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DISTANCE SPECTRA AND DISTANCE ENERGIES OF ITERATED LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The distance or D-eigenvalues of a graph G are the eigenvalues of its distance matrix. The distance or D-energy $E_D(G)$ of the graph G is the sum of the absolute values of its D-eigenvalues. Two graphs G_1 and G_2 are said to be D-equienergetic if $E_D(G_1) = E_D(G_2)$. Let F_1 be the 5-vertex path, F_2 the graph obtained by identifying one vertex of a triangle with one end vertex of the 3-vertex path, F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle and F_4 be the graph obtained by identifying one end vertex of a 4-vertex star with a middle vertex of a 3-vertex path. In this paper we show that if G is r-regular, with $\operatorname{diam}(G) \leqslant 2$, and F_i, $i = 1, 2, 3, 4$, are not induced subgraphs of G, then the k-th iterated line graph $L^k(G)$ has exactly one positive D-eigenvalue. Further, if G is r-regular, of order n, $\operatorname{diam}(G) \leqslant 2$, and G does not have F_i, $i = 1, 2, 3, 4$, as an induced subgraph, then for $k \geqslant 1$, $E_D(L^k(G))$ depends solely on n and r. This result leads to the construction of non D-cospectral, D-equienergetic graphs having same number of vertices and same number of edges.

1. Introduction

Let G be a simple, undirected graph without loops and multiple edges. Let n be the number of vertices and m the number of edges of G. The vertices of G are labelled as $v_1, v_2, \dots, v_n$. The distance between the vertices v_i and v_j is the length of a shortest path between them, and is denoted by $d(v_i, v_j)$. The diameter of G, denoted by $\operatorname{diam}(G)$, is the maximum distance between any pair of vertices of G [**6, 15**].

The adjacency matrix of a graph G is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of the adjacency matrix $A(G)$ are the (adjacency or ordinary) eigenvalues of G, forming the (adjacency or ordinary) spectrum of G [**8**]. These will be labelled as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

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The distance matrix of a graph G is the square matrix $D = D(G) = [d_{ij}]$, in which d_{ij} is the distance between the vertices v_i and v_j in G. The eigenvalues of the distance matrix $D(G)$, labelled as $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$, are said to be the distance or D-eigenvalues of G and to form the distance or D-spectrum of G [**6, 7**].

Two connected graphs G and H are said to be D-cospectral if they have same D-spectra. The characteristic polynomial and eigenvalues of the distance matrix of graphs have been considered in [**9, 10, 11, 16, 17, 18, 19, 34**].

The distance or D-energy of a connected graph G is defined as

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i| \; . \tag{1}$$

The D-energy was first time introduced by Indulal et al. [**19**], and was conceived in full analogy with the ordinary graph energy $E(G)$, defined as [**12, 13, 14**]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \; . \tag{2}$$

Various bounds for the D-energy have been communicated in [**19, 27**].

Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$. A large number of constructions of non-cospectral, equienergetic graphs was recently reported [**1, 2, 3, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 32, 33**].

Two graphs G_1 and G_2 are said to be D-equienergetic if $E_D(G_1) = E_D(G_2)$. Of course D-cospectral graphs are D-equienergetic. We are interested in non D-cospectral, D-equienergetic graphs having same number of vertices. Recently Indulal et al. [**19**] constructed pairs of D-equienergetic graphs on n vertices for $n \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 6$. In this paper we give one more class of D-equienergetic graphs.

We need the following results.

THEOREM 1. [**8**] *If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

THEOREM 2. [**7, 19**] *Let G be an r-regular graph of order n and $\operatorname{diam}(G) \leqslant 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its D-eigenvalues are $2n-r-2$ and $-(\lambda_i + 2)$, $i = 2, 3, \ldots, n$.*

2. On the line graph of a regular graph

The line graph of G will be denoted by $L(G)$ [**15**]. For $k = 1, 2, \ldots$, the k-th iterated line graph of G is $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$.

The line graph of a regular graph is a regular graph. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (r_0 n_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [**4, 5**]:

$$n_k = \tfrac{1}{2}\, r_{k-1} n_{k-1} \qquad \text{and} \qquad r_k = 2r_{k-1} - 2$$

where n_i and r_i stand for the order and degree of $L^i(G)$, $i = 1, 2, \ldots$. Therefore,

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{3}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = n_0 \prod_{i=0}^{k-1} (2^{i-1} r_0 - 2^i + 1) \ . \tag{4}$$

THEOREM 3. [**31**] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \quad i = 1, 2, \ldots, n, \qquad \textit{and}$$
$$-2, \quad n(r-2)/2 \quad \textit{times.}$$

For any set S of vertices (edges) of G, the induced subgraph $< S >$ is the maximal subgraph of G with vertex set (edge set) S.

Let F_1 be the 5-vertex path, F_2 the graph obtained by identifying a vertex of a triangle with one end vertex of the 3-vertex path, F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle and F_4 be the graph obtained by identifying one end vertex of a 4-vertex star with a middle vertex of a 3-vertex path, see Fig. 1.

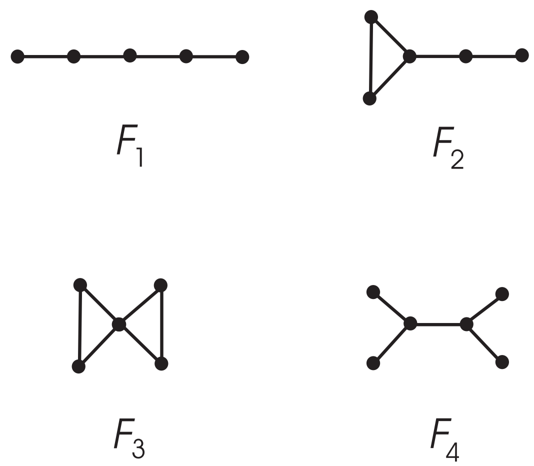

FIGURE 1. The forbidden subgraphs.

THEOREM 4. *If* $\mathrm{diam}(G) \leqslant 2$ *and if none of the three graphs F_1, F_2 and F_3 of Fig.* 1 *is an induced subgraph of G, then* $\mathrm{diam}(L(G)) \leqslant 2$.

PROOF. The results can be easily verified for $n \leqslant 4$. We thus assume that $n > 4$.

Let $e_1, e_2, \dots, e_m$ be the edges of a graph G. These are the vertices of $L(G)$. If the edges e_i and e_j are adjacent in G, then the vertices e_i and e_j are adjacent in $L(G)$. Therefore the distance between e_i and e_j in $L(G)$ is 1.

If the edges e_i and e_j are not adjacent in G, then we consider two cases.

Case 1: Suppose $\mathrm{diam}(G) = 1$. Then $G \cong K_n$ a complete graph on n vertices. In K_n, if the edges e_i and e_j are not adjacent then, because every vertex of K_n is adjacent to the remaining vertices, there exists an edge e_k in K_n adjacent to both e_i and e_j. Therefore in $L(K_n)$, $d(e_i, e_j) = d(e_i, e_k) + d(e_k, e_j) = 1 + 1 = 2$. Hence $\mathrm{diam}(L(G)) = 2$, for all $n \geqslant 4$.

Case 2: Suppose $\mathrm{diam}(G) = 2$. Consider the conditions under which $L(G)$ will have diameter greater than 2. For this G must possess two independent edges say $e_i = (uv)$ (connecting the vertices u and v) and $e_j = (xy)$ (connecting the vertices x and y), such that neither u nor v are adjacent to either x or y. If so, then because G has diameter 2, there must exist a vertex w adjacent to u and x. If w is not adjacent to either v or y, then G has F_1 as induced subgraph (spanned by the vertices u, v, x, y and w). If w is adjacent to v, but not to y (or, what is the same, adjacent to y but not to v), then G has F_2 as induced subgraph. If w is adjacent to both v and y, then G has F_3 as induced subgraph. Hence, if none of F_i $i = 1, 2, 3$, is an induced subgraph of G, then $\mathrm{diam}(L(G)) \leqslant 2$. □

Theorem 5. *If* $\mathrm{diam}(G) \leqslant 2$ *and if none of the four graphs of Fig.* 1 *is an induced subgraph of* G*, then none of the four graphs of Fig.* 1 *is an induced subgraph of* $L(G)$.

Proof. If $\mathrm{diam}(G) \leqslant 2$ and none of the four graphs of Fig. 1 is an induced subgraph of G, then any 5-edge subset of the edges of G induces one of the graphs depicted in Fig. 2.

None of the line graphs of graphs depicted in Fig. 2 has F_i, $i = 1, 2, 3, 4$, as an induced subgraph. Hence the proof. □

Combining Theorems 4 and 5 we have following Theorem.

Theorem 6. *If* $\mathrm{diam}(G) \leqslant 2$ *and if none of the four graphs of Fig.* 1 *is an induced subgraph of* G*, then for* $k \geqslant 1$*,*

(i) $\mathrm{diam}(L^k(G)) \leqslant 2$ *and*

(ii) *none of the four graphs of Fig.* 1 *is an induced subgraph of* $L^k(G)$.

Theorem 7. *If* G *is an* r*-regular graph of order* n *with* $\mathrm{diam}(G) \leqslant 2$ *and if none of the three graphs* F_1*,* F_2 *and* F_3 *of Fig.* 1 *is an induced subgraph of* G*, then* $L(G)$ *has exactly one positive* D*-eigenvalue, equal to* $nr - 2r$.

Proof. Let $r, \lambda_2, \lambda_3, \dots, \lambda_n$ be the adjacency eigenvalues of a regular graph G. Then from Theorem 3, the adjacency eigenvalues of of $L(G)$ are

$$\begin{aligned} &\lambda_i + r - 2, \quad i = 1, 2, \dots, n, \qquad \text{and} \\ &-2, \quad n(r-2)/2 \quad \text{times} \end{aligned} \tag{5}$$

The graph G is regular of degree r and has order n. Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r - 2$. As $\mathrm{diam}(G) \leqslant 2$ and none of the three

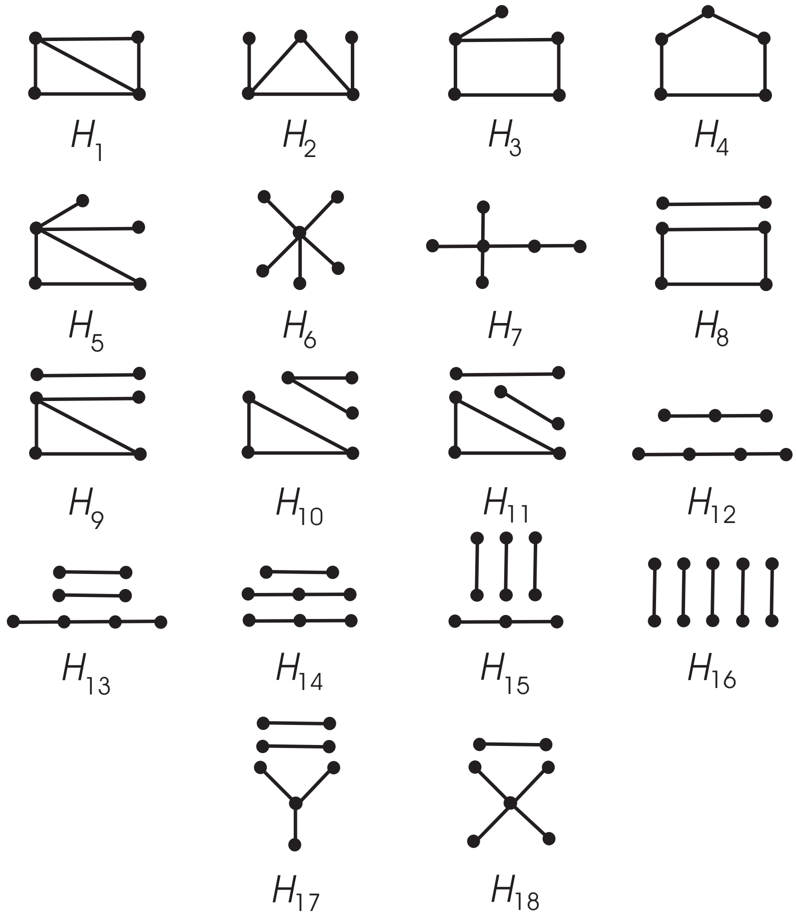

FIGURE 2. 5-edge graphs.

graphs F_1, F_2 and F_3 of Fig. 1 is an induced subgraph of G, from Theorem 4, $\mathrm{diam}(L(G)) \leqslant 2$. Therefore from Theorem 2 and Eq. (5), the D-eigenvalues of $L(G)$ are

$$\begin{cases} nr - 2r, & \text{and} \\ -(\lambda_i + r), & i = 2, 3, \ldots, n \qquad \text{and} \\ 0, & n(r-2)/2 \quad \text{times}\,. \end{cases} \tag{6}$$

All eigenvalues of a regular graph of degree r satisfy the condition $-r \leqslant \lambda_i \leqslant r$ [**8**]. Therefore $\lambda_i + r \geqslant 0$, $i = 1, 2, \ldots, n$. The theorem follows from Eq. (6). $\square$

COROLLARY 7.1. *Let G be an r-regular graph on n vertices with* $\mathrm{diam}(G) \leqslant 2$ *and let none of the four graphs of Fig.* 1 *be an induced subgraph of G. Let n_k and r_k be the order and degree, respectively, of the k-th iterated line graph $L^k(G)$ of G,*

$k \geqslant 1$. Then $L^k(G)$ has exactly one positive D-eigenvalue equal to

$$n_{k-1}r_{k-1} - 2r_{k-1} = 2n_k - r_k - 2 = 2n \prod_{i=0}^{k-1} [2^{i-1}r - 2^i + 1] - [2^k r - 2^{k+1} + 4].$$

3. Distance energy

The D-energy $E_D(G)$ of a graph G is defined via Eq. (1).

THEOREM 8. *If G is an r-regular graph of order n with* $\operatorname{diam}(G) \leqslant 2$ *and if none of the three graphs F_1, F_2 and F_3 of Fig.* 1 *is an induced subgraph of G, then $E_D(L(G)) = 2nr - 4r$.*

PROOF. Bearing in mind Theorem 7 and Eq. (6), the D-energy of $L(G)$ is computed as:

$$E_D(L(G)) = nr - 2r + \sum_{i=2}^{n} (\lambda_i + r) + |0| \times \frac{n(r-2)}{2} = 2nr - 4r$$

since $\sum\limits_{i=2}^{n} \lambda_i = -r$. □

COROLLARY 8.1. *Let G be an r_0-regular graph of order n_0 with* $\operatorname{diam}(G) \leqslant 2$ *and let none of the four graphs of Fig.* 1 *be an induced subgraph of G. Let n_k and r_k be the order and degree, respectively, of the k-th iterated line graph $L^k(G)$ of G, $k \geqslant 1$. Then*

$$E_D(L^k(G)) = 2n_{k-1}r_{k-1} - 4r_{k-1} = 4n_k - 2r_k - 4.$$

COROLLARY 8.2. *Under the same conditions as in the previous corollary,*

$$E_D(L^k(G)) = 4n_0 \prod_{i=0}^{k-1} (2^{i-1}r_0 - 2^i + 1) - 2(2^k r_0 - 2^{k+1} + 4).$$

From Corollary 8.2 we see that the D-energy of the k-th iterated line graph of a regular graph G of diameter less than or equal to 2, that does not contain F_i, $i = 1, 2, 3, 4$, as an induced subgraph is fully determined by the order n_0 and degree r_0 of G.

4. Distance-equienergetic graphs

LEMMA 9. *Let G_1 and G_2 be two regular graphs of the same order and of the same degree. Let* $\operatorname{diam}(G_i) \leqslant 2$*, and none of the four graphs of Fig. 1 be an induced subgraph of G_i, $i = 1, 2$. Then for any $k \geqslant 1$ the following holds:*

(i) $L^k(G_1)$ and $L^k(G_2)$ are of the same order, same degree and have the same number of edges.

(ii) $L^k(G_1)$ and $L^k(G_2)$ are D-cospectral if and only if G_1 and G_2 are cospectral.

PROOF. Statement (i) follows from Eqs. (3) and (4), and the fact that the number of edges of $L^k(G)$ is equal to the number of vertices of $L^{k+1}(G)$. Statement (ii) follows from Eqs. (5) and (6), and Theorem 6. □

THEOREM 10. *Let G_1 and G_2 be two non D-cospectral regular graphs of the same order and of the same degree. Let* $\mathrm{diam}(G_i) \leqslant 2$ *and let none of the four graphs of Fig.* 1 *be an induced subgraphs of* G_i, $i = 1, 2$. *Then for any* $k \geqslant 1$, *the iterated line graphs* $L^k(G_1)$ *and* $L^k(G_2)$ *form a pair of non D-cospectral, D-equienergetic graphs of equal order and of equal number of edges.*

PROOF. follows from Lemma 9 and Corollary 8.2. □

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