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A NOTE ON SUNS IN CONVEX METRIC SPACES

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Abstract. We prove that in a convex metric space (X, d), an existence set K having a lower semi continuous metric projection is a δ-sun and in a complete M-space, a Chebyshev set K with a continuous metric projection is a γ-sun as well as almost convex.

1. Introduction

One of the most outstanding open problem of approximation theory is: Whether every Chebyshev set in a Hilbert space is convex? Several partial answers to this problem are known in the literature (see e.g.[1], [2], [3], [6], [10], and [12]) but the problem is still unsolved. While making an attempt in this direction, Effimov and Steckin [4] introduced the concept of a sun and Vlasov [13] introduced the concepts of α-, β-, γ- , δ-suns and almost convex sets in Banach spaces. These concepts were extended to convex metric spaces in [8] and some of the results proved by Vlasov [13] in Banach spaces were also proved in convex metric spaces. Continuing the study taken up in [8] and [9], we prove that in a convex metric space (X, d), an existence set K having a lower semi continuous metric projection is a δ-sun and in a complete M-space, a Chebyshev set K with a continuous metric projection is a γ-sun as well as almost convex.

2. Definitions and notations

We recall a few definitions and set up some notations. Let (X, d) be a metric space and $x, y, z \in X$. We say that z is *between* x and y if $d(x, z)+d(z, y) = d(x, y)$. For any two points $x, y \in X$, the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is called *metric segment* and is denoted by $G[x, y]$. The set $G[x, y, -]$ denotes the half ray starting from x and passing through y, and $G(x, y) \equiv G[x, y] \setminus \{x, y\}$.

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A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. A metric space (X, d) with a convex structure is called a *convex metric space* [11]. A convex metric space (X, d) is called an *M-space* [5] if for every two points $x, y \in X$ with $d(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that $B[x, r] \cap B[y, \lambda - r] = \{z_r\}$, where $B[x, r] = \{y \in X : d(x, y) \leqslant r\}$. If (X, d) is a convex metric space, then for each two distinct points $x, y \in X$ and for every λ $(0 < \lambda < 1)$ there exists a point $z \in X$ such that $d(x, z) = (1 - \lambda) d(x, y)$ and $d(z, y) = \lambda d(x, y)$. For M-spaces such a z is always unique. A normed linear space need not be an M-space. For examples of convex metric spaces and M-spaces one may refer to [5], [7] and [11].

Let V be a nonempty subset of a metric space (X, d) and $x \in X$. An element $v_0 \in V$ is called a *best approximation* to x if $d(x, v_0) = \operatorname{dist}(x, V) \equiv d_V(x)$. We denote by $P_V(x)$ the set of all best approximants to x in V. The set V is said to be *proximinal* or an *existence set* if $P_V(x) \neq \emptyset$ for each $x \in X$ and is said to be *Chebyshev* if $P_V(x)$ is exactly singleton for each $x \in X$. The set-valued mapping $P_V : X \to 2^V \equiv$ the set of all subsets of V, which associates with each $x \in X$, the set $P_V(x)$ is called the *metric projection* or the *best approximation map* or the *nearest point map*. For Chebyshev sets V, the map P_V is single-valued.

For two topological spaces X and Y, a set-valued mapping $\phi : X \to 2^Y$ is called *lower semi-continuous* [13] if $\{x \in X : \phi(x) \subset G\}$ is closed in X for each closed set G in Y.

A nonempty closed subset K of an M-space (X, d) is called

i. a *δ-sun* if for all $x \notin K$ there exists a sequence $\langle z_n \rangle$ for which $z_n \neq x$, $z_n \to x$ and $(d_K(z_n) - d_K(x))/d(z_n, x) \to 1$.

ii. a *γ-sun* if for all $x \notin K$ and for all $R > 0$ there exists z_n such that $d_K(z_n) - d_K(x) \to R$, $d(z_n, x) = R$ for all n.

iii. *almost convex* if for any ball V with $\rho(V, K) \equiv \inf\{d(x, K) : x \in V\} > 0$ there exists a sufficiently large ball $V' \supset V$ with $\rho(V', K) > 0$.

3. Suns in convex metric spaces

The following lemma will be used in the sequel:

Lemma 3.1. *Let K be an existence set in a convex metric space (X, d) with metric projection P and suppose that $x, x', v \in X$, are such that $x' \in Px$ and $x \in G(x', v)$. Then*

$$0 \leqslant 1 - \frac{d(v, K) - d(x, K)}{d(v, x)} \leqslant \frac{d(x', Pv)}{d(x, K)} \tag{1}$$

Proof. Since $x \in G(x', v)$, we can find α, $0 < \alpha < 1$ such that $d(x', x) = (1 - \alpha) d(x', v)$ and $d(x, v) = \alpha d(x', v)$ i.e., $x = W(x', v, \alpha)$. Consider

$$\begin{aligned} d(x,x') \leqslant d(x,Pv) = d(W(x',v,\alpha),Pv) &\leqslant \alpha d(x',Pv) + (1-\alpha)d(v,Pv) \\ &= \alpha d(x',Pv) + (1-\alpha)d(v,K). \end{aligned}$$

This implies

$$\begin{aligned} d(v,K) &\geqslant \frac{1}{1-\alpha}d(x,x') - \frac{\alpha}{1-\alpha}d(x',Pv) \\ &= d(x',v) - \frac{d(x,v)}{d(x,x')}d(x',Pv) = d(x',x) + d(x,v) - \frac{d(x,v)}{d(x,x')}d(x',Pv) \\ &= d(x',x) + d(x,v)\Big[1 - \frac{d(x',Pv)}{d(x,x')}\Big] = d(x,K) + d(x,v)\Big[1 - \frac{d(x',Pv)}{d(x,K)}\Big] \end{aligned}$$

i.e.,

$$\frac{d(v,K)-d(x,K)}{d(x,v)} \geqslant \Big[1 - \frac{d(x',Pv)}{d(x,K)}\Big].$$

This gives

$$\frac{d(x',Pv)}{d(x,K)} \geqslant 1 - \frac{d(v,K)-d(x,K)}{d(x,v)}. \tag{2}$$

Also $d(v,K) \leqslant d(v,x) + d(x,K)$ implies $(d(v,K)-d(x,K))/d(v,x) \leqslant 1$ i.e.,

$$1 - \frac{d(v,K)-d(x,K)}{d(v,x)} \geqslant 0. \tag{3}$$

Therefore (2) and (3) give the desired result. $\square$

Using the above lemma, we prove

THEOREM 3.1. *In a convex metric space (X,d), an existence set K having a lower semi-continuous metric projection P is a δ-sun.*

PROOF. Above lemma implies

$$0 \leqslant 1 - \frac{d(v,K)-d(x,K)}{d(v,x)} \leqslant \frac{d(x',Pv)}{d(x,K)}. \tag{4}$$

We claim that for all $x' \in Px$, $d(x',Pv) \to 0$ as $v \to x$. Suppose this is not true. Then there exists $x' \in Px, \varepsilon > 0$ and $v_n \to x$ such that $d(x',Pv_n) \geqslant \varepsilon$ for all n. Consider the set $F = \{z \in X : d(z,x') \geqslant \varepsilon\}$. Then $Pv_n \subseteq F$ and F is closed. By the lower semicontinuity of P, the set $F_1 = \{z \in X : Pv_n \subseteq F\}$ is closed. As $Pv_n \subseteq F$, $v_n \in F_1$ and $v_n \to x$, $x \in F_1$ i.e., $Px \subseteq F$ and so $x' \in F$. This gives $d(x',x') \geqslant \varepsilon$ which is absurd. Therefore for all $x' \in Px$, $d(x',Pv) \to 0$ as $v \to x$ and so (4) gives $(d(v,K)-d(x,K))/d(v,x) \to 1$ i.e., K is a δ-sun. $\square$

COROLLARY 3.1. *In a convex metric space (X,d) a Chebyshev set K with a continuous metric projection P is a δ-sun.*

Aliter Proof. As K is Chebyshev, $x' = Px$. The continuity of P implies $\lim_{v\to x} d(x', Pv) = d(x', Px) = d(x', x') = 0$. So (1) implies

$$\lim_{v\to x} \frac{d(v,K) - d(x,K)}{d(v,x)} = 1$$

i.e., K is a δ-sun. □

Since in a complete M-space, every δ-sun is a γ-sun [9], and is almost convex [8], we have:

Corollary 3.2. *In a complete M-space, a Chebyshev set with a continuous metric projection is a γ-sun and almost convex.*

Remark 1. For Banach spaces, the above results were proved by Vlasov [13].

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