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# PARASTROPHICALLY EQUIVALENT QUASIGROUP EQUATIONS

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Abstract. Fedir M. Sokhats'kyi recently posed four problems concerning parastrophic equivalence between generalized quasigroup functional equations. Sava Krstić in his PhD thesis established a connection between generalized quadratic quasigroup functional equations and connected cubic graphs. We use this connection to solve two of Sokhats'kyi's problems, giving also complete characterization of parastrophic cancellability of quadratic equations and reducing the problem of their classification to the problem of classification of connected cubic graphs. Further, we give formulas for the number of quadratic equations with a given number of variables. Finally, we solve all equations with two variables.

## 1. Introduction

We study generalized quadratic functional equations on quasigroups. These are equations $s = t$, where each variable appears exactly twice in $s = t$ and each operational symbol is assumed to be a quasigroup operation on a (fixed) set.

A fundamental problem in this class of equations is to investigate their structure and classify them accordingly. Our main tool in this endeavor is a cubic graph representation of the equations. In the first part of the paper we consider parastrophic equivalence as one criterion to classify quadratic equations. The second part of the paper begins a systematic classification of the equations with one and two variables based on their corresponding graphs.

The paper is organized as follows. We review necessary definitions and facts about quasigroups, quasigroup functional equations and graphs in Sections 2, 3 and

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4, respectively. In Section 5 we consider a connection between generalized quadratic quasigroup functional equations and connected cubic graphs. The four problems of Sokhats'kyi are presented in Section 6. In this section we also prove our main result about the full characterization of parastrophic (un)cancellability of equations. Section 7 is devoted to the calculation of the number of generalized quadratic equations for a given number of variables. Section 8 starts with a brief discussion of the degenerate case of the equation with one variable, and then proceeds to a full treatment of the equations with two variables giving general solutions to all nine of these equations.

## 2. Quasigroups

A quasigroup is a natural generalization of the concept of group. Quasigroups differ from groups in that they need not be associative.

Definition 2.1. We say that a groupoid $(S;\cdot)$ is a *quasigroup* if for all $a, b \in S$ there are unique solutions $x, y \in S$ to the equations $x \cdot a = b$ and $a \cdot y = b$.

A *loop* is a quasigroup with an identity element $e$, which satisfies the identities $e \cdot x = x \cdot e = x$. An associative quasigroup is a *group*.

Quasigroups are important algebraic (combinatorial, geometric) structures arising in various areas of mathematics and other disciplines. We mention just a few of their applications:

- in combinatorics (as latin squares, see Dénes and Keedwell [**4**])
- in geometry (as nets/webs, see Belousov [**3**] and [**4**])
- in statistics (see Fisher [**6**] and [**4**])
- in coding theory and cryptography (see [**4**])
- in special theory of relativity (see Ungar [**20**])

It is well known (see Belousov [**3**]) that a net can be coordinatized by an ortogonal system of quasigroups. Closure conditions in nets correspond to some (systems of) equations in their coordinate quasigroups. The equations are not always quadratic, the case we are particularly interested in, but when they are, they can be solved using methods developed mainly by Krstić [**14**] and described in this paper.

The other typical application of generalized quasigroup equations is within the theory of quasigroups. Let $s = t$ be a quadratic equation expressing a property of a quasigroup $\cdot$. We consider the generalized version of $s = t$ in which each occurrence of operation $\cdot$ (or $\backslash, /$) is replaced by a new operational symbol so that no symbol appears more than once. This new equation often gives us important information about $\cdot$ – usually that it is isotopic to some group.

* * *

Whenever unambiguous, a term like $x \cdot y$ is shortened to $xy$. Also, if many quasigroup operations are defined on the base set $S$, they are denoted by capital letters.

A quasigroup operation $\cdot$ is often considered together with its *inverse operations*: left ($\backslash$) and right ($/$) division. The inverse operations are defined by: $xy = z$

iff $x\backslash z = y$ iff $z/y = x$. Both of the inverse operations are also quasigroups. However, the inverse operations of a loop (group) operation need not be loops (groups).

It is often convenient to say that the operation $\cdot$ itself is a quasigroup, assuming the underlying base set $S$ and the division operations.

Definition 2.2. A triple groupoid $(S; \cdot, \backslash, /)$ is an *equational quasigroup* (also known as *equasigroup* or *primitive quasigroup*) if it satisfies the following axioms:

$$x\backslash xy = y, \qquad xy/y = x,$$
$$x(x\backslash y) = y\ , \qquad (x/y)y = x$$

If it further satisfies $x\backslash x = y/y$ (i.e., if the operation $\cdot$ is a loop operation) we have an *equational loop*.

The systems of quasigroups (loops) and equational quasigroups (loops) are equivalent, but the advantage of the latter is that it defines a variety.

Definition 2.3. The *dual operations* of $\cdot$, $\backslash$, $/$ are:

$$x * y = yx, \qquad x\backslash\!\!\backslash y = y\backslash x, \qquad x/\!\!/y = y/x$$

These are also quasigroup operations, and the six operations $\cdot$, $\backslash$, $/$, $*$, $\backslash\!\!\backslash$, $/\!\!/$ are said to be *parastrophes* (or conjugates) of each other.

We use the *notation* $x \circ y$ so that the symbol $\circ$ stands for either one of the operations $\cdot$ or $*$. Similarly, in $x \diamond y$ the symbol $\diamond$ stands for one of the operations $\cdot$, $\backslash$, $/$, $*$, $\backslash\!\!\backslash$, or $/\!\!/$.

When we use the prefix notation for operations and a quasigroup operation is $A$, we define: $A(x_1, x_2) = x_3$ iff $A^{(1)}(x_1, x_2) = x_3$ iff $A^{(12)}(x_2, x_1) = x_3$ iff $A^{(13)}(x_3, x_2) = x_1$ iff $A^{(23)}(x_1, x_3) = x_2$ iff $A^{(123)}(x_2, x_3) = x_1$ iff $A^{(132)}(x_3, x_1) = x_2$. In general, $A(x_1, x_2) = x_3$ iff $A^{\sigma}(x_{\sigma(1)}, x_{\sigma(2)}) = x_{\sigma(3)}$ for $\sigma \in S_3$ (symmetric group in three elements).

Definition 2.4. If $(S; \cdot)$ and $(T; \times)$ are quasigroups and $f, g, h : S \to T$ are bijections such that $f(xy) = g(x) \times h(y)$, then we say that $(S; \cdot)$ and $(T; \times)$ are *isotopic* and that $(f, g, h)$ is an *isotopy*.

Isotopy is a generalization of isomorphism. The isotopic image of a quasigroup is again a quasigroup. Every quasigroup is isotopic to some loop. A loop isotopic to a group is isomorphic to it. If two quasigroups are isotopic, so are their corresponding parastrophes.

Definition 2.5. Two quasigroups are *isostrophic* if one of them is isotopic to a parastrophe of the other.

All these relations are equivalences between quasigroups. Isomorphism is a finer relation than isotopy, which in turn is finer than isostrophy.

## 3. Functional equations on quasigroups

We use (object) variables $x_1, x_2, \ldots$. However, we also use $x$, $y$, $z$, $u$, $v$, $w$ in formulas with a small number of variables. Operation symbols (i.e. functional variables) are $F_1, F_2, \dots,$ but we use $A, B, C, \dots$ in formulas with a small number of operation symbols. We assume that all operation symbols represent quasigroup operations and that if a symbol $A$ is used, we also have symbols for the parastrophes of $A$.

Definition 3.1. A *functional equation* is an equality $s = t$, where $s$ and $t$ are terms with symbols of unknown operations occurring in at least one of them.

We write $Eq[F_1, \dots, F_n]$ to emphasize that all operation symbols of the equation $Eq$ are among $F_1, \dots, F_n$.

Definition 3.2. A *solution* to the functional equation $Eq[F_1, \dots, F_n]$ on a set $S$ is a sequence $Q_1, \dots, Q_n$ of quasigroup operations on $S$ such that $Eq[Q_1, \dots, Q_n]$ is identically true on $S$.

A *general solution* to the equation $Eq$ is a sequence of formulas

$$F_i = t_i(p_1, \dots, p_m), \quad (1 \leqslant i \leqslant n),$$

with parameters $p_1, \dots, p_m$, such that $Eq[t_1, \dots, t_n]$ is identically true on $S$ and such that every solution to the equation $Eq$ can be obtained by specifying the values of parameters.

The equation $Eq$ is *consistent* if it has at least one solution. Obviously, every functional equation has a solution on any one–element set. This solution is called *trivial* and except for proving consistency it is quite uninteresting. We further assume that solutions of functional equations are algebras of quasigroups on a given but otherwise unspecified set $S$ with more than one element.

Definition 3.3. The *length* $|t|$ of the term $t$ is the number of occurrences of object variables in it. Formally:

- If $t$ is a variable, then $|t| = 1$.
- If $t = t_1 \diamond t_2$, then $|t| = |t_1| + |t_2|$.

The length of the equation $s = t$ is $|s| + |t|$.

We define an order between terms that contain only the operation symbol $\cdot$.

Definition 3.4. The order $\lhd$ between terms is defined as follows:

- If $|s| < |t|$, then $s \lhd t$.
- For variables $x_i$ and $x_j$, $x_i \lhd x_j$ iff $i < j$.
- If $|s| = |t|$, $s = s_1 \cdot s_2$, $t = t_1 \cdot t_2$ and $s_1 \lhd t_1$, then $s \lhd t$.
- If $|s| = |t|$, $s = s_1 \cdot s_2$, $t = s_1 \cdot t_2$ and $s_2 \lhd t_2$, then $s \lhd t$.

Additionally, if we use $x, y, z, u, v, w$ as variables, we assume $x \lhd y \lhd z \lhd u \lhd v \lhd w$. Note that, if we restrict ourselves to one type of variables (either from the set $\{x_1, x_2, \dots\}$ or from the set $\{x, y, z, u, v, w\}$), the relation $\lhd$ is a total order.

Definition 3.5. The term $t$ is *linear* if every object variable appears exactly once in $t$. The functional equation $s = t$ is *linear* if both $s$ and $t$ are linear.

Definition 3.6. The functional equation $s = t$ is *quadratic* if every object variable appears exactly twice in $s = t$. The equation is *balanced* if every object variable appears exactly once in $s$ and once in $t$.

Obviously, a quadratic functional equation is balanced iff it is linear.

DEFINITION 3.7. Functional equation $s = t$ is *generalized* if every functional variable $F$ of $s = t$ (including all parastrophes of $F$) appears only once in $s = t$.

EXAMPLE 3.1. The following are various functional equations.

| | |
|---|---|
| $xy \cdot z = x \cdot yz$ | (associativity) |
| $xy \cdot zu = xz \cdot yu$ | (mediality) |
| $xy \cdot zu = (xz \cdot y)u$ | (pseudomediality) |
| $x \cdot yz = xy \cdot xz$ | (left distributivity) |
| $xy \cdot yz = xz$ | (transitivity) |
| $A(B(x,y),z) = C(x,D(y,z))$ | (generalized associativity) |
| $A(B(x,y),C(z,u)) = D(E(x,z),F(y,u))$ | (generalized mediality) |
| $A(B(x,y),C(z,u)) = D(E(F(x,z),y),u)$ | (generalized pseudomediality) |
| $A(x,B(y,z)) = D(E(x,y),F(x,z))$ | (generalized left distributivity) |
| $A(B(x,y),C(y,z)) = D(x,z)$ | (generalized transitivity) |

Associativity, mediality and pseudomediality (generalized or not) are balanced, transitivity is quadratic but not balanced, and left distributivity is not even quadratic.

Investigation of generalized balanced quasigroup equations was initiated in the important paper [**1**] by Aczél, Belousov and Hosszú where equations of generalized associativity and mediality were solved. Alimpić in [**2**] gave formulas of general solution to any generalized balanced quasigroup equation. Quadratic quasigroup equations were defined in Krapež [**11**] where a fairly wide class of them were solved. The complete solution to quadratic equations was given in Krstić [**14**].

DEFINITION 3.8. Let $Eq[F_1, \dots, F_n]$ be a generalized quadratic functional equation on quasigroups. We write $F_i \sim F_j$ $(1 \leqslant i, j \leqslant n)$ and say that $F_i$ and $F_j$ are *necessarily isostrophic* if in every solution $Q_1, \dots, Q_n$ of $Eq$ the operations $Q_i$ and $Q_j$ are isostrophic.

An operational symbol $F_i$ is {loop, group, abelian} if $Q_i$ is always isostrophic to a {loop, group, abelian group} operation.

DEFINITION 3.9. A $\sim$–class with one or two elements is called *small*, otherwise it is *big*.

DEFINITION 3.10. Two equations $Eq$ and $Eq'$ are *parastrophically equivalent* (denoted $Eq\,\mathrm{PE}\,Eq'$) if one of them can be obtained from the other by applying a finite number of the following steps:

(1) Renaming object and/or functional variables.
(2) Replacing $s = t$ by $t = s$.
(3) Replacing equation $A(t_1, t_2) = t_3$ by one of the following equations: $A^\sigma(t_{\sigma(1)}, t_{\sigma(2)}) = t_{\sigma(3)}$ for some permutation $\sigma \in S_3$.

(4) Replacing a subterm $A(t_1, t_2)$ of $s$ or $t$ by $A^{(12)}(t_2, t_1)$.
(5) Replacing a subterm $A(x, t_2)$ by a new variable $y$ and simultaneously replacing all other occurrences of $x$ by either $A^{(13)}(y, t_2)$ or $A^{(123)}(t_2, y)$.
(6) Replacing a subterm $A(t_1, x)$ by a new variable $y$ and simultaneously replacing all other occurrences of $x$ by either $A^{(23)}(t_1, y)$ or $A^{(132)}(y, t_1)$.

Theorem 3.1 (Krstić [**14**]). *Let equations $Eq[F_1, \ldots, F_n]$ and $Eq'[G_1, \ldots, G_n]$ be parastrophically equivalent. For all $i$ $(1 \leqslant i \leqslant n)$ let $G_i$ be obtained from $F_i$ by transformations described in Definition 3.10, and let $Q_1, \ldots, Q_n$ and $R_1, \ldots, R_n$ be solutions on a set $S$ of $Eq, Eq'$, respectively. Then the operations $Q_i$ and $R_i$ $(1 \leqslant i \leqslant n)$ are mutually isostrophic.*

This theorem shows why the notion of parastrophic equivalence is so important – namely, if we have a solution to a quadratic equation, then we can easily produce solutions to all equations parastrophically equivalent to it.

## 4. Graphs

Following Krstić [**14**], functional equations are represented by multigraphs. We use standard graph–theoretic notions and facts, which we review next for the sake of completeness.

A *multigraph* is a triple $(V, E; I)$, where $V$ and $E$ are disjoint sets whose elements are called *vertices* and *edges*, respectively, while $I$ is an *incidence* relation $I \subseteq V \times E$. We also assume that for every edge $e$ there are one or two vertices incident to $e$. If there is a unique vertex $v$ incident to an edge $e$, then $e$ is called a *loop* (which should not be confused with a loop as a quasigroup with an identity, see Section 2). A *simple graph* is a multigraph with no loops and no multiple edges. In this paper we shall use shorter term *graph* for multigraph and assume that all graphs are finite ($V$ and $E$ are both finite) and nontrivial ($V$ and $E$ are both nonempty).

Definition 4.1. A graph $(W, F; J)$ is a *subgraph* of a graph $(V, E; I)$ if $W \subseteq V$, $F \subseteq E$ and $J \subseteq I \cap (W \times F)$.

Definition 4.2. Two graphs $(V, E; I)$ and $(W, F; J)$ are *isomorphic* ($(V, E; I) \simeq (W, F; J)$) if there are bijections $f : V \to W$ and $g : E \to F$ such that vertices $v_1, v_2 \in V$ are incident to an edge $e \in E$ iff the vertices $f(v_1), f(v_2)$ are incident to the edge $g(e)$.

Two vertices in a graph are *adjacent* if there is an edge such that both are incident to it. Two edges are *adjacent* if there is a vertex incident to both of them. A *path* (from $v_0$ to $v_n$) in a graph is an alternating (vertex–edge) sequence $v_0, e_1, v_1, \ldots, e_n, v_n$ such that for $i = 1, \ldots, n$ the vertices $v_{i-1}$ and $v_i$ are the endvertices of the edge $e_i$. If $v_0 = v_n$ the path is *closed.* A path is *simple* if it has no subpath which is closed. A *cycle* is a simple closed path. Two paths, one from $v_1$ to $v_2$ and the other from $v_3$ to $v_4$, are disjoint if they have neither common edges nor common vertices except perhaps $v_1, v_2, v_3, v_4$.

A graph is *connected* if for every two vertices there is a path from one to the other. A *bridge* of a graph $G$ is an edge whose removal disconnects $G$. A pair of edges is a *bridge-couple* of $G$ if neither is a bridge and the removal of both disconnects $G$. A *connectivity* $c(G)$ of a graph $G$ is the smallest number such that removal of some $c(G)$ edges disconnects $G$.

THEOREM 4.1 (Menger). *For any two vertices $v_1, v_2$ of a graph $G$ there are at least $c(G)$ disjoint paths from $v_1$ to $v_2$.*

Obviously, $c(G) = 1$ iff there is a bridge in $G$ and $c(G) = 2$ iff there is no bridge in $G$ but $G$ contains a bridge-couple.

We are particularly interested in cubic graphs. A graph is *cubic* if for every vertex $v$ there are exactly three edges to which $v$ is incident, provided that if an edge is a loop it is counted twice. In a cubic graph $G$, $c(G) \leqslant 3$.

LEMMA 4.1. *If $(V, E; I)$ is a cubic graph, then there is a positive integer $n$ such that $|V| = 2n$ and $|E| = 3n$.*

DEFINITION 4.3. Two vertices $v_1$ and $v_2$ of a graph $G$ are 3-*edge-connected* if $v_1 = v_2$ or we need to remove at least three edges in $G$ to disconnect $v_1$ and $v_2$.

The 3-edge-connectivity relation between vertices of $G$ is an equivalence relation denoted by the symbol $\equiv$. To see transitivity, if $v_1 \equiv v_2$ and $v_2 \equiv v_3$, then there are at least three edge-disjoint paths from $v_1$ to $v_2$ and three edge-disjoint paths from $v_2$ to $v_3$. But then if we remove any two edges from the graph, we may disconnect at most two of the three edge-disjoint paths from $v_1$ to $v_2$ and at most two of the three edge-disjoint paths from $v_2$ to $v_3$. Thus, $v_1$ and $v_3$ remain connected by one path from $v_1$ to $v_2$ and one path from $v_2$ to $v_3$.

The vertices of a graph $G$ are partitioned by this relationship into equivalence classes called $\equiv$-classes.

DEFINITION 4.4. A $\equiv$-class with one or two elements is called *small*, otherwise it is *big*.

DEFINITION 4.5. A graph $G$ is 3-*edge-connected* if $c(G) \geqslant 3$, i.e., we need to remove at least three edges from $G$ to make $G$ disconnected.

Since we are only interested in the notion of egde-connectivity in a graph (as oposed to the vertex-connectivity), we call the 3-edge-connectivity property simply 3-connectivity. By Menger Theorem, a graph $G$ is 3-connected iff for any two vertices $v_1$ and $v_2$ of $G$ there are at least three edge-disjoint paths in $G$ from $v_1$ to $v_2$. Obviously, a cubic graph $G$ is 3-connected iff $c(G) = 3$ iff the relation $\equiv$ is the full relation on the vertices of $G$.

DEFINITION 4.6. Let vertices $v_1$ and $v_2$ be incident to an edge $e$ in a graph $G$. A new graph is said to be obtained by *the subdivision of the edge* $e$ if it is obtained from $G$ by the addition of a new vertex $v$ and the replacement of the edge $e$ by two new edges $e_1$ and $e_2$ such that $v_1$ and $v$ are incident to $e_1$ whereas $v_2$ and $v$ are incident to $e_2$.

DEFINITION 4.7. A graph $G'$ is a *subdivision* of a graph $G$ iff there is a sequence $G_1, \ldots, G_n$ of graphs such that $G = G_1$, $G' = G_n$, and $G_i$ $(1 < i \leqslant n)$ is obtained from $G_{i-1}$ by the subdivision of some edge of $G_{i-1}$.

DEFINITION 4.8. Two graphs $G$ and $H$ are *homeomorphic* iff there is an isomorphism from some subdivision of $G$ to some subdivision of $H$.

DEFINITION 4.9. A graph $G$ is *homeomorphically embeddable* into a graph $H$ iff there is a subgraph $H^{'}$ of $H$ homeomorphic to $G$.

DEFINITION 4.10. A graph $G$ is *homeomorphically embeddable into a graph $H$ within a subgraph $H'$ of $H$* iff it is homeomorphically embeddable into $H'$.

DEFINITION 4.11. A graph $G$ is *planar* if it can be represented by points (for vertices) and lines (for edges) in the Euclidean plane so that lines intersect only at vertex points.

Figure 1 shows planar cubic graph $K_4$, nonplanar noncubic graph $K_5$, and nonplanar cubic graph $K_{3,3}$.

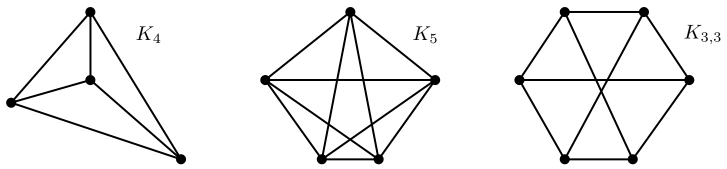


FIGURE 1. The graphs $K_4$, $K_5$, and $K_{3,3}$.

Embeddability of these graphs is an important condition for graphs relevant to the properties of associated equations (see the next section).

THEOREM 4.2 (Krstić [**14**]). *A graph $G$ consists of small $\equiv$–classes iff $K_4$ cannot be homeomorphically embedded in $G$.*

THEOREM 4.3 (Kuratowski). *A graph $G$ is planar iff neither $K_5$ nor $K_{3,3}$ can be homeomorphically embedded in $G$.*

## 5. Functional equations and their graphs

In this section we establish a connection between generalized quadratic quasigroup functional equations and connected cubic graphs. The results are mainly from Krstić [**14**] with occasional improvements. See also Krapež and Taylor [**13**].

DEFINITION 5.1. Let $s = t$ be a generalized quadratic quasigroup functional equation. The *Krstić graph* $\mathrm{K}(s = t)$ of the equation $s = t$ is a graph $(V, E; I)$ given by:

- The vertices of $\mathrm{K}(s = t)$ are operation symbols from $s = t$.
- The edges of $\mathrm{K}(s = t)$ are subterms of $s$ and $t$, including $s$ and $t$ which are considered a single edge. Likewise, any variable (which appears twice in $s = t$) is taken to be a single edge.
- If $A(p, q)$ is a subterm of $s$ or $t$, then the vertex $A$ is incident to edges $p, q, A(p, q)$ and no other.

Note that if an equation $s = t$ has a subterm of the form $A(x, x)$, then there is a corresponding loop in the graph $\mathrm{K}(s = t)$.

LEMMA 5.1. *For every generalized quadratic functional equation* $s = t$, *the graph* $\mathrm{K}(s = t)$ *is a connected cubic graph.*

Observe that Lemma 4.1 implies that if the quadratic equation $s = t$ has $n$ variables, then the graph $\mathrm{K}(s = t)$ has $2(n - 1)$ vertices and $3(n - 1)$ edges.

EXAMPLE 5.1. The parastrophically equivalent equations of generalized associativity and generalized transitivity have the same Krstić graph $K_4$, and the Krstić graph of generalized mediality is $K_{3,3}$.

The conclusion that the parastrophically equivalent equations have the same (i.e., isomorphic) Krstić graphs holds also in general.

LEMMA 5.2. *Let* $s = t$ *and* $s' = t'$ *be two parastrophically equivalent generalized quadratic functional equations. Then* $\mathrm{K}(s = t)$ *and* $\mathrm{K}(s' = t')$ *are isomorphic graphs.*

DEFINITION 5.2. Given a connected cubic graph $G = (V, E; I)$, we construct its functional equation $\mathrm{QE}(G)$ as follows.

Let $F_v$ $(v \in V)$ be operation symbols and $x_e$ $(e \in E)$ variables related to $G$. For every vertex $v$ write $F_v(x_p, x_q) = x_r$ if $vIp, vIq, vIr$ (we could use any $F_v^\sigma$ instead). Choose $v_1 \in V$, define $V_1 = V \setminus \{v_1\}$ and establish the quasiidentity $(\bigwedge_{v\in V_1} F_v(x_{p_v}, x_{q_v}) = x_{r_v}) \Rightarrow F_{v_1}(x_{p_1}, x_{q_1}) = x_{r_1}$. Denote this quasiidentity by $(\bigwedge_{v\in V_1} F_v(x_{p_v}, x_{q_v}) = x_{r_v}) \Rightarrow s_1 = t_1$.

Next, given $(\bigwedge_{v\in V_i} F_v(x_{p_v}, x_{q_v}) = x_{r_v}) \Rightarrow s_i = t_i$, choose a variable $y$ with just one occurrence in $s_i = t_i$ (there is always one such because $G$ is connected). There is a $v_{i+1} \in V_i$ such that $F_{v_{i+1}}(x_{p_i}, x_{q_i}) = x_{r_i}$ and $y$ is one of $x_{p_i}, x_{q_i}, x_{r_i}$. Then $y = F^\sigma_{v_{i+1}}(x, z)$ for $\{x, y, z\} = \{x_{p_i}, x_{q_i}, x_{r_i}\}$ and some $\sigma \in S_3$. Replace $y$ in $s_i = t_i$ by $F^\sigma_{v_{i+1}}(x, z)$to obtain $s_{i+1} = t_{i+1}$. Define $V_{i+1} = V_i \setminus \{v_{i+1}\}$. We have $(\bigwedge_{v\in V_{i+1}} F_v(x_{p_v}, x_{q_v}) = x_{r_v}) \Rightarrow s_{i+1} = t_{i+1}$.

The equation $\mathrm{QE}(G)$ is $s_{|V|} = t_{|V|}$.

LEMMA 5.3. *Let* $G$ *be a connected cubic graph. Then* $\mathrm{QE}(G)$ *is a generalized quadratic functional equation.*

Note that we can ensure the uniqueness of $\mathrm{QE}(G)$ if we prescribe the choice of $F_1$ first, and then if we take variable $y$ in $s_i = t_i$ with the smallest index. However, in view of the next lemma, it is not necessary to do so.

LEMMA 5.4. *Let* $G$ *and* $H$ *be two isomorphic connected cubic graphs. Then* $\mathrm{QE}(G)$ *and* $\mathrm{QE}(H)$ *are parastrophically equivalent equations.*

Together, Lemmas 5.2 and 5.4 give the following theorem:

THEOREM 5.1. *Generalized quadratic quasigroup functional equations* $Eq$ *and* $Eq'$ *are parastrophically equivalent iff their Krstić graphs* $\mathrm{K}(Eq)$ *and* $\mathrm{K}(Eq')$ *are isomorphic.*

THEOREM 5.2 (Krstić [**14**]). *Let* $Eq[F_1, \dots, F_n]$ *be a generalized quadratic functional equation. Then:*

- $F_i \sim F_j$ *in* $Eq$ *iff* $F_i \equiv F_j$ *in* $\mathrm{K}(Eq)$.
- *Every* $F_i$ *is a loop symbol.*
- *A symbol* $F_i$ *is a group symbol iff* $F_i/\sim$ *is big iff* $F_i/\equiv$ *is big iff* $K_4$ *is homeomorphically embeddable in* $\mathrm{K}(Eq)$ *within* $F_i/\equiv$.
- *A symbol* $F_i$ *is abelian iff the subgraph of* $\mathrm{K}(Eq)$ *defined by* $F_i/\equiv$ *is not planar iff* $K_{3,3}$ *is homeomorphically embeddable in* $\mathrm{K}(Eq)$ *within* $F_i/\equiv$.

Therefore Krstić graphs can be used to determine if equations are parastrophically equivalent or not, but also whether some or all operations occurring in an equation are necesarilly isostrophic to each other and to some (abelian) groups. Other questions on quadratic equations can be also answered using corresponding graph notions (see the next section).

## 6. The problems of Sokhats'kyi

In this and the next section we use the following convention:

- The difference between operation symbols will not be significant. We shall therefore use only one, the infix binary symbol $\cdot$, to denote any of them.
- All products will be assumed to associate to the left.

For example, using this convention, the equation $A(B(x, y), z) = C(x, D(y, z))$ of generalized associativity is represented by the equation $xyz = x \cdot yz$.

* * *

Sokhats'kyi formulated in [**17**] some problems concerning quasigroup functional equations. We cite verbatim:

PROBLEM 6.1. Construct a complete classification of uncancellable quadratic functional equations with an arbitrary number of object variables.

PROBLEM 6.2. For parastrophically uncancelablle quadratic equations, determine visual properties that distinguish equations parastrophically equivalent to the general identity of mediality from equations parastrophically equivalent to the general identity of pseudomediality.

PROBLEM 6.3. Construct a complete classification of cancellable quadratic equations.

PROBLEM 6.4. Find applications of the results obtained to the investigation of identities on quasigroup algebras, i.e., on algebras whose signature is composed of quasigroup operations.

Some partial results on the Problem 6.1 were given by Duplák [**5**] (uncancellable equations with three variables), Sokhats'kyi [**15**]–[**18**] (uncancellable equations with four variables) and Koval' [**7**]–[**10**] (uncancellable equations with five variables). The Problem 6.2 is solved in Krapež, Simić and Tošić [**12**].

The following definition of (parastrophic) cancellability is used in the formulation of the above problems.

DEFINITION 6.1 (Sokhats'kyi [**17**]). A quasigroup functional equation is *cancellable* if it has a self-sufficient sequence of subwords (a sequence of subwords of an equation is called self-sufficient if it contains all appearances of all its variables in the equation). Otherwise it is *uncancellable*.

An equation is *parastrophically cancellable* if it is parastrophically equivalent to a cancellable equation. Otherwise it is *parastrophically uncancellable*.

The definition becomes more transparent if we take into account the following lemmas.

LEMMA 6.1 (Sokhats'kyi [**17**]). *If an object variable $x$ has exactly two appearances in the functional equation $s = t$, then this equation is parastrophically equivalent to an equation $x = xt_0 \dots t_n$ for some subterms $t_0, \dots, t_n$ of $s, t$.*

The sequence $t_0, \dots, t_n$ is called *the edging* of the variable $x$ in the equation $s = t$.

For example, the equation $A(B(x,y),z) = C(x, D(y,z))$ of generalized associativity is equivalent to the equation $x = C^{(13)}(A(B(x,y),z), D(y,z))$, i.e., parastrophically equivalent to the equation (represented by) $x = xyz \cdot yz$. Since the difference between $C$ and $C^{(13)}$ disappears, it would be, perhaps, more appropriate to use the symbol $\diamond$ introduced in Section 2, instead of $\cdot$, but we keep the notation as defined by Sokhats'kyi.

LEMMA 6.2 (Sokhats'kyi [**17**]). *A cyclic permutation of an edging of the variable $x$ is also an edging of this variable in some functional equation parastrophically equivalent to the given one.*

A subsequence $t_{i+1}, \dots, t_{i+j}$, where $+$ is the addition modulo $n+1$, is called an *edging arc*. If an arc contains all appearances of all of its variables, then it is called *self-sufficient*.

LEMMA 6.3 (Sokhats'kyi [**17**]). *If a variable of the equation $s = t$ has a self-sufficient edging arc, then this equation is parastrophically cancellable.*

Finally, building on work of Sokhats'kyi [**17**] and Krstić [**14**] we are able to look at parastrophic (un)cancellability of equations from different perspective.

THEOREM 6.1. *The following statements are mutually equivalent:*

(1) *A generalized quadratic quasigroup functional equation $Eq$ is parastrophically uncancellable.*
(2) *The relation $\equiv$ is the full relation on vertices of* $\mathrm{K}(Eq)$.
(3) *The relation $\sim$ is the full relation on operation symbols of $Eq$.*
(4) $c(\mathrm{K}(Eq)) = 3$*, i.e.,* $\mathrm{K}(Eq)$ *is* 3*-connected.*

PROOF. ($1 \Leftrightarrow 2$) To show $2 \Rightarrow 1$, assume that the equation $Eq$ is cancellable. Then $Eq$ is parastrophically equivalent to some equation $x = xt_0 \cdots t_n$ with a self-sufficient edging arc $t_{i+1}, \dots, t_n$ ($i < n$). The equivalent equation is just a shorthand for the equation pictured in Figure 2.

The Krstić graph $\mathrm{K}(Eq)$ of $Eq$ is shown in Figure 3, where $T_m$ are graphs of terms $t_m$ ($0 \leqslant m \leqslant n$).

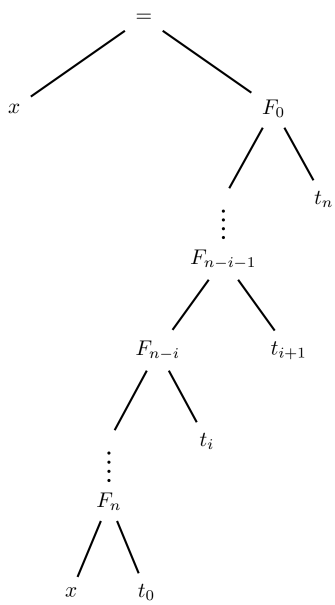


FIGURE 2. The tree of $x = xt_0 \cdots t_n$.

Note that some $T_j$ and $T_k$ are connected via common variables of $t_j$ and $t_k$ but, because of the self-sufficiency of $t_{i+1}, \dots, t_n$, this never happens for $0 \leqslant j \leqslant i < k \leqslant n$. Therefore, there are only two disjoint paths in $\mathrm{K}(Eq)$ from $F_0$ to $F_n$: the first one via $x$ and the second through the vertices $F_1, F_2, \dots, F_{n-i-1}, F_{n-i}, \dots, F_{n-1}$.

Actually, there are other paths from $F_0$ to $F_n$ (via some of $T_m$'s) but they all contain edge $y$, so cannot be disjoint from the path through the vertices $F_1, F_2, \dots, F_{n-i-1}, F_{n-i}, \dots, F_{n-1}$. Consequently, $F_0 \equiv F_n$ is not true and so $\equiv$ is not the full relation.

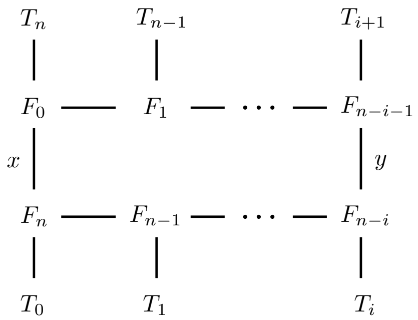


FIGURE 3. The graph $\mathrm{K}(Eq)$.

To show the other direction $1 \Rightarrow 2$, assume that an equation $Eq$ is given such that the relation $\equiv$ on vertices of $\mathrm{K}(Eq)$ is not full. Then $\mathrm{K}(Eq)$ has either a bridge or a bridge-couple. We show that in both cases the equation $Eq$ is parastrophically cancellable.

Case (i): There is a bridge in $\mathrm{K}(Eq)$.

If we construct $\mathrm{QE}(\mathrm{K}(Eq))$ using Definition 5.2 starting from the bridge, then $Eq$ is parastrophically equivalent to an equation $s = t$ such that the sets of variables of $s$ and $t$ are disjoint. Since $Eq$ is quadratic, $s$ must be a product $pq$. Without loss of generality we may assume that there is a variable in $p$ which also occurs in $q$.

If this is not the case, then all variables of $p$ and $q$ are disjoint and equation $pq = t$ is equivalent to $p = q/\!\!/t$. Thus it is parastrophically equivalent to $p = qt$ and variables of $p$ and $qt$ are also disjoint. This procedure can be repeated if necessary, until, since regression must be finite, we get the product with a variable in both factors.

So let us assume that a variable $x$ occurs in both $p$ and $q$. Equation $Eq$ is parastrophically equivalent to the equation $xs_1 \dots s_i \cdot xs_{i+1} \dots s_j = t$ for some subterms $s_1, \dots, s_j$ $(0 \leqslant i \leqslant j)$ of $s$. The last equation is parastrophically equivalent to $x = xs_{i+1} \dots s_j t s_i \dots s_1$ and therefore $t$ is self-sufficient edging arc of the variable $x$. By the Lemma 6.3 the equation $s = t$ is parastrophically cancellable and so is $Eq$.

Case (ii): There is no bridge but there is a bridge-couple (with edges $x, y$) in $\mathrm{K}(Eq)$.

Making $\mathrm{QE}(\mathrm{K}(Eq))$ as in Definition 5.2 and starting from the variable $y$, we get the equation $s = t$, parastrophically equivalent to $Eq$ and such that the variable $x$ occurs in both $s$ and $t$, while all other variables of $s$ and $t$ are disjoint. We can rewrite $s = t$ in the form $xs_1 \dots s_i = xt_1 \dots t_j$ for some subterms $s_1, \dots, s_i$ of $s$ and $t_1, \dots, t_j$ of $t$. Consequently, the equation $s = t$ is parastrophically equivalent to the equation $x = xt_1 \dots t_j s_i \dots s_1$ with the sequence $t_1, \dots, t_j$ being a self-sufficient edging arc for $x$. This proves that both $s = t$ and $Eq$ are parastrophically cancellable.

$(2 \Leftrightarrow 3)$ That the relation $\sim$ is full iff $\equiv$ is full follows from Theorem 5.2.

$(2 \Leftrightarrow 4)$ Obviously, a cubic graph $G$ is 3-connected iff $c(G) = 3$ iff the relation $\equiv$ is the full relation on $V$. □

This is our main result. Together with Theorem 5.1, it gives the full characterization of parastrophic (un)cancellability of equations. Therefore, it solves Sokhat'skyi problems 6.1 and 6.3.

## 7. How many generalized quadratic functional equations are there?

In this section we give a count of all generalized quadratic equations with $n$ variables, as well as a count of their normal subset. Normal equations (defined below) avoid repetitions of equations with nonessential differences such as variable substitution. The numbers of generalized and normal generalized quadratic equations with $n$ variables are denoted by $E_n$ and $e_n$, respectively.

We also pose the problem of finding a general formula for the sequence of numbers $\pi_n$ of parastrophically nonequivalent generalized quadratic quasigroup equations with $n$ variables.

THEOREM 7.1. *The total number of generalized quadratic quasigroup functional equations with n variables is*

$$E_n = \frac{(4n-2)!}{2^n(2n-1)!}$$

PROOF. Let $s = t$ be a quadratic equation with $n$ variables. The equation is fully determined by the binary tree of the equation $s = t$ and the order in which variables occur in the equation. The tree and the order are independent of each other.

i) It is well known that the number of different binary trees with $n$ leaves is $C_{n-1}$, where $C_n$ ($n \geqslant 0$) is the sequence of *Catalan numbers.* The sequence satisfies the formula: $C_n = (2n)!/(n+1)!n!$ and is denoted by A000108 in Sloane's "The On-Line Encyclopedia of Integer Sequences" [**19**]. The first ten members of the sequence $C_n$ ($0 \leqslant n \leqslant 9$) are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862.

The tree of the equation $s = t$ is a binary tree with the root '=', its left subtree being the tree of $s$ and its right subtree being the tree of $t$. Therefore there are $T_n = C_{2n-1}$ different trees of quadratic equations with $n$ variables. The first five members of the sequence $T_n$ ($n \geqslant 1$) are 1, 5, 42, 429, 4862. This is the Sloane sequence A024492.

ii) The order of variables in $s = t$ is determined by the word of length $2n$ in which every one of the letters $x_1, \dots, x_n$ appears exactly twice. Let us denote by $W_n$ the number of such words. It is easy to see that the sequence $W_n$ satisfies the recurrence relation:

$$W_1 = 1, \quad W_n = n(2n-1)W_{n-1}$$

$W_n$ is the Sloane sequence A000680, and the general formula is $W_n = (2n)!/2^n$. The first five members of $W_n$ are 1, 6, 90, 2520, 113400.

iii) As noted before, the number of all quadratic equations with $n$ variables is $E_n = T_nW_n = C_{2n-1}W_n = (4n-2)!/2^n(2n-1)!$. The recurrence relation is $E_{n+1} = 2(16n^2 - 1)E_n$ and the first five members of this sequence are 1, 30, 3780, 1081080, 551350800. This sequence of numbers is not listed among the Sloane sequences. □

There is a vast number of repetitions among above equations – the difference being just the order of variables which is not essential since any permutation of variables gives basically the same equation. Likewise, the equations $s = t$ and $t = s$ are essentially the same so we can choose just one. Note that we cannot delete equations of the type $t = t$ although they are obviously equivalent to $x = x$. This is because our equations just *represent* generalized equations. For example, the equation $xy = xy$ stands for the equation $A(x, y) = B(x, y)$ which is not equivalent to $x = x$. Instead, it states certain relationship between operations $A$ and $B$.

DEFINITION 7.1. A generalized quadratic quasigroup functional equation $s = t$ is called *normal* if:

- For $1 \leqslant i < j \leqslant n$, the first occurrence of $x_i$ in $s = t$ appears before the first occurrence of $x_j$.
- If the terms $s, t$ are not identical, then $t \lhd s$.

THEOREM 7.2. *The total number of normal generalized quadratic quasigroup functional equations with n variables is*

$$e_n = \frac{(2n)!}{2^{n+1}n!}(C_{2n-1} + C_{n-1})$$

PROOF. The calculations are similar to those in the previous theorem.

i) The sequence $t_n$ $(n \geqslant 1)$ of numbers of trees we get if we exclude all trees of equations $s = t$, where $s \lhd t$ and the formula is: $t_n = (C_{2n-1} + C_{n-1})/2$. We get it from the following formula for Catalan numbers: $C_n = \sum_{i=1}^{n} C_i C_{n-i}$ and the observation that for $t_n$ we use only cases with $i \geqslant n - i$.

The first five members of $t_n$ are 1, 3, 22, 217, 2438. The sequence is not listed among the Sloane sequences.

ii) We get the number $w_n$ of words of length $2n$ by taking only words in which variables occur in fixed order: $x_1, \ldots, x_n$ (ignoring the repetitions). Therefore $w_n = W_n/n! = (2n)!/2^n n!$. This sequence is the Sloane sequence A001147 and the first five members are 1, 3, 15, 105, 945.

iii) The number $e_n$ $(n \geqslant 1)$ of normal equations is

$$e_n = t_n w_n = \frac{(2n)!}{2^{n+1}n!}(C_{2n-1} + C_{n-1})$$

The first five members of the sequence $e_n$ are 1, 9, 330, 22785, 2303910. This sequence is not present in the Sloane list of sequences. $\square$

We also define the sequence $\pi_n$ $(n \geqslant 1)$, where $\pi_n$ is the number of classes of parastrophically nonequivalent generalized quadratic quasigroup equations with $n$ variables. According to Theorem 5.1 $\pi_n$ is also the number of nonisomorphic connected cubic graphs with $2(n-1)$ vertices and $3(n-1)$ edges. By the definition (see section 8), $\pi_1 = 1$. We prove that $\pi_2 = 2$. We know that $\pi_3 = 5$ and $\pi_4 = 17$, but the proof will be published elsewhere. We do not know a general formula for $\pi_n$. Therefore:

PROBLEM 7.1. Find a general formula for the sequence $\pi_n$ $(n \geqslant 1)$.

## 8. Equations with one and two variables

As an example of an application of general results, we give solutions of all normal generalized quadratic quasigroup equations with (one and) two variables. As for equations with more variables, we can solve any such *particular equation* (using methods provided by Krstić), but the complexities of connected cubic graphs prevent us from producing a *formula in closed form* giving rise to general solutions of all such equations. This problem requires further investigation.

* * *

The special case of equations with one variable is easy, since there is only one such equation: $x = x$. However, there are no operation symbols in $x = x$ so it does not fit our definition of functional equation. Despite this we *define* $\pi_1$ to be 1.

The case of equations with two variables is much more interesting. There are 30 generalized quadratic quasigroup functional equations with two variables, nine of them normal. They are:

$$A(x,x) = B(y,y) \tag{8.1}$$

$$A(x,y) = B(x,y) \tag{8.2}$$

$$A(x,y) = B(y,x) \tag{8.3}$$

$$A(x,B(x,y)) = y \tag{8.4}$$

$$A(x,B(y,x)) = y \tag{8.5}$$

$$A(x,B(y,y)) = x \tag{8.6}$$

$$A(B(x,x),y) = y \tag{8.7}$$

$$A(B(x,y),x) = y \tag{8.8}$$

$$A(B(x,y),y) = x. \tag{8.9}$$

The following lemmas are useful in solving these equations.

LEMMA 8.1. *Let $S$ be a nonempty set, $e \in S$ and $\sigma$ a permutation of $S$. A general solution to the equation*

$$\sigma A(x,x) = e \tag{8.10}$$

*in the quasigroup $A$ on $S$ is given by:*

$$A(x,y) = \sigma^{-1}\alpha L^{(23)}(\sigma x, \sigma y)$$

*where:*

- *$L$ is an arbitrary loop on $S$ with the identity $e$*
- *$\alpha$ is an arbitrary permutation of $S$ such that $\alpha e = e$.*

PROOF. We prove first that the above formulas always give a solution to the equation (8.10).

Since $L$ is a loop, we have $L(x,e) = x$, so $L^{(23)}(x,x) = e$. It follows that $\sigma A(x,x) = \sigma\sigma^{-1}\alpha L^{(23)}(\sigma x, \sigma x) = \alpha e = e$.

Next, we prove that every solution to equation (8.10) is of the form given in the statement of the lemma.

Let $A$ be a particular quasigroup on $S$ which satisfies (8.10). Pick any $p \in S$ and define: $a = A(p,p)$, $\alpha x = \sigma A(a, \sigma^{-1}x)$ and $L(x,y) = \sigma A^{(23)}(\sigma^{-1}x, \sigma^{-1}\alpha y)$. It follows that $\sigma a = \sigma A(p,p) = e$, function $\alpha$ is permutation and

$$\alpha e = \sigma A(a, \sigma^{-1}e) = \sigma A(a,a) = e.$$

The operation $L$ is a quasigroup since it is an isostrophe of the quasigroup $A$. Moreover, $L$ is a loop with identity $e$. To see this, first observe that

$$\begin{aligned} L(e,x) &= \sigma A^{(23)}(\sigma^{-1}e, \sigma^{-1}\alpha x) = \sigma A^{(23)}(a, \sigma^{-1}\sigma A(a, \sigma^{-1}x)) \\ &= \sigma A^{(23)}(a, A(a, \sigma^{-1}x)) = \sigma\sigma^{-1}x = x. \end{aligned}$$

On the other hand, $A(\sigma^{-1}x, \sigma^{-1}x) = \sigma^{-1}e = a$ implies $A^{(23)}(\sigma^{-1}x, a) = \sigma^{-1}x$ and thus

$$\begin{aligned} L(x,e) &= \sigma A^{(23)}(\sigma^{-1}x, \sigma^{-1}\alpha e) = \sigma A^{(23)}(\sigma^{-1}x, \sigma^{-1}e) \\ &= \sigma A^{(23)}(\sigma^{-1}x, a) = \sigma\sigma^{-1}x = x. \end{aligned}$$

From the definition of $L$ it follows that $A(x,y) = \sigma^{-1}\alpha L^{(23)}(\sigma x, \sigma y)$. This completes the proof. □

The proofs of the next two lemmas are similar, so we skip them.

LEMMA 8.2. *Let $S$ be a nonempty set, $q \in S$ and $\sigma$ a permutation of $S$. A general solution to the equation*

$$\sigma A(x,q) = x$$

*in the quasigroup $A$ on $S$ is given by:*

$$A(x,y) = \sigma^{-1}L(x, \alpha y)$$

*where:*

- *$L$ is an arbitrary loop on $S$ with identity $e$*
- *$\alpha$ is an arbitrary permutation of $S$ such that $\alpha q = e$.*

LEMMA 8.3. *Let $S$ be a nonempty set, $p \in S$ and $\sigma$ a permutation of $S$. A general solution to the equation*

$$\sigma A(p,x) = x$$

*in the quasigroup $A$ on $S$ is given by:*

$$A(x,y) = \sigma^{-1}L(\alpha x, y)$$

*where:*

- *$L$ is an arbitrary loop on $S$ with identity $e$*
- *$\alpha$ is an arbitrary permutation of $S$ such that $\alpha p = e$.*

The graph $\mathrm{K}(Eq)$ of an equation $Eq$ with 2 variables has 2 vertices and 3 edges. There are only two such non-isomorphic graphs shown in Figure 4: the dumbbell graph and the dipole $D_3$ graph. Therefore, $\pi_2 = 2$.

The dumbbell graph corresponds to equations (8.1), (8.6) and (8.7). The $\sim$-classes are singletons. General solutions to these equations are given in the next three theorems.

the dumbbell graph the dipole $D_3$ graph

FIGURE 4. Two non-isomorphic graphs with 2 vertices and 3 edges.

THEOREM 8.1. *A general solution to the equation* (8.1) *on a set $S$ is given by:*

$$A(x,y) = \alpha L_1^{(23)}(x,y)$$
$$B(x,y) = \beta L_2^{(23)}(x,y)$$

*where:*

- *$L_1$ and $L_2$ are arbitrary loops on $S$ with a common identity $e$.*
- *$\alpha$ and $\beta$ are arbitrary permutations on $S$ such that $\alpha e = e$, $\beta e = e$.*

PROOF. The equation (8.1) is equivalent to the system

$$A(x,x) = e, \quad B(y,y) = e$$

for some $e \in S$. Both equations are special cases of (8.10) for $\sigma = \mathrm{Id}$, where $\mathrm{Id}(x) = x$ is the identity function on $S$. By Lemma 8.1, the general solution to $A(x,x) = e$ is given by $A(x,y) = \alpha L_1^{(23)}(x,y)$, where $L_1$ is an arbitrary loop on $S$ with identity $e$ and $\alpha$ is a permutation of $S$ such that $\alpha e = e$. Analogously, $B(x,y) = \beta L_2^{(23)}(x,y)$, where $L_2$ is an arbitrary loop on $S$ with identity $e$ and $\beta$ is a permutation of $S$ such that $\beta e = e$.

Combined together, the last two statements complete the proof. □

Proofs of general solutions to equations (8.6) and (8.7) are similar.

THEOREM 8.2. *A general solution to the equation* (8.6) *on a set $S$ is given by:*

$$A(x,y) = L_1(x,\alpha y),$$
$$B(x,y) = \alpha^{-1}\beta L_2^{(23)}(\alpha x,\alpha y),$$

*where:*

- *$L_1$ and $L_2$ are arbitrary loops on $S$ with a common identity $e$.*
- *$\alpha$ and $\beta$ are arbitrary permutations on $S$ such that $\beta e = e$.*

THEOREM 8.3. *A general solution to the equation* (8.7) *on a set $S$ is given by:*

$$A(x,y) = L_1(\alpha x, y)$$
$$B(x,y) = \alpha^{-1}\beta L_2^{(23)}(\alpha x,\alpha y)$$

*where:*

- *$L_1$ and $L_2$ are arbitrary loops on $S$ with a common identity $e$*
- *$\alpha$ and $\beta$ are arbitrary permutations on $S$ such that $\beta e = e$.*

As the representative equation for this class of parastrophically equivalent equations we take the equation (8.1).

The dipole $D_3$ graph corresponds to the rest of the equations: (8.2)–(8.5), (8.8) and (8.9). The relation $\sim$ is the full relation, so $A \sim B$. General solutions to these equations are given in the next theorems.

THEOREM 8.4. *A general solution to the equation* (8.2) *on a set $S$ is given by:*

$$A(x, y) = L(A_1x, A_2y)$$
$$B(x, y) = L(B_1x, B_2y)$$

*where:*

- *$L$ is an arbitrary loop on $S$*
- *$A_1, A_2, B_1, B_2$ are arbitrary permutations on $S$ such that $A_1 = B_1$, $A_2 = B_2$.*

The proof of this theorem is only slightly more complicated then the proof of Theorem 8.1. However, in this and the remaining cases there is a much simpler form of a general solution – note that equations are just requirements for $B$ to be a certain parastrophe of $A$. Therefore, the following theorem is true.

THEOREM 8.5. *A general solution to the equation* {(8.2), (8.3), (8.4), (8.5), (8.8), (8.9)} *on a set $S$ is given by:*

- $A = Q$
- $B = Q^\sigma$ $\{\sigma = (1),\ \sigma = (12),\ \sigma = (23),\ \sigma = (132),\ \sigma = (123),\ \sigma = (13)\}$.

*where $Q$ is an arbitrary quasigroup on $S$.*

We take equation (8.2) as the representative one for this class of parastrophically equivalent equations.

The results concerning equations with two variables are summarized in Table 1.

| PE–class | Graph | Number of $\sim$–classes | Number of equations | Representative equation |
|---|---|---|---|---|
| 1 | dumbbell | 2 | 3 | (8.1) |
| 2 | $D_3$ | 1 | 6 | (8.2) |

TABLE 1. Summary results on equations with two variables

Neither Krapež [**11**] nor Krstić [**14**] considered equations (8.1)–(8.9). If they had done, their solutions would have been similar to those given in Theorem 8.4. There are two novelties in our approach:

- The use of unipotent quasigroups instead of loops in solutions of (8.1), (8.6) and (8.7)
- The use of parastrophes of the quasigroup $Q$ in remaining equations

with a result of increased simplicity.

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