$, if $\iota_\tau(u)$ is in $\mathcal{G}_\tau^\infty(\mathbb{R}^d)$, then u is in $\mathcal{O}_M(\mathbb{R}^d)$.

Proof. Take $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\iota_\tau(u)$ is in $\mathcal{G}_\tau^\infty(\mathbb{R}^d)$. Then $(u * \rho_\varepsilon)_\varepsilon$ is in $\mathcal{M}_{\mathcal{O}_M}^\infty(\mathbb{R}^d)$. Recall that

$$u \in \mathcal{O}_M(\mathbb{R}^d) \Leftrightarrow \widehat{u} \in \mathcal{O}'_C(\mathbb{R}^d) \Leftrightarrow \forall \psi \in \mathcal{S}(\mathbb{R}^d),\ \widehat{u} * \psi \in \mathcal{S}(\mathbb{R}^d).$$

So consider $\psi \in \mathcal{S}(\mathbb{R}^d)$. We are going to show that $\langle\cdot\rangle^m \widehat{u} * \psi(\cdot)$ is bounded for all $m \in \mathbb{N}$. We have

$$\widehat{u} * \psi = (\widehat{u}(1-\widehat{\rho_\varepsilon})) * \psi + (\widehat{u}\,\widehat{\rho_\varepsilon}) * \psi. \tag{5.1}$$

Recalling that $\rho_\varepsilon = \varepsilon^{-d}\rho(\cdot/\varepsilon)$, we easily get that $\widehat{\rho_\varepsilon}(\cdot) = \widehat{\rho}(\varepsilon\cdot)$. Note also that $\widehat{\rho}(0) = \int \rho(x)\,dx = 1$. Thus

$$1 - \widehat{\rho_\varepsilon}(x) = -\varepsilon \int_0^1 \nabla\widehat{\rho}(\varepsilon x t)\cdot x\,dt = \varepsilon B(\varepsilon, x).$$

As $\widehat{\rho}$ is rapidly decreasing, there exists $C > 0$ such that $|\nabla\widehat{\rho}(\varepsilon x t)\cdot x| \leqslant C\langle x\rangle$ for all $(\varepsilon, x, t) \in (0,1] \times \mathbb{R}^d \times [0,1]$. The same holds for the derivatives with respect to x and, thus, for the function B and its derivatives. From this, for example by using the structure of elements of $\mathcal{S}'(\mathbb{R}^d)$, it can be shown that $\widehat{u}(1-\widehat{\rho_\varepsilon}) * \psi$ satisfies

$$\forall x \in \mathbb{R}^d \ \ |((\widehat{u}(1-\widehat{\rho_\varepsilon})) * \psi)(x)| \leqslant C_0 \varepsilon \langle x\rangle^q, \tag{5.2}$$

for some $C_0 > 0$ and q not depending on ε.

Consider $l \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ with $|\beta| = l$. We have, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}(\imath x)^\beta((\widehat{u}\,\widehat{\rho_\varepsilon}) * \psi)(x) &= (\imath x)^\beta \mathcal{F}\left((u * \rho_\varepsilon)\,\mathcal{F}^{-1}(\psi)\right)(x)\\ &= \mathcal{F}\left((\partial^\beta(u * \rho_\varepsilon))\,\mathcal{F}^{-1}(\psi)\right)(x).\end{aligned}$$

Applying the definition of $\mathcal{M}_{\mathcal{O}_M}^\infty(\mathbb{R}^d)$ for $(\rho_\varepsilon * u)_\varepsilon$ with $\varphi = \langle\cdot\rangle^{(d+1)/2}\mathcal{F}^{-1}(\psi)$, we get the existence of N (only depending on $(\rho_\varepsilon * u)_\varepsilon$ and ψ) and $C_1 > 0$ such that

$$\forall y \in \mathbb{R}^d \ \left|(\partial^\beta(u * \rho_\varepsilon))(y)\mathcal{F}^{-1}(\psi)(y)\right| \leqslant C_1 \langle y\rangle^{-(d+1)/2}\varepsilon^{-N} \ \text{ for } \varepsilon \text{ small enough.}$$

Thus, we get the existence of $C_2 > 0$ such that

$$\left|(\imath x)^\beta((\widehat{u}\,\widehat{\rho_\varepsilon}) * \psi)(x)\right| = \left|\mathcal{F}((\partial^\beta(u * \rho_\varepsilon))\,\mathcal{F}^{-1}(\psi))(x)\right| \leqslant C_2\,\varepsilon^{-N}$$

for ε small enough and all $x \in \mathbb{R}^d$. Using a classical argument, we get a constant $C_3 > 0$ such that

$$\forall x \in \mathbb{R}^d \ \langle x\rangle^l \big|(\widehat{u}\,\widehat{\rho_\varepsilon}) * \psi)(x)\big| \leqslant C_3\,\varepsilon^{-N} \ \text{ for } \varepsilon \text{ small enough.}$$

Fix $m \in \mathbb{N}$ and take $l = m + (m+q)N$. Writing the previous inequality in the form $\langle x\rangle^m\,|(\widehat{u}\,\widehat{\rho_\varepsilon}) * \psi)(x)| \leqslant C_3(\varepsilon\langle x\rangle^{m+q})^{-N}$, using (5.2) and finally inserting these intermediates steps in (5.1), we get

$$\forall x \in \mathbb{R}^d \ |\langle x\rangle^m(\widehat{u} * \psi)(x)| = C\big(\varepsilon\langle x\rangle^{m+q} + (\varepsilon\langle x\rangle^{m+q})^{-N}\big) = T(\varepsilon\langle x\rangle^{m+q})$$

for ε smaller than some ε_0 and some $C > 0$. Thus, for x such that $\langle x\rangle^{m+q} \geqslant \varepsilon_0^{-1}$, take ε_x such that $\varepsilon_x = \langle x\rangle^{-m-q}$ to obtain that $|\langle x\rangle^m(\widehat{u} * \psi)(x)| \leqslant T(1)$. From this, it follows that the function $|\langle\cdot\rangle^m(\widehat{u} * \psi)|$ is bounded on $\mathbb{R}^d$, as claimed. □

5.2. Regularities for temperate generalized functions. As in the presheaf $\mathcal{G}_\tau(\cdot)$ the localization principle (F_1) is not fulfilled, we are not in the situation to apply the results of [**14**] concerning singular supports and their properties. Indeed, following the notations of the quoted paper, we need a presheaf $\mathcal{A}(\cdot)$ (of vector spaces, of algebras,…) with localization principle and a subpresheaf $\mathcal{B}(\cdot)$ of $\mathcal{A}(\cdot)$ to define the $\mathcal{B}$-singular support of a section $u \in \mathcal{A}(\Omega)$. Thus, as it is done in [**13**] for the definition of the presheaf $\mathcal{G}^L(\cdot)$, we shall start from the sheaf $\mathcal{G}(\cdot)$ and define some regular subpresheaves of it. More precisely, for the two cases $\mathcal{B}(\cdot) = \mathcal{G}_\tau(\cdot)$, $\mathcal{G}_\tau^\infty(\cdot)$, we set $\mathcal{N}^\sharp_{\mathcal{O}_M,*}(\cdot) = \mathcal{N}(\cdot) \cap \mathcal{M}^\sharp_{\mathcal{O}_M}(\cdot)$, where the symbol $\sharp$ means successively the blank character and ∞. According to the results recalled in Section 2 and to the inclusion $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) \subset \mathcal{M}_{\mathrm{C}^\infty}(\mathbb{R}^d))$, $\mathcal{G}^\sharp_{\tau,*}(\cdot) = \mathcal{M}^\sharp_{\mathcal{O}_M}(\cdot)/\mathcal{N}^\sharp_{\mathcal{O}_M,*}(\cdot)$ is a subpresheaf of $\mathcal{G}(\cdot)$. Using the framework and the results of [**14**], we say that the elements of $\mathcal{G}^\sharp_{\tau,*}(\Omega)$ are *$\mathcal{G}^\sharp_\tau$-regular elements* of $\mathcal{G}(\Omega)$. For $u \in \mathcal{G}(\Omega)$, we can define $\mathcal{O}^\sharp_\tau(u)$, the set of all $x \in \Omega$ such that u is *$\mathcal{G}^\sharp_\tau$-regular* at x, that is

$$\mathcal{O}^\sharp_\tau(u) = \big\{x \in \Omega \ \exists V \in \mathcal{V}_x : u|_V \in \mathcal{G}^\sharp_{\mathcal{O}_M,*}(V)\big\}$$

($\mathcal{V}_x$ being the family of all open neighborhoods of x.) The *$\mathcal{G}^\sharp_\tau$-singular support* of u is the well defined set $\mathcal{S}^\sharp_\tau(u) = \operatorname{singsupp}_{\mathcal{G}^\sharp_\tau} u = \Omega \smallsetminus \mathcal{O}^\sharp_\tau(u)$ and has the following properties [**14**]:

Proposition 5.3. *Consider $u, v \in \mathcal{G}(\Omega)$, α in $\mathbb{N}^d$ and g in $\mathcal{G}^\sharp_{\tau,*}(\Omega)$. We have:*

$$\begin{array}{llll} \text{(i)} & \mathcal{S}^\sharp_\tau(u+v) \subset \mathcal{S}^\sharp_\tau(u) \cup \mathcal{S}^\sharp_\tau(v); & \text{(iii)} & \mathcal{S}^\sharp_\tau(\partial^\alpha u) \subset \mathcal{S}^\sharp_\tau(u); \\ \text{(ii)} & \mathcal{S}^\sharp_\tau(uv) \subset \mathcal{S}^\sharp_\tau(u) \cup \mathcal{S}^\sharp_\tau(v); & \text{(iv)} & \mathcal{S}^\sharp_\tau(gu) \subset \mathcal{S}^\sharp_\tau(u). \end{array}$$

From these properties, one easily gets:

Corollary 5.1. *Let $P(\partial) = \sum_{|\alpha| \leqslant m} C_\alpha \partial^\alpha$ be a differential polynomial with coefficients in $\mathcal{G}_{\tau,*}(\Omega)$ (resp. $\mathcal{G}^\infty_{\tau,*}(\Omega)$). For any $u \in \mathcal{G}(\Omega)$, we have*

$$\mathcal{S}_\tau(P(\partial)u) \subset \mathcal{S}_\tau(u) \quad (\textit{resp. } \mathcal{S}^\infty_\tau(P(\partial)u) \subset \mathcal{S}^\infty_\tau(u)).$$

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Laboratoire Analyse, Optimisation, Contrôle
Faculté des sciences – Université des Antilles et de la Guyane
Pointe à Pitre
Guadeloupe (France)
antoine.delcroix@univ-ag.fr