PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 87(101) (2010), 121–128 DOI: 10.2298/PIM1001121D

ON THE COPRIMALITY OF SOME ARITHMETIC FUNCTIONS

Jean-Marie De Koninck and Imre Kátai

Dedicated to Professor Aleksandar Ivić on the occasion of his 60^{th} *anniversary*

Abstract. Let φ stand for the Euler function. Given a positive integer n, let $\sigma(n)$ stand for the sum of the positive divisors of n and let $\tau(n)$ be the number of divisors of n. We obtain an asymptotic estimate for the counting function of the set $\{n : \gcd(\varphi(n), \tau(n)) = \gcd(\sigma(n), \tau(n)) = 1\}$. Moreover, setting $l(n) := \gcd(\tau(n), \tau(n+1))$, we provide an asymptotic estimate for the size of $\#\{n \leqslant x : l(n) = 1\}$.

1. Introduction

Let φ stand for the Euler function. Given a positive integer n, let $\sigma(n)$ stand for the sum of the positive divisors of n and let $\tau(n)$ be the number of divisors of n. This last function has been extensively studied by A. Ivić in his book on the Riemann Zeta-Function [**6**].

Given an arithmetical function f and a large number x, examining the number of positive integers $n \leqslant x$ for which $\gcd(n, f(n)) = 1$, has been the focus of several papers. For instance, Paul Erdős [**4**] established that

$$\#\{n \leqslant x : \gcd(n, \varphi(n)) = 1\} = (1 + o(1))\frac{e^{-\gamma}x}{\log\log\log x},$$

where γ is the Euler constant. A similar result can be obtained if one replaces $\varphi(n)$ by $\sigma(n)$. Similarly, letting $\Omega(n)$ stand for the number of prime factors of n counting their multiplicity, Alladi [**1**] proved that the probability that n and $\Omega(n)$ are relatively prime is equal to $6/\pi^2$ by examining the size of $\{n \leqslant x : \gcd(n, \Omega(n)) = 1\}$. Let $K(x)$ stand for the number of positive integers $n \leqslant x$ such that $\gcd(n\tau(n), \sigma(n)) = 1$. Some fifty years ago, Kanold [**7**] showed that there exist positive constants $c_1 < c_2$ and a positive number x_0 such that

$$c_1 < K(x)/\sqrt{x/\log x} < c_2 \qquad (x \geqslant x_0).$$

2010 *Mathematics Subject Classification*: 11A05, 11A25, 11N37.
Key words and phrases: Arithmetic functions, number of divisors, sum of divisors.

In 2007, the authors [**2**] proved that there exists a positive constant c_3 such that $K(x) = c_3(1+o(1))\sqrt{x/\log x}$ $(x \to \infty)$. The analogue problem for counting the number of positive integers n for which

$$\gcd(n\tau(n), \varphi(n)) = 1 \tag{1.1}$$

is trivial. Clearly (1.1) holds for $n = 1, 2$. But these are the only solutions. Indeed, assume that (1.1) holds for some $n \geqslant 3$. Then n is squarefree and it must therefore have an odd prime divisor p, in which case $2 \mid \varphi(n)$ and $2 \mid \tau(n)$, implying that $\gcd(n\tau(n), \varphi(n)) > 1$, thereby proving our claim.

In this paper, we obtain asymptotic estimates for the counting functions

$$R(x) := \#\big\{n \leqslant x : \gcd(\varphi(n), \tau(n)) = \gcd(\sigma(n), \tau(n)) = 1\big\}$$
$$N(x) := \#\{n \leqslant x : l(n) = 1\},$$

where $l(n) := \gcd(\tau(n), \tau(n+1))$.

From here on, $\gcd(a, b)$ will be written simply as (a, b). In what follows, we shall denote the logarithmic integral of x by $\operatorname{li}(x)$, that is $\operatorname{li}(x) := \int_2^x \frac{dt}{\log t}$. Moreover, given an integer $n \geqslant 2$, we shall let $\omega(n)$ stand for the number of distinct prime factors of n, with $\omega(1) = 0$. Finally, the letters $c_1, c_2, \ldots$ will stand for positive constants, while the letters p and q, with or without subscripts, will always stand for prime numbers.

2. Main results

THEOREM 1. *As $x \to \infty$, we have $R(x) = c_4(1+o(1))\frac{x}{\sqrt{\log x}}$, where c_4 is a suitable positive constant.*

THEOREM 2. *As $x \to \infty$, we have $N(x) = c_5(1+o(1))\sqrt{x}$ for some positive constant c_5.*

3. Preliminary results

To prove our results we shall need the following lemmas.

LEMMA 1. *Let $f(n) := An^2 + Bn + C$ be a primitive polynomial with integer coefficients. Let $\rho(m)$ be the number of solutions of $f(n) \equiv 0 \pmod{m}$. Let D be the discriminant of f and assume that $D \neq 0$. Then ρ is a multiplicative function whose values on the prime powers satisfy*

$$\rho(p^\alpha) \begin{cases} = \rho(p) & \text{if } p \nmid D, \\ \leqslant 2D^2 & \text{if } p \mid D. \end{cases}$$

Finally, setting

$$M_f(x, y) := \#\big\{n \leqslant x : \exists p > y \text{ such that } p^2 | f(n)\big\},$$

then

$$\lim_{y\to\infty} \limsup_{x\to\infty} \frac{M_f(x, y)}{x} = 0.$$

PROOF. For a proof of this result, see Chapter 4 in the book of Hooley [**5**]. □

Lemma 2. *As $x \to \infty$, we have*

$$\sum_{m \leqslant x} |\mu(m)| \cdot |\mu(m^2-1)| = \xi_1(1+o(1))x,$$
$$\sum_{m \leqslant x} |\mu(m)| \cdot |\mu(m^2+1)| = \xi_2(1+o(1))x,$$

where ξ_1 and ξ_2 are positive constants.

Proof. The proof is a simple application of the Sieve of Eratosthenes and we shall therefore skip it. □

4. The proof of Theorem 1

Let R be the set of those integers n for which

$$(\varphi(n), \tau(n)) = (\sigma(n), \tau(n)) = 1. \tag{4.1}$$

Clearly, we can ignore all solutions of (4.1) which are powers of 2 (namely the even powers of 2). Hence, we only need to consider those solutions n of (4.1) such that $p|n$ for some odd prime p. In this case $\varphi(n)$ must be even, meaning that $\tau(n)$ must be odd, implying that $n = u^2$ for some positive integer u. Now, the size of the set of those integers $n = u^2 \leqslant x$ for which u is a squarefull number and with n satisfying (4.1) is small since it is clearly no larger than $cx^{1/4}$ for some constant $c > 0$. Ignoring these integers n, we may assume that $3|\tau(n)$ and consequently that 3 does not divide $\varphi(u^2) = u\varphi(u)$.

Let us now write $u = Kv$, where K is squarefull and v is squarefree, with $(K, v) = 1$. Assume that $v > 1$. Then we have

$$(\varphi(n), \tau(n)) = (\varphi(K^2)\varphi(v^2), 3\tau(K^2)),$$
$$(\sigma(n), \tau(n)) = (\sigma(K^2)\varphi(v^2), 3\tau(K^2)).$$

For each squarefull integer K, let R_K be the set of those $n = u^2 \in R$ for which $u = Kv$ and let $R_K(x) = \{n \leqslant x : n \in R_K\}$. It is clear that $R_K(x) \leqslant \frac{\sqrt{x}}{K}$, implying that

$$\sum_{K > \log^2 x} R_K(x) \leqslant \sqrt{x} \sum_{K > \log^2 x} \frac{1}{K} \ll \frac{\sqrt{x}}{\log x}. \tag{4.2}$$

It follows from this that we only need to consider those squarefull numbers $K \leqslant \log^2 x$.

Let $n \in R_K$. Then, $n = v^2K^2 \leqslant x$, where v is a squarefree number whose prime factors are $\equiv -1 \pmod 3$. Hence,

$$v \leqslant \frac{\sqrt{x}}{K} \quad \text{with} \quad \left(v, \prod_{\substack{p \leqslant \sqrt{x} \\ p \equiv 1 \pmod 3}} p \right) = 1.$$

Therefore, by standard sieve techniques, one can easily establish that, for some positive constant c_6,

$$R_K(x) \leqslant c_6 \frac{\sqrt{x}}{K\sqrt{\log x}}. \tag{4.3}$$

Since $\sum_{K \text{ squarefull}} \frac{1}{K} < +\infty$, it follows from (4.3) that

$$\sum_{K>y} R_K(x) \leqslant o(1) \cdot c_6 \frac{\sqrt{x}}{\sqrt{\log x}} \qquad (y \to \infty) \tag{4.4}$$

Let us now estimate $R_K(x)$ for a fixed squarefull number K. We separate the different squarefull K's into two classes:

$$\text{Class I} = \{K : \tau(K^2) = \text{ power of } 3\},$$
$$\text{Class II} = \{K : \tau(K^2) \neq \text{ power of } 3\}.$$

But first consider the case $K = 1$. In this case $u = v \leqslant \sqrt{x}$, and the prime factors p of u satisfy $p \equiv 1 \pmod 3$. On the other hand $(u, 3) = 1$. Hence, letting $u = q_1 q_2 \cdots q_r$, with $5 \leqslant q_1 < q_2 < \cdots < q_r$, it follows that

$$\tau(u^2) = 3^r, \quad \varphi(u^2) = u \prod_{j=1}^{r} (q_j - 1), \quad \sigma(u^2) = \prod_{j=1}^{r} (1 + q_j + q_j^2).$$

Since $(\varphi(u^2), 3) = 1$ and $(\sigma(u^2), 3) = 1$, it follows that $u^2 \in R_1$.

Hence, $R_1(x) = \#\big\{u \leqslant \sqrt{x} : u \text{ squarefree}, (p, u) = 1 \text{ if } p \equiv -1 \pmod 3\big\}$. Since

$$\sum_{u^2 \in R_1} \frac{1}{u^s} = \prod_{p \equiv -1 \pmod 3} \Big(1 + \frac{1}{p^s}\Big),$$

one can use the classical method of Landau (see his book [**9**, pp. 641–649]) and deduce that

$$R_1(x) = c_7 \sqrt{\frac{x}{\log x}} \left(1 + O\Big(\frac{1}{\log\log x}\Big)\right), \tag{4.5}$$

for some positive constant c_7.

Now, assume that $K \in$ class I, in which case $(\sigma(K^2), 3) = 1$ and $(\varphi(K^2), 3) = 1$. Then $n = K^2 v^2 \leqslant x$, with $(K, v) = 1$, belongs to R_K if and only if v is squarefree and all its prime factors p satisfy $p \equiv -1 \pmod 3$, in which case

$$\sum_{v} \frac{1}{v^s} = \prod_{\substack{p \equiv -1 \pmod 3 \\ (p,K)=1}} \Big(1 + \frac{1}{p^s}\Big) = \prod_{p \mid K} \Big(1 + \frac{1}{p^s}\Big)^{-1} \prod_{p \equiv -1 \pmod 3} \Big(1 + \frac{1}{p^s}\Big).$$

It follows that, for $K \in$ class I,

$$R_K(x) = c_7 \prod_{p \mid K} \Big(1 + \frac{1}{p}\Big)^{-1} \frac{1}{K} \sqrt{\frac{x}{\log x}} \left(1 + O\Big(\frac{1}{\log\log x}\Big)\right),$$

implying that, for some constant $c_8 > 0$,

$$\sum_{K\in\text{class I}} R_K(x) = c_8\sqrt{\frac{x}{\log x}}\left(1+O\Big(\frac{1}{\log\log x}\Big)\right). \tag{4.6}$$

Consider now $K \in$ class II, $K \leqslant y$. Let $q|\tau(K^2)$, $q \neq 3$. In this case, $q \leqslant y$. If $n \in R_K$, then $n = K^2v^2$ and $(3, \varphi(v)) = 1$. Consequently, $p|v$ implies that $p \not\equiv 1 \pmod 3$ and $p \not\equiv 1 \pmod q$. By using the Selberg sieve, we obtain that, for some positive constant c_9,

$$\begin{aligned} R_K(x) &\leqslant c_9\frac{\sqrt{x}}{K}\prod_{\substack{p\equiv 1 \pmod 3\\ \text{or } p\equiv 1 \pmod q}}\Big(1-\frac{1}{p}\Big) \leqslant \frac{c_9}{K}\frac{\sqrt{x}}{\sqrt{\log x}}\prod_{\substack{p\equiv -1 \pmod 3\\ \text{and } p\equiv 1 \pmod q}}\Big(1-\frac{1}{p}\Big)\\ &\leqslant \frac{c_9}{K}\frac{\sqrt{x}}{\sqrt{\log x}}\exp\Big\{-\frac{1}{2(q-1)}\log\log x\Big\} = \frac{c_9}{K}\frac{\sqrt{x}}{\sqrt{\log x}}\cdot\frac{1}{(\log x)^{1/(2(q-1))}}. \end{aligned}$$

From this last estimate, it is clear that we can ignore those $K \in$ class II. Hence the main contributions to $R(x)$ comes from (4.5) and (4.6), thus completing the proof of Theorem 1.

5. The proof of Theorem 2

Let $\mathcal{N} := \{n \in \mathbb{N} : l(n) = 1\}$. If $n \in \mathcal{N}$, then one of the numbers $\tau(n)$ and $\tau(n+1)$ must be odd, implying that either n or $n+1$ is a square. So let us set

$$\begin{aligned} N_0(x) &:= \#\{n \leqslant x : n \in \mathcal{N},\ n = \text{square}\},\\ N_1(x) &:= \#\{n \leqslant x : n \in \mathcal{N},\ n+1 = \text{square}\}, \end{aligned}$$

so that $N(x) = N_0(x) + N_1(x)$. We shall therefore consider two cases, namely the case when $\tau(n)$ is odd, and thereafter the one when $\tau(n+1)$ is odd.

We start with the first case. In this case, $l(n) = 1$ implies that $n = u^2$, so that $\tau(n+1) = \tau(u^2+1)$. Write $u = Km$, where K is squarefull and m is squarefree, with $(K, m) = 1$. The contribution of the case $m = 1$ to $N_0(x)$ is clearly $O(x^{1/4})$, since in this case $n = K^2m^2 = K^2 \leqslant x$, that is $K \leqslant \sqrt{x}$. Similarly, write $n + 1 = R\nu$, where R is squarefull and ν is squarefree, with $(R, \nu) = 1$, in which case, $\tau(n + 1) = \tau(R)2^{\omega(\nu)}$. As above, the contribution of the case $\nu = 1$ to $N_1(x)$ is no more than $O(x^{1/4})$. Hence, from here on, we will assume that $m > 1$ and $\nu > 1$.

Given squarefull numbers K and R, we set

$$\begin{aligned} U(x|K,R) &:= \#\big\{n \leqslant x : n \in \mathcal{N},\ n = K^2m^2,\ m > 1,\ n+1 = R\nu\big\},\\ V(x|K,R) &:= \#\big\{1 < m \leqslant \sqrt{x}/K : K^2m^2 + 1 \equiv 0 \pmod R\big\}. \end{aligned}$$

Note that we clearly have $U(x|K,R) \leqslant V(x|K,R)$. Hence, our first goal will be to prove

$$\sum_{\max(K,R)\geqslant T} V(x|K,R) = o\big(\sqrt{x}\,\big) \qquad (T \to \infty). \tag{5.1}$$

Assume first that K is arbitrary and fixed. We shall sum over those positive integers m, ν for which $R \geqslant K$. We will find an upper bound for the number of solutions of

$$n^2 + 1 = R\nu, \qquad R \geqslant T, \qquad n \leqslant \sqrt{x}. \tag{5.2}$$

First we consider the contribution of those R in the above which have a squarefull divisor S such that $T \leqslant S \leqslant \sqrt{x}$. In this case, $n^2 + 1 \equiv 0 \pmod{R}$ implies that $n^2 + 1 \equiv 0 \pmod{S}$. Adding up the contributions of all such S's, (5.2) yields at most

$$2 \sum_{S \geqslant T} \frac{\sqrt{x}}{S} \rho(S) \ll \frac{\sqrt{x}}{\sqrt{T}} \qquad \text{solutions,}$$

where we used the trivial bound $\rho(S) \ll S^{\varepsilon}$.

It remains to estimate the number of solutions $n \leqslant \sqrt{x}$ in (5.2) for which the corresponding squarefull number $R \leqslant x$ has no squarefull divisor $S \leqslant \sqrt{x}$. If R has at least two prime divisors, say p and q, then $p^2 q^2 | R$ and $\min(p^2, q^2) < \sqrt{x}$, which is impossible. This means that $R = p^{\alpha}$ for some integer $\alpha \geqslant 2$. If $\alpha \geqslant 4$, then $S = p^2 < \sqrt{x}$, again a contradiction. This means that we only have two possibilities, namely $R = p^2, p^3$. In the case $R = p^2$, we have $p^2 | n^2 + 1$, $p \geqslant \sqrt{x}$; thus, applying Lemma 1 with $f(n) = n^2 + 1$, the assertion is proved. If $R = p^3$, the result follows even more directly.

For fixed K, there are no more than $\sqrt{x}/K$ integers for which $(Km)^2 \leqslant x$. Summing on K, we get no more than $\sqrt{x} \sum_{K \geqslant T} \frac{1}{K} \ll o(\sqrt{x})$ $(T \to \infty)$, thus completing the proof of (5.1).

Now further define $\mathcal{K}_1 := \{(K, R) : (K, R) = 1 \text{ and } (3\tau(K^2), 2\tau(R)) = 1\}$. Note that the condition $(3\tau(K^2), 2\tau(R)) = 1$ is a necessary condition for $K^2 m^2 + 1 = R\nu$, with $m > 1$, to satisfy $l(K^2 m^2) = 1$.

Now let T be a large number. Since $U(x|K, R) \leqslant V(x|K, R)$, it follows from (5.1) that

$$\sum_{\max(K,R) \geqslant T} U(x|K, R) = o(\sqrt{x}) \qquad (T \to \infty).$$

In particular, we have

$$U(x|1, 1) = \sum_{m \leqslant \sqrt{x}} |\mu(m^2 + 1)| \cdot |\mu(m)|,$$

so that by Lemma 2, $U(x|1, 1) = \xi_2 (1 + o(1)) \sqrt{x}$.

Now we have

$$N_0(x) = \sum_{K, R \in \mathcal{K}_1} U(x|K, R), \tag{5.3}$$

where

$$U(x|K, R) = \sum_{\substack{(\delta_1, K) = 1 \\ (\delta_2, R) = 1}} \mu(\delta_1) \mu(\delta_2) Q(K, R; \delta_1, \delta_2), \tag{5.4}$$

with

$$
\begin{aligned}
Q(K,R;\delta_1,\delta_2) &= \#\big\{K^2\delta_1^4 m_1^2+1 = R\delta_2^2\nu_1 \leqslant x,\ (\nu_1,R)=1,(m_1,K)=1\big\}\\
&= \sum_{d_1\mid R}\mu(d_1)\sum_{d_2\mid K}\mu(d_2)\cdot\#\big\{K^2\delta_1^4 d_2^2 m_2^2+1 = R\delta_2^2 d_1\nu_2\leqslant x\big\}, \qquad (5.5)
\end{aligned}
$$

where this last expression was obtained by setting $\nu_1 = d_1\nu_2$ and $m_1 = d_2 m_2$. Now let $E_0 = K\delta_1^2 d_2$ and $F_0 = R\delta_2^2 d_1$, so that

$$
\#\big\{K^2\delta_1^4 d_2^2 m_2^2+1 = R\delta_2^2 d_1\nu_2\leqslant x\big\} = V(x|E_0,F_0).
$$

Since $R, K \leqslant T$, it follows that $d_1, d_2 \leqslant T$. But as we have seen earlier, the contribution of those $V(x|E_0,F_0)$ for which $\max(\delta_1,\delta_2)\geqslant T$, is small.

In light of this observation and using (5.5), relation (5.4) can be replaced by

$$
U(x|K,R) = \sum_{\substack{(\delta_1,K)=1\\(\delta_2,R)=1\\ \delta_1\leqslant T,\,\delta_2\leqslant T}} \mu(\delta_1)\mu(\delta_2)Q(K,R;\delta_1,\delta_2) + o\big(\sqrt{x}\big) \qquad (T\to\infty). \tag{5.6}
$$

If E_0 and F_0 are bounded,

$$
V(X|E_0,F_0) = \frac{\sqrt{x}}{E_0F_0}\rho(F_0) + O(\rho(F_0)).
$$

Consequently, (5.6) becomes

$$
\begin{aligned}
U(x|K,R) \quad = \quad & \sum_{\substack{(\delta_1,K)=1\\ \delta_1\leqslant T}}\ \sum_{\substack{(\delta_2,R)=1\\ \delta_2\leqslant T}}\ \sum_{d_1\mid R}\mu(d_1)\sum_{d_2\mid K}\mu(d_2)\left(\frac{\sqrt{x}\rho(R\delta_2^2 d_1)}{K\delta_1^2 d_2 R\delta_2^2 d_1} + O(\rho(R\delta_2^2 d_1))\right)\\
& + o(\sqrt{x}) \qquad (T\to\infty). \qquad (5.7)
\end{aligned}
$$

Setting

$$
C(K,R) := \frac{1}{KR}\sum_{(\delta_1,K)=1}\ \sum_{(\delta_2,R)=1}\ \sum_{d_1\mid R}\sum_{d_2\mid K}\frac{\mu(\delta_1)\mu(\delta_2)\mu(d_1)\mu(d_2)\rho(R\delta_2^2 d_1)}{\delta_1^2\delta_2^2 d_1 d_2} \tag{5.8}
$$

and noticing that the right hand side of (5.8) represents a finite quantity, we may conclude that $C(K,R)$ is a nonnegative (actually positive) constant. Hence, in light of this last observation, (5.7) and (5.8) yield

$$
U(x|K,R) = C(K,R)(1+o(1))\sqrt{x}. \tag{5.9}
$$

Since $\sum_{(K,R)\in\mathcal{K}_1} C(K,R)$ is convergent, it follows, combining (5.3) and (5.9), that

$$
N_0(x) = \sum_{K,R\in\mathcal{K}_1} U(x|K,R) = (1+o(1))c_{10}\sqrt{x} \qquad (x\to\infty),
$$

where $c_{10} = \sum_{(K,R)\in\mathcal{K}_1} C(K,R)$ is a constant which is positive because $C(1,1)$ is positive by Lemma 2.

It remains to consider the second case, namely the one where $\tau(n+1)$ is odd, in which case $n+1$ is a square. In this case, $l(n)=1$ implies that $n+1 = K^2m^2$, where K is squarefull, $m>1$ squarefree, $(K,m)=1$, $n = R\nu$, $(\nu,R)=1$, R squarefull and ν squarefree. Now, $l(n)=1$ also implies that $(2\tau(R),3\tau(K^2))=1$. Hence, let

$\mathcal{K}_2$ stand for the set of all pairs of squarefull integers K, R, with $(K, R) = 1$, for which

$$(2\tau(R), 3\tau(K^2)) = 1. \tag{5.10}$$

Observe that $K = R = 1$ satisfies (5.10) and that we have

$$N_1(x) = \sum_{K,R\in\mathcal{K}_2} \#\big\{R\nu \leqslant x : K^2m^2 - 1 = R\nu,\ m > 1,\\ (K,m) = (R,\nu) = 1,\ \mu^2(m) = \mu^2(\nu) = 1\big\}.$$

Proceeding along the same lines as in the first case yields the estimate

$$N_1(x) = (1 + o(1))c_{11}\sqrt{x} \qquad (x \to \infty),$$

for some positive constant c_{11}. Since the rest of the proof is similar, we shall omit it. This completes the proof of Theorem 2.

References

[1] K. Alladi, *On the probability that n and $\Omega(n)$ are relatively prime*, Fib. Quart. **19** (1981), 228-233.

[2] J. M. De Koninck and I. Kátai, *On an estimate of Kanold*, Int. J. Math. Anal. 5/8 (2007), no. 1–12, 9–18.

[3] P. D. T. A. Elliott, *Probabilistic Number Theory I, Mean Value Theorems*, Springer-Verlag, Berlin, 1979.

[4] P. Erdős, *Some asymptotic formulas in number theory*, J. Indian Math. Soc. (N. S.) **12** (1948), 75–78.

[5] C. Hooley, *Applications of Sieve Methods to the Theory of Numbers*, Cambridge Tracts in Mathemtics,

[6] A. Ivić, *The Riemann Zeta-Function*, Dover, New York, 1985.

[7] H. J. Kanold, *Über das harmonische Mittel der Teiler einer natürlichen Zahl II*, Math. Annalen **134** (1958), 225–231.

[8] I. Kátai, *On an application of the large sieve: shifted prime numbers, which have no prime divisors from a given arithmetical progression*, Acta Math. Acad. Sci. Hung. **21**(1–2) (1970), 151–173.

[9] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd II, Teubner, Leipzig, 1909.

(Received 09 11 2009)
(Revised 26 02 2010)

Dép. de mathématiques et de statistique
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katai@compalg.inf.elte.hu