PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 86(100) (2009), 123–131 DOI: 10.2298/PIM0900123S

FACTORIZATION PROPERTIES OF SUBRINGS IN TRIGONOMETRIC POLYNOMIAL RINGS

Tariq Shah and Ehsan Ullah

Communicated by Žarko Mijajlović

ABSTRACT. We explore the subrings in trigonometric polynomial rings and their factorization properties. Consider the ring S' of complex trigonometric polynomials over the field $\mathbb{Q}(i)$ (see [**11**]). We construct the subrings S'_1, S'_0 of S' such that $S'_1 \subseteq S'_0 \subseteq S'$. Then S'_1 is a Euclidean domain, whereas S'_0 is a Noetherian HFD. We also characterize the irreducible elements of S'_1, S'_0 and discuss among these structures the condition: Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$.

1. Introduction

Factorization properties of integral domains have been a common interest of algebraists, particularly for polynomial rings. In this study we investigate the factorization properties of the subrings of S' (see [**11**]). The basic concepts, notions and terminology are standard, as in [**7**].

For the factorization of exponential polynomials, J. F. Ritt developed: "If $1 + a_1e^{\alpha_1 x} + \cdots + a_ne^{\alpha_n x}$ is divisible by $1 + b_1e^{\beta_1 x} + \cdots + b_re^{\beta_r x}$ with no $b = 0$, then every β is a linear combination of $\alpha_1, \ldots, \alpha_n$ with rational coefficients" [**9**, Theorem].

Getting inspired by this, G. Picavet and M. Picavet [**7**] investigated some factorization properties in trigonometric polynomial rings. Following [**7**], when we replace all α_k above by im, with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \right\},$$

$$T = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \right\}$$

2000 *Mathematics Subject Classification*: 13A05, 13B30, 12D05, 42A05.
Key words and phrases: trigonometric polynomial, HFD, subrings, condition 1, irreducible.

are trigonometric polynomial rings.

Following Cohn [**4**], an integral domain D is atomic if each nonzero nonunit of D is a product of irreducible elements (*atoms*) of D, and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist any infinite strictly ascending chain of principal integral ideals of D. Every PID, UFD and Noetherian domain satisfy ACCP and a domain satisfying ACCP is atomic. Grams [**6**] and Zaks [**13**] provided examples of atomic domains, which do not satisfy ACCP. Following [**12**], an integral domain D is said to be a *half-factorial domain* (HFD) if D is *atomic* and whenever $x_1 \dots x_m = y_1 \dots y_n$, where $x_1, x_2 \dots x_m$, $y_1, y_2 \dots y_n$ are irreducibles in D, then $m = n$. A UFD is obviously an HFD, but the converse fails, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [**12**], but not a UFD. Moreover a polynomial extension of an HFD is not an HFD, for example, $\mathbb{Z}\big[\sqrt{-3}\,\big][X]$ is not an HFD, as $\mathbb{Z}\big[\sqrt{-3}\,\big]$ is an HFD but not integrally closed [**5**]. Following [**2**], an integral domain D is a *finite factorization domain* (FFD) if each nonzero nonunit of D has only a finite number of non-associate divisors and hence only a finite number of factorizations up to order and associates. In general,

$$\textbf{UFD} \Longrightarrow \textbf{HFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic},$$
$$\textbf{UFD} \Longrightarrow \textbf{FFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic}.$$

But none of the above implications is reversible.

In [**7**, Theorems 2.1 and 3.1], G. Picavet and M. Picavet demonstrated that T' is a Euclidean domain and T is a Dedekind half-factorial domain. Moreover, in [**11**] we extended the study of factorization properties of trigonometric polynomials with coefficients from the field $\mathbb{Q}$ and its algebraic extension $\mathbb{Q}(i)$, instead of $\mathbb{R}$ and $\mathbb{C}$, that is we study

$$S' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}(i) \right\},$$
$$S = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q} \right\}.$$

where S' is a Euclidean domain and S is a Dedekind finite factorization domain (see [**11**, Theorem 1 & Theorem 2]).

Again following [**7**], $\operatorname{Sin}^2 x = (1-\operatorname{Cos} x)(1+\operatorname{Cos} x)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $\operatorname{Cos} kx$ and $\operatorname{Sin} kx$ the two functions $x \mapsto \operatorname{Cos} kx$ and $x \mapsto \operatorname{Sin} kx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N} \setminus \{1\}$, $\operatorname{Cos} nx$ represents a polynomial in $\operatorname{Cos} x$ with degree n and $\operatorname{Sin} nx$ represents the product of $\operatorname{Sin} x$ and a polynomial in $\operatorname{Cos} x$ with degree $n-1$. Conversely by linearization formulas, it follows that any product $\operatorname{Cos}^n x \operatorname{Sin}^p x$ can be written as:

$$\sum_{k=0}^{q}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx), \text{ where } q \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{Q}.$$

Hence $S = \mathbb{Q}[\operatorname{Cos} x, \operatorname{Sin} x] \subseteq \mathbb{R}[\operatorname{Cos} x, \operatorname{Sin} x] = T$ and $S' = \mathbb{Q}(i)[\operatorname{Cos} x, \operatorname{Sin} x] \subseteq \mathbb{C}[\operatorname{Cos} x, \operatorname{Sin} x] = T'$.

We continue the investigations to find the factorization properties in trigonometric polynomial rings, begun in [**7**] and extended in [**11**]. In other words we extend this study towards finding factorization properties of subrings of trigonometric polynomial rings, by establishing S'_0 and S'_1 as subrings.

In Section 2 we explore S'_1 and S'_0, and demonstrate that the ring S'_1 is Euclidean domain ($\backsimeq (\mathbb{Q}[X])_X$), whereas S'_0 is a Notherian HFD ($\backsimeq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$). In Section 3 we discus Condition 1 (see [**8**, p. 661]) among the rings S'_1, S'_0 and S'. We also extend the Condition 1, as Condition 2.

2. The Subrings of $\mathbb{Q}(i)[\operatorname{Cos} x, \operatorname{Sin} x]$

A Construction of S'_1. We consider

$$S'_1 = \left\{\sum_{k=0}^{n}(a_k \operatorname{Cos} kx + ib_k \operatorname{Sin} kx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}\right\}.$$

Let $z = \sum_{k=0}^{n}(a_k \operatorname{Cos} kx + ib_k \operatorname{Sin} kx) \in S'_1$. As $\operatorname{Cos} x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\operatorname{Sin} x = \frac{1}{2i}(e^{ix} - e^{-ix})$, so

$$\begin{aligned} z &= \sum_{k=0}^{n}\Big\{\Big(\frac{a_k+b_k}{2}\Big)e^{ikx} + \Big(\frac{a_k-b_k}{2}\Big)e^{-ikx}\Big\} \\ &= e^{-inx}\left[\sum_{k=0}^{n}\Big\{\Big(\frac{a_k+b_k}{2}\Big)e^{i(n+k)x} + \Big(\frac{a_k-b_k}{2}\Big)e^{i(n-k)x}\Big\}\right], \end{aligned}$$

where $(a_k + b_k)/2, (a_k - b_k)/2 \in \mathbb{Q}$. Therefore any element z is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}[X]$ and $\deg(P) \leqslant 2n$.

Conversely, for $\alpha_k \in \mathbb{Q}$, $0 \leqslant k \leqslant 2n$, we have

$$e^{-inx}P(e^{ix}) = e^{-inx}\Big(\sum_{k=0}^{2n}\alpha_k e^{ikx}\Big) = \sum_{k=0}^{n-1}\big(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}\big) + \alpha_n.$$

As $e^{ix} = \operatorname{Cos} x + i \operatorname{Sin} x$, so

$$\begin{aligned} e^{-inx}P(e^{ix}) &= \sum_{k=0}^{n-1}\big\{\alpha_k(\operatorname{Cos}(n-k)x - i\operatorname{Sin}(n-k)x) \\ &\qquad + \alpha_{2n-k}(\operatorname{Cos}(n-k)x + i\operatorname{Sin}(n-k)x)\big\} + \alpha_n \\ &= \sum_{k=0}^{n-1}\big\{(\alpha_k + \alpha_{2n-k})\operatorname{Cos}(n-k)x \\ &\qquad + i(\alpha_{2n-k} - \alpha_k))\operatorname{Sin}(n-k)x\big\} + \alpha_n, \end{aligned}$$

where $\alpha_k + \alpha_{2n-k}$, $\alpha_{2n-k} - \alpha_k \in \mathbb{Q}$. Therefore S_1' contains all the elements that are of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}[X]$ has degree at most $2n$.

CONCLUSION 1. A consequence of the above construction is : $S_1' = \{e^{-inx}P(e^{ix}),\ n \in \mathbb{N},$ where $P(X) \in \mathbb{Q}[X]$ and $\deg(P) \leqslant 2n\}$. So we have an isomorphism $f : (\mathbb{Q}[X])_X \to S_1'$ through the substitution morphism $X \to e^{ix}$. Therefore $S_1' \backsimeq (\mathbb{Q}[X])_X$.

THEOREM 2.1. *S_1' is a Euclidean domain having nonzero elements of $\mathbb{Q}$ as units and irreducible elements, up to units, trigonometric polynomials of the form $\operatorname{Cos} x + i \operatorname{Sin} x - a$, where $a \in \mathbb{Q} \setminus \{0\}$.*

PROOF. $(\mathbb{Q}[X])_X$ is a localization of $\mathbb{Q}[X]$ by a multiplicative system generated by a prime because X is a prime in $\mathbb{Q}[X]$ [**1**, Example 1.8 (b)]. Also $\mathbb{Q}[X]$ is a Euclidean domain. Therefore $(\mathbb{Q}[X])_X$ is a Eucledean domain [**10**, Proposition 7]. Now use the isomorphism $S_1' \backsimeq (\mathbb{Q}[X])_X$ in Conclusion 1. □

A Construction of S_0'. Let $z = \sum_{k=0}^{n}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx)$, $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{Q}(i)$, such that $a_n = \alpha + \gamma + i\beta$ and $b_n = -\beta + i(\alpha - \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{Q}$; obviously $z \in S'$. We define S_0' to be the set of all the polynomials of the form $\sum_{k=0}^{n}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx)$, $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{Q}(i)$ and $a_n = \alpha + \gamma + i\beta$, $b_n = -\beta + i(\alpha - \gamma)$. Let z be a polynomial from S_0'. We may write

$$\begin{aligned} z &= a_0 + \sum_{k=1}^{n-1}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) + \big\{(\alpha+\gamma+i\beta)\operatorname{Cos} nx + (-\beta + i(\alpha-\gamma))\operatorname{Sin} nx\big\} \\ &= a_0 + \sum_{k=1}^{n-1}\Big\{\Big(\frac{a_k' + b_k'' + i(a_k'' - b_k')}{2}\Big)e^{ikx} + \Big(\frac{a_k' - b_k'' + i(a_k'' + b_k')}{2}\Big)e^{-ikx}\Big\} \\ &\qquad + (\alpha + i\beta)e^{inx} + \gamma e^{-inx}, \end{aligned}$$

where $a_k = a_k' + i a_k''$, $b_k = b_k' + i b_k''$ and $a_k', a_k'', b_k', b_k'' \in \mathbb{Q}$, $a_0 \in \mathbb{Q}(i)$. Setting $\alpha_k' = \frac{1}{2}\big(a_k' + b_k'' + i(a_k'' - b_k')\big)$ and $\beta_k' = \frac{1}{2}\big(a_k' - b_k'' + i(a_k'' + b_k')\big)$, we have

$$z = e^{-inx}\Big[a_0 e^{inx} + \sum_{k=1}^{n-1}\big\{\alpha_k' e^{i(n+k)x} + \beta_k' e^{i(n-k)x}\big\} + (\alpha + i\beta)e^{i2nx} + \gamma\Big],$$

where $\alpha_k', \beta_k', a_0 \in \mathbb{Q}(i)$ and $\alpha, \beta, \gamma \in \mathbb{Q}$. So z is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{Q}(i)[X]$ and $\deg(P) \leqslant 2n$.

Conversely, for $\alpha_0 \in \mathbb{Q}$, and $\alpha_k \in \mathbb{Q}(i)$, $1 \leqslant k \leqslant 2n$, we have

$$\begin{aligned} e^{-inx}P(e^{ix}) &= e^{-inx}\big(\alpha_0 + \alpha_1 e^{ix} + \cdots + \alpha_{2n} e^{i2nx}\big) \\ &= \alpha_0 e^{-inx} + \sum_{k=1}^{2n-1} \alpha_k e^{-i(n-k)x} + \alpha_{2n} e^{inx} \\ &= \alpha_0 e^{-inx} + \alpha_{2n} e^{inx} + \sum_{k=1}^{n-1}\big(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k} e^{i(n-k)x}\big) + \alpha_n \\ &= \alpha_0(\operatorname{Cos} nx - i \operatorname{Sin} nx) + \alpha_{2n}(\operatorname{Cos} nx + i \operatorname{Sin} nx) \end{aligned}$$

$$+\sum_{k=1}^{n-1}\big\{\alpha_k\big(\operatorname{Cos}(n-k)x - i\operatorname{Sin}(n-k)x\big) \\ + \alpha_{2n-k}(\operatorname{Cos}(n-k)x + i\operatorname{Sin}(n-k)x)\big\} + \alpha_n.$$

Take $\alpha_k = \alpha'_k + i\alpha''_k$, $\alpha_{2n-k} = \alpha'_{2n-k} + i\alpha''_{2n-k}$ and $\alpha_{2n} = \alpha'_{2n} + i\alpha''_{2n}$. Thus

$$\begin{aligned} e^{-inx}P(e^{ix}) &= (\alpha_0 + \alpha'_{2n} + i\alpha''_{2n})\operatorname{Cos} nx + (-\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0))\operatorname{Sin} nx \\ &\quad + \sum_{k=1}^{n-1}\big\{(\alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}))\operatorname{Cos}(n-k)x \\ &\qquad + (\alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k))\operatorname{Sin}(n-k)x\big\} + \alpha_n \\ &= a_n\operatorname{Cos} nx + b_n\operatorname{Sin} nx + \sum_{k=1}^{n-1}\big\{a_k\operatorname{Cos}(n-k)x + b_k\operatorname{Sin}(n-k)x\big\} + \alpha_n, \end{aligned}$$

where

$$\begin{aligned} a_n &= \alpha_0 + \alpha'_{2n} + i\alpha''_{2n}, & a_k &= \alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}), \\ b_n &= -\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0), & b_k &= \alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k). \end{aligned}$$

So, every element of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{Q}(i)[X]$ and $\deg(P) \leqslant 2n$ is in S'_0.

Conclusion 2. A consequence of above construction is: $S'_0 = \big\{e^{-inx}P(e^{ix}),\ n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{Q}+X\mathbb{Q}(i)[X] \text{ and } \deg(P) \leqslant 2n\big\}$. So we have an isomorphism $f : (\mathbb{Q}+X\mathbb{Q}(i)[X])_X \to S'_0$ through the substitution morphism $X \to e^{ix}$. Therefore $S'_0 \backsimeq (\mathbb{Q} + X\mathbb{Q}(i)[X])_X$.

Theorem 2.2. *The integral domain S'_0 is a Noetherian HFD having nonzero elements of $\mathbb{Q}(i)$ as units and trigonometric polynomials $\operatorname{Cos} x + i\operatorname{Sin} x - a$, where $a \in \mathbb{Q}(i) \smallsetminus \{0\}$ are irreducible elements, up to units.*

Proof. Since X is a prime in $\mathbb{Q} + X\mathbb{Q}(i)[X]$ [**1**, Example 1.8(b)], we have that $(\mathbb{Q}+X\mathbb{Q}(i)[X])_X$ is a localization of $\mathbb{Q}+X\mathbb{Q}(i)[X]$ by a multiplicative system generated by a prime. Also $\mathbb{Q} + X\mathbb{Q}(i)[X]$ is a Notherian HFD [**3**, Theorem 4], [**2**, Proposition 3.1]. Therefore $(\mathbb{Q} + X\mathbb{Q}(i)[X])_X$ is an HFD [**1**, Corollary 2.5] and Notherian [**14**, Corollary 1, p. 224]. Hence the isomorphism $S'_0 \backsimeq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$ in Conclusion 2 gives the result. □

The following is an analogue of [**11**, Corollary 1] and gives a factorization in S'_0 instead of S'.

Corollary 2.1. *Let $z = \sum_{k=0}^{n}(a_k\operatorname{Cos} kx + b_k\operatorname{Sin} kx)$, $n \in \mathbb{N} \smallsetminus \{1\}$, $a_k, b_k \in \mathbb{Q}(i)$ with $(a_n, b_n) \neq (0,0)$, such that $a_n = \alpha+\gamma+i\beta$ and $b_n = -\beta+i(\alpha-\gamma)$, where $\alpha, \beta, \gamma \in \mathbb{Q}$. Let d be a common divisor of the integers k such that $(a_k, b_k) \neq (0,0)$. Then z has a unique factorization*

$$\lambda(\operatorname{Cos} nx - i\operatorname{Sin} nx)\prod_{j=1}^{2n/d}(\operatorname{Cos} dx + i\operatorname{Sin} dx - \alpha_j),\ \text{where } \lambda, \alpha_j \in \mathbb{Q}(i) \smallsetminus \{0\}.$$

Proof. Since $S'_0 \subset S'$, the proof follows by [**11**, Corollary 1]. $\square$

Remark 2.1. The factorization in S'_1 is an analogue of Corollary 2.1.

Now onwards the symbol $\cap$ in all diagrams will represent the inclusion $\subseteq$.

Remark 2.2. $\mathbb{Q} + X\mathbb{Q}(i)[X]$ is a Noetherian HFD wedged between two Euclidean domains $\mathbb{Q}[X]$ and $\mathbb{Q}(i)[X]$, that is $\mathbb{Q}[X] \subseteq Q + X\mathbb{Q}(i)[X] \subseteq \mathbb{Q}(i)[X]$ and the localization of all these by a multiplicative system generated by X preserves their factorization properties as

$$\begin{array}{ccccc} \mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\ \cap & & \cap & & \cap \\ (\mathbb{Q}[X])_X & \subseteq & (\mathbb{Q} + X\mathbb{Q}(i)[X])_X & \subseteq & (\mathbb{Q}(i)[X])_X. \end{array}$$

Using Conclusion 1, Conclusion 2 and [**11**, Theorem 1], we have

$$\begin{array}{ccccc} \mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\ \cap & & \cap & & \cap \\ S'_1 & \subseteq & S'_0 & \subseteq & S', \end{array}$$

where S'_0 is a Noetherian HFD wedged between two Euclidean domains S'_1 and S'.

Remark 2.3. (a) Consider the domain extension $\mathbb{Q}[X] \subseteq (Q[X])_X$. As $X\mathbb{Q}[X]$ is a maximal ideal of $\mathbb{Q}[X]$ and $X\mathbb{Q}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}[X])^e = (\mathbb{Q}[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}[X])^e \backsimeq S'_1$ by Conclusion 1.

(b) If we consider the domain extension $\mathbb{Q}+X\mathbb{Q}(i)[X] \subseteq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$. We observe that $X\mathbb{Q}(i)[X]$ is a maximal ideal of $\mathbb{Q}+X\mathbb{Q}(i)[X]$ and $X\mathbb{Q}(i)[X]\cap(X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}(i)[X])^e = (\mathbb{Q} + X\mathbb{Q}(i)[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}(i)[X])^e \backsimeq S'_0$ by Conclusion 2.

(c) On the same lines we can apply the same result to the domain extension $\mathbb{Q}(i)[X] \subseteq (Q(i)[X])_X$. In this case $X\mathbb{Q}(i)[X]$ is a maximal ideal of $\mathbb{Q}(i)[X]$ and $X\mathbb{Q}(i)[X]\cap(X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}(i)[X])^e = (\mathbb{Q}(i)[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}(i)[X])^e \backsimeq S'$ by [**11**, Theorem 1].

Definition 2.1. Let J' be the subset of S'_1 defined by

$$J' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + ib_k \operatorname{Sin} kx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q} \text{ and } a_n = b_n \right\}.$$

Definition 2.2. Let I' be the subset of S'_0 defined by

$$I' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}(i) \text{ and } a_n = \alpha + i\beta, b_n = -\beta + i\alpha \right\}.$$

Lemma 2.1. *For the maximal ideal $X\mathbb{Q}[X]$ (respectively $X\mathbb{Q}(i)[X]$) of $\mathbb{Q}[X]$ (respectively $\mathbb{Q} + X\mathbb{Q}(i)[X]$) we have $(X\mathbb{Q}[X])_X \backsimeq J'$ (respectively $(X\mathbb{Q}(i)[X])_X \backsimeq I'$).*

Proof. Follows by Conclusion 1 (respectively Conclusion 2). $\square$

3. Conditions satisfied by ring extensions

In this section we discuss two special conditions. First one, known as Condition 1, is borrowed from [**8**] and the second one is derived from Condition 1. Moreover, we study a few interesting results about these conditions and trigonometric polynomial ring extensions satisfying them.

CONDITION 1. Let $A \subseteq B$ be a unitary (commutative) ring extension. For every $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$ [**8**, page 661].

EXAMPLE 3.1. Following [**8**, Example 1.1]; (a) If the ring extension $A \subseteq B$ satisfies Condition 1, then the ring extension $A+XB[X] \subseteq B[X]$ (or $A+XB[[X]] \subseteq B[[X]]$) also satisfies Condition 1.

(b) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, then so does the ring extension $A \subseteq C$.

(c) If B is a fraction ring of A, then the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfies Condition 1 is the generalization of localization.

(d) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1.

CONDITION 2. Let A, A_1, B and B_1 be unitary (commutative) rings such that

$$\begin{array}{ccc} A & \subseteq & B \\ \cap & & \cap \\ A_1 & \subseteq & B_1 \end{array}.$$

Then for each $x \in B_1$ there exist $x' \in U(B)$ and $x'' \in A_1$ such that $x = x'x''$.

LEMMA 3.1. *Let $A \subseteq B$ be a unitary (commutative) ring extension which satisfies Condition* 1*. If N is a multiplicative system in A, then the ring extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition* 2*.*

PROOF. Since the ring extension $A \subseteq B$ satisfies Condition 1. Therefore for each $a \in B$ there exist $b \in U(B)$ and $c \in A$ such that $a = bc$. Obviously $N^{-1}A \subseteq N^{-1}B$. Let $x = \frac{a}{s} \in N^{-1}B$, where $a \in B$, $s \in N$. This implies $x = \frac{bc}{s} = b\frac{c}{s}$, where $b \in U(B)$ and $\frac{c}{s} \in N^{-1}A$. $\square$

EXAMPLE 3.2. (a) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 2, then so does the ring extension $A \subseteq C$.

(b) By Lemma 3.1 the ring extensions $S_1' \subseteq S_0'$ and $S_0' \subseteq S'$ satisfy Condition 2 so does the ring extension $S_1' \subseteq S'$.

(c) If the ring extension $A \subseteq B$ satisfies Condition 1, then obviously it satisfies Condition 2.

PROPOSITION 3.1. *Let $A \subseteq B$ and $A_1 \subseteq B_1$ be unitary (commutative) ring extensions, where $A \subseteq A_1$ and $B \subseteq B_1$. Let M be a commen ideal of A, B, A_1 and B_1 for which the extension $A_1/M \subseteq B_1/M$ satisfies Condition* 2*. Assume for each $\alpha \in U(B_1/M)$ there exists $a \in U(B)$ such that $p(a) = \alpha$, where $p : B_1 \to B_1/M$ is the canonical surjection; then $A_1 \subseteq B_1$ satisfies Condition* 2*.*

Proof. Let $b \in B_1$. We represent the class of b by $\hat{b}$ in B_1/M. Using Condition 2, we have $\hat{b} = \hat{b}'\hat{b}''$, with $\hat{b}' \in U(B/M)$, $\hat{b}'' \in A_1/M$. By hypothesis $b' \in U(B)$, since $\hat{b}'' \in A_1/M$, for $b'' \in A_1$, we have $b = b'b'' + m = b'(b'' + b'^{-1}m)$ with $m \in M$. Thus $b'' + b'^{-1}m \in A_1$. □

Lemma 3.2. *Let $A \subseteq B$ and $A_1 \subseteq B_1$ be unitary (commutative) ring extensions, where $A \subseteq A_1$ and $B \subseteq B_1$. Let M be an ideal of A_1 that is also an ideal in B_1. If for each $b \in B_1 \setminus M$ there exists $m \in M$ such that $b + m \in U(B)$, then the extension $A \subseteq B$ satisfies Condition 2.*

Proof. If $b \in M$, then $b = 1.b$. Let $b \in B_1 \setminus M$, then there exists $m \in M$ with $b + m \in U(B)$. So we can write, $b = (b + m)(b + m)^{-1}b$ and $(b + m)^{-1}b \in A_1$, because $(b + m)^{-1}b = 1 + m'$ with $m' \in M$. □

Proposition 3.2. *Let $A \subseteq B_1 \subseteq B_2$ be a unitary (commutative) ring extension such that $A \subseteq B_2$ satisfies Condition 2. If for each $x \in U(B_1)$, we have $x \in A$ or $x^{-1} \in A$ then $B_1 = N^{-1}A$, where $N = U(B_1) \cap A$.*

Proof. The inclusion $N^{-1}A \subseteq B_1$ is obvious. Let $x \in B_2$. We can write $x = x'x''$, where $x' \in U(B_1)$, $x'' \in A$. If $x' \in A$ then $x' \in A \cap U(B_1) = N$ and $x \in A$. If $x'^{-1} \in A$ then $x'^{-1} \in A \cap U(B_1) = N$ and $x = \frac{x''}{x'^{-1}} \in N^{-1}A$. □

Remark 3.1. Consider the following commutative inclusion diagram which follows from Remark 2.2.

$$\begin{array}{ccccc} \mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\ \cap & \searrow & \cap & \searrow & \cap \\ S_1' & \subseteq & S_0' & \subseteq & S'. \end{array}$$

Now the following table concludes our discussion on Condition 1 and Condition 2 among trigonometric polynomial ring extensioins.

Ring Extension	Condition 1	Condition 2
$\mathbb{Q}[X] \subseteq \mathbb{Q} + X\mathbb{Q}(i)[X]$	No	No
$\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq \mathbb{Q}(i)[X]$	Yes	Yes
$\mathbb{Q}[X] \subseteq S_1'$	Yes	Yes
$\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq S_0'$	Yes	Yes
$\mathbb{Q}(i)[X] \subseteq S'$	Yes	Yes
$S_1' \subseteq S_0'$	No	Yes
$S_0' \subseteq S'$	No	Yes

By transitivity the domain extensions $\mathbb{Q}[X] \subseteq S_0'$, $\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq S'$ and $S_1' \subseteq S'$ also satisfy Condition 2.

References

[1] D. Anderson, D. F. Anderson and M. Zafrullah, *Factorization in integral domains II*, J. Algebra 152 (1992), 78–93

[2] D. Anderson, D. F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra 69 (1990), 1–19

[3] J. Brewer and E. A. Rutter, *$D + M$ constructions with general overrings*, Michigan Math. J. 23 (1976) 33–42
[4] P. M. Cohn, *Bezout rings and their subrings*, Proc. Camb. Phil. Soc. 64 (1968), 251–264
[5] J. Coykendall, *A characterization of polynomial rings with the Half-Factorial Property, Factorization in integral domains*, Lect. Notes Pure Appl. Math. 189, Marcel Dekker, New York 1997, 291–294
[6] A. Grams, *Atomic domains and the ascending chain condition for principal ideals*, Proc. Camb. Phil. Soc. 75 (1974), 321–329
[7] G. Picavet and M. Picavet, *Trigonometric polynomial rings. Commutative ring theory*, Lect. Notes Pure Appl. Math. 231, Marcel Dekker, 2003 , 419–433
[8] N. Radu, S. O. I. Al-Salihi, and T. Shah, *Ascend and descend of factorization properties*, Rev. Roum. Math. Pures Appl. 45 (2000), 659–669
[9] J. F. Ritt, *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$, Trans. Amer. Math. Soc. 29 (1987), 584–596
[10] P. Samuel, *About Euclidean rings*, J. Algebra 19 (1971), 282–301
[11] T. Shah and E. Ullah, *On trigonometric polynomial rings*, submitted
[12] A. Zaks, *Half-factorial domains*, J. Israel Math. 37 (1980), 281–302
[13] A. Zaks, *Atomic rings without a.c.c. on principal ideals*, J. Algebra 74 (1982), 223–231
[14] O. Zariski and P. Sammuel, *Commutative algebra*, Vol. 1, Springer-Verlag, New York, Heidelberg, Berlin, 1958

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
stariqshah@gmail.com

(Received 29 02 2008)
(Revised 30 11 2008)

Fakultät für Informatik und Mathematik
Passau Universität
Passau, Germany
ehsanu01@stud.uni-passau.de