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AR(1) TIME SERIES WITH APPROXIMATED BETA MARGINAL

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Abstract. We consider the AR(1) time series model $X_t - \beta X_{t-1} = \xi_t$, $\beta^{-p} \in \mathbb{N} \setminus \{1\}$, when X_t has Beta distribution $\mathrm{B}(p,q)$, $p \in (0,1]$, $q > 1$. Special attention is given to the case $p = 1$ when the marginal distribution is approximated by the power law distribution closely connected with the Kumaraswamy distribution $\mathrm{Kum}(p,q)$, $p \in (0,1]$, $q > 1$. Using the Laplace transform technique, we prove that for $p = 1$ the distribution of the innovation process is uniform discrete. For $p \in (0,1)$, the innovation process has a continuous distribution. We also consider estimation issues of the model.

1. Introduction

In the standard time series analysis one assumes that its marginal distribution is Gaussian. However, a Gaussian distribution will not always be appropriate. In earlier works stationary non-Gaussian time series models have been developed for variables with positive and highly skewed distributions. There still remain some situations where Gaussian marginals are inappropriate, i.e., where the marginal time-series variable being modeled, although not skewed or inherently positive valued, has a large kurtosis and long-tailed distributions. There are plenty real situations that can not be modeled by Gaussian distribution like in hydrology, meteorology, information theory, economics, etc. Simple models with exponential marginals or mixed exponential marginals are considered in [**9, 29, 1, 2, 13**], while another marginals have been discussed like Gamma [**9, 19, 7**], Laplace [**19**], uniform [**15, 21**] and Weibull [**19, 7**]. Finally, we point out autoregressive processes PBAR and NBAR autoregressive process models constructed by McKenzie's [**10**] for positively and respectively, negatively correlated pairs of Beta random variables employing certain properties of the $\mathrm{B}(p,q)$ distribution.

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In this paper, we introduce an autoregressive first order time series model with the marginal Beta distribution $\mathrm{B}(p,q)$, $p \in (0,1]$, $q > 1$ of which the Laplace transform is approximated when the transformation argument is large. The resulting approximation determines a new distribution and for $p = 1$ results in a discrete uniform distribution for the innovation process. For $0 < p < 1$ the distribution of the innovation process is continuous. Special attention is given to the case $p = 1$, when the approximation becomes a kind of power law distribution, characterized by the related probability density function (PDF) $f(x) = q(1-x)^{q-1}$, $x \in [0,1]$. This distribution coincides with the Beta distribution $\mathrm{B}(1,q)$. On the other hand, the $\mathrm{B}(1,q)$ generates Kumaraswamy distribution $\mathrm{Kum}(p,q)$, $p \in (0,1]$, $q > 1$ introduced in [**24**]. This distribution is very important in applications when double bounded random processes arise in practice [**27, 28**], e.g., it has been applied in modelling the storage volume of a reservoir and system design [**26**] and turns out to be very important in many hydrological problems. For a complete account of the properties of the Kumaraswamy distribution, consult [**16**].

A random variable $X_{p,q}$ defined on some standard probability space $(\Omega, \mathfrak{F}, P)$ having the $\mathrm{Kum}(p,q)$ distribution possesses the PDF [**8**, pp. 169–170] (in its simplest form):

$$f(x) = pqx^{p-1}(1-x^p)^{q-1} \cdot \mathbf{1}_{[0,1]}(x).$$

The cumulative distribution function's (CDF) non-constant part is

$$F(x) = \big(1 - (1-x^p)^q\big) \cdot \mathbf{1}_{[0,1]}(x),$$

where $\mathbf{1}_{[0,1]}(x)$ stands for the indicator function of the closed unit interval. Furthermore, we have

$$X_{p,q}^p \stackrel{d}{=} Y_{1,q}, \tag{1.1}$$

where $Y_{1,q}$ has Beta distribution $\mathrm{B}(1,q)$, $q > 1$. Note that both $\mathrm{Kum}(1,1)$ and $\mathrm{B}(1,1)$ represent the uniform $\mathcal{U}(0,1)$ distribution.

The technical part of the research begins with the derivation of the Laplace transform (LT) of the autoregressive model. The resulting LT function is expressed in terms of the Wright hypergeometric function Φ. Therefore we are faced with extremely hard calculations in inverting the Laplace transform. So, we approximate the derived LT of the model, obtaining the inverse LT *mutatis mutandis* the distribution of the innovation sequence $\{\xi_t : t \in \mathbb{Z}\}$. The resulting approximate distribution will be referred to as the *approximated Beta* ($\mathrm{AB}_{p,q}$) and as the *approximated power law* (APL) for $p = 1$. Therefore, considering initially a Linear time series model with $\mathrm{AB}_{p,q}$, we arrive at a new model called $\mathrm{LAB}_{p,q}\,\mathrm{AR}(1)$, which becomes $\mathrm{LAPLAR}_q(1)$ for $p = 1$.

Finally, the unknown parameter will be estimated using conditional least squares estimator.

2. Approximated Laplace transform of $\mathbf{B}(p,q)$

Consider a random variable $Y_{p,q}$ defined on a standard probability space $(\Omega, \mathfrak{F}, P)$; having the $\mathrm{B}(p,q)$ distribution. The related PDF is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\, x^{p-1}(1-x)^{q-1}\mathbf{1}_{[0,1]}(x)\,. \tag{2.1}$$

In what follows $\Gamma(p)$ stands for the familiar Euler's gamma function. We write the Laplace transform of some suitable function f as

$$\mathcal{L}_\lambda[f] = \int_0^\infty e^{-\lambda x} f(x)\,dx =: \varphi(\lambda),$$

while

$$\mathcal{L}_x^{-1}[\varphi] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{x\lambda}\varphi(\lambda)\,d\lambda =: f(x)$$

stands for the inverse Laplace-transform pair of $\varphi(\lambda)$.

The Laplace transform of the PDF (2.1) equals

$$\varphi_{Y_{p,q}}(\lambda) = Ee^{-\lambda Y_{p,q}} = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\int_0^1 e^{-\lambda x}x^{p-1}(1-x)^{q-1}dx = F(p, q+p, -\lambda),$$

where F denotes the familiar *Kummer function of the first kind*, that is, the confluent hypergeometric function ${}_1F_1$ defined by the series

$$F(a,b,z) := \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\,\frac{z^n}{n!}.$$

Here $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$, $(a)_0 = 1$ is the *Pochhammer symbol.*

Let us prove the formula

$$\varphi_{Y_{p,q}}(\lambda) \sim \frac{\Gamma(p+q)}{\Gamma(q)\lambda^p}\big(1 - e^{-p(q-1)\lambda^p}\big), \qquad \lambda\to\infty, \tag{2.2}$$

for all $q > 1$, where $\sim$ denotes the asymptotic equality. Indeed, considering the transformation of the integrand in

$$\begin{aligned}\varphi_{Y_{p,q}}(\lambda) &= \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\int_0^1 e^{-\lambda x}x^{p-1}(1-x)^{q-1}dx\\ &= \frac{\lambda^{-p}}{\mathrm{B}(p,q)}\int_0^\lambda e^{-x}x^{p-1}\Big(1-\frac{x}{\lambda}\Big)^{q-1}dx,\end{aligned}$$

gives the asymptotics

$$\begin{aligned}\varphi_{Y_{p,q}}(\lambda) &\sim \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)\lambda^p}\int_0^\infty e^{-x}x^{p-1}\Big(1-\frac{(q-1)x}{\lambda}\Big)dx\\ &= \frac{\Gamma(p+q)}{\Gamma(q)\lambda^p}\Big(1-\frac{p(q-1)}{\lambda}\Big) \sim \frac{\Gamma(p+q)}{\Gamma(q)\lambda^p}\,\exp\Big\{-\frac{p(q-1)}{\lambda^p}\Big\}\\ &\sim \frac{\Gamma(p+q)}{\Gamma(q)\lambda^p}\big(1-\exp\big\{-p(q-1)\lambda^p\big\}\big),\end{aligned} \tag{2.3}$$

for λ large enough. Hence the asserted result (2.2).

An important special case $p = 1$ of (2.2) is $\varphi_{Y_{1,q}}(\lambda) \sim \frac{q}{\lambda}\big(1 - e^{-(q-1)\lambda}\big)$, $\lambda \to \infty$.

3. The LAPLAR$_q$(1) and LAB$_{p,q}$(1) models

In this section we introduce a new autoregressive model of the first order. Since the LT of Beta distribution one expresses as special function, for the derivation of distribution of innovation sequence $\{\xi_t \colon t \in \mathbb{Z}\}$, we will use approximated LT (2.2). In this way, marginal distribution of the model (3.1) is replaced by the AB$_{p,q}$ such that is generated by the model. The LAB$_{p,q}$AR(1) model is given with the following equations:

$$X_t - \beta X_{t-1} = \xi_t, \;\; \beta^{-p} \in \mathbb{N} \setminus \{1\}, \; p \in (0,1]. \tag{3.1}$$

Here X_t is $\mathrm{B}(p,q), p \in (0,1], q > 1$. Also, we make the following assumptions:

(i) X_t is stationary; (ii) $\{X_t\}$ and $\{\xi_s\}$ are independent for $t < s$,

where stationarity is ensured by $\beta \in (0,1)$.

Let us define a positive integer $\kappa(p) = \big[\beta^{-p}\big]$, where $[A]$ stands for the integer part of some A. In what follows, we consider such parameters $p \in (0,1]$ and $\beta \in (0,1)$ that $\kappa(p) \in \mathbb{N}$.

Now, we derive our principal result-the distribution (or a related approximation) of the innovation sequence ξ_t.

Theorem 3.1. *Consider the* LAPLAR$_q$(1) *times series model*

$$X_t - \beta X_{t-1} = \xi_t, \;\; \beta = \frac{1}{n}, \; n \in \mathbb{N} \setminus \{1\},$$

such that possesses marginal APL *distribution, let $q > 1$. Then the distribution of the i.i.d. sequence $\{\xi_t : t \in \mathbb{Z}\}$ is approximated by the uniform discrete distribution*

$$P\Big\{\xi_t = \frac{(q-1)j}{n}\Big\} = 1/n, \qquad j = \overline{0, n-1}.$$

Proof. Let us find the LT of the model (3.1). Using the assumption (ii) upon the independent X_t and ξ_s, $t < s$, we have:

$$\varphi_{X_t}(\lambda) = Ee^{-\lambda X_t} = Ee^{-(\beta X_{t-1} + \xi_t)\lambda} = Ee^{-\beta X_{t-1}\lambda} Ee^{-\xi_t\lambda} = \varphi_{X_{t-1}}(\beta\lambda)\, \varphi_{\xi_t}(\lambda).$$

With regard to the assumed stationarity (i) of X_t we can conclude $\varphi_X(\lambda) = \varphi_X(\beta\lambda)\varphi_\xi(\lambda)$. Using asymptotic formula (2.2), we obtain

$$\begin{aligned}\varphi_\xi(\lambda) &= \frac{\varphi_X(\lambda)}{\varphi_X(\beta\lambda)} \sim \frac{1}{n} \cdot \frac{1 - e^{-(q-1)\lambda}}{1 - e^{-(q-1)\beta\lambda}} = \frac{1}{n} \cdot \frac{1 - (e^{-(q-1)\frac{1}{n}\lambda})^n}{1 - e^{-(q-1)\frac{1}{n}\lambda}} \\ &= \frac{1}{n} \cdot \sum_{j=0}^{n-1} e^{-(q-1)\frac{j}{n}\lambda} =: A_\lambda(\varphi_\xi), \qquad \lambda \to \infty.\end{aligned}$$

Now, by inverse LT we easily deduce that the distribution of i.i.d. sequence $\{\xi_t : t \in \mathbb{Z}\}$. The inverse LT of the approximation $A_\lambda(\varphi_\xi)$ we calculate directly:

$$\mathcal{L}_x^{-1}\big[A_\lambda(\varphi_\xi)\big] = \frac{1}{n}\mathcal{L}_x^{-1}\bigg[\sum_{j=0}^{n-1} e^{-(q-1)\frac{j}{n}\lambda}\bigg] = \frac{1}{n}\sum_{j=0}^{n-1} \delta\Big(x - \frac{(q-1)j}{n}\Big),$$

where δ is the well-known Dirac function. Hence, we conclude that the distribution of ξ_t is approximated by discrete probability law

$$\underbrace{\begin{pmatrix} 0 & (q-1)/n & \cdots & (q-1)(n-1)/n \\ n^{-1} & n^{-1} & \cdots & n^{-1} \end{pmatrix}}_{n}, \tag{3.2}$$

which is the assertion of the theorem. □

Remark 3.1. The same model, (3.1), but with the uniform $\mathcal{U}(0,1)$ marginal distribution has been studied in [**15**]. In this case it has been proved that innovation sequence of (3.1) coincides with discrete distribution (3.2) for $q = 2$.

Remark 3.2. By equality (1.1) one deduces $X_t\colon \mathrm{Kum}(p,q) \stackrel{d}{=} X_t^{1/p}\colon \mathrm{B}(1,q)$, that links our results to Kumaraswamy $\mathrm{Kum}(p,q)$, $q > 1$ distribution.

Let us introduce the hypergeometric function $\Phi(a,\alpha;z)$ by E. M. Wright defined in the whole complex plane $\mathbb{C}$, for complex $a \in \mathbb{C}$ and real $0 < \alpha < 1$, by the series

$$\Phi(a,-\alpha;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(a-\alpha n)\, n!}, \tag{3.3}$$

see e.g. [**6**]. It is not hard to show that

$$\mathcal{L}_x^{-1}\big[\lambda^{-\tau} e^{-T\lambda^p}\big] = x^{\tau-1}\Phi\big(\tau,-p\,;-Tx^{-p}\big), \qquad T > 0, \tag{3.4}$$

is equivalent to the Humbert–Pollard inverse LT formula [**23**], [**11**, Eq. (4)]. Indeed, taking certain $p \in (0,1)$, we deduce (3.4) having

$$\begin{aligned} \lambda^{-\tau} e^{-T\lambda^p} &= \sum_{n=0}^{\infty} \frac{(-T)^n\, \lambda^{-\tau+np}}{n!} = \sum_{n=0}^{\infty} \frac{(-T)^n}{n!} \cdot \frac{1}{\Gamma(\tau-np)} \int_0^\infty e^{-\lambda x} x^{\tau-np-1} dx \\ &= \int_0^\infty e^{-\lambda x} x^{\tau-1} \Big\{ \sum_{n=0}^{\infty} \frac{(-Tx^{-p})^n}{\Gamma(\tau-np)\, n!} \Big\} dx \\ &= \int_0^\infty e^{-\lambda x} x^{\tau-1}\, \Phi\big(\tau,-p\,;-Tx^{-p}\big)\, dx. \end{aligned}$$

Theorem 3.2. *Consider the* $\mathrm{LAB}_{p,q}\,\mathrm{AR}(1)$ *times series model*

$$X_t - \beta X_{t-1} = \xi_t \quad \beta^p = \frac{1}{n},\ n \in \mathbb{N} \setminus \{1\},$$

that possesses the marginal $\mathrm{AB}_{p,q}$. *Assume that* $p \in (0,1)$, $q > 1$ *and* $\kappa(p) \in \mathbb{N}$. *Then the distribution of the i.i.d. sequence* $\{\xi_t : t \in \mathbb{Z}\}$ *is approximated by continious distribution having PDF*

$$f_\xi(p;x) = \frac{x^{-1}}{\|\Phi_p\|} \sum_{j=0}^{\kappa(p)-1} \Phi\Big(0,-p\,;-\frac{p(q-1)\beta^p j}{x^p}\Big), \tag{3.5}$$

where

$$\|\Phi_p\| := \sum_{j=0}^{\kappa(p)-1} \int_0^\infty x^{-1}\Phi\Big(0,-p\,;-\frac{p(q-1)\beta^p j}{x^p}\Big) dx. \tag{3.6}$$

PROOF. Using similar lines to the proof of Theorem 3.1, with the aid of the asymptotic identity (2.3) for large enough λ, we conclude that

$$\varphi_\xi(\lambda) = \frac{\varphi_X(\lambda)}{\varphi_X(\beta\lambda)} \sim \beta^p \sum_{j=0}^{\kappa(p)-1} \exp\{-p(q-1)\beta^p j\lambda^p\}. \tag{3.7}$$

Denote the right-hand side of this expression by $B_{\lambda,p}(\varphi_\xi)$, say. Now, setting $\tau = 0$ in the Humbert–Pollard formula (3.4) and employing it to the addends in (3.7), one deduces

$$\mathcal{L}_x^{-1}\big[B_{\lambda,p}(\varphi_\xi)\big] = \beta^p \sum_{j=0}^{\kappa(p)-1} x^{-1}\Phi\Big(0,-p\,;-\frac{p(q-1)\beta^p j}{x^p}\Big), \tag{3.8}$$

where the Wright function series converges, because $p \in (0,1)$.

The behavior of $\mathcal{L}_x^{-1}\big[B_{\lambda,p}(\varphi_\xi)\big]$ near the origin is controlled by the upper bound

$$\big|x^{-1}\Phi(0,-p\,;-x^{-p})\big| \leqslant \frac{1}{p\pi}\,\frac{\Gamma(1/p)}{\cos^{1/p}(p\pi/2)}, \quad x \geqslant 0,$$

such that is the special case of the majorant reported in [**6**, Eq. (32)].

It remains only to show the non-negativity of $\mathcal{L}_x^{-1}\big[B_{\lambda,p}\big(\varphi_\xi\big)\big]$. Indeed, in [**11**] it was shown that $x^{-1}\Phi\big(0,-p\,;-Tx^{-p}\big) > 0$, $T > 0$, $p \in (0,1)$, almost everywhere, when $x > 0$. [**6**, Theorem 8] strengthened his result to the strict positivity.

Finally, normalizing (3.8), hence the proof of the assertion (3.5). $\square$

Asymptotic behavior of the PDF of the $\mathrm{AB}_{p,q}, p \in (0,1)$, when x approaches nil or infinity, follows from the adopted Mikusiński's results [**14**, Eqs (4), (5)]

$$x^{-1}\Phi\Big(0,-p\,;-\frac{T}{x^p}\Big) \sim \begin{cases} \dfrac{Q_T}{x^{p/(2(1-p))}}\,\exp\big\{-R_T x^{-p/(1-p)}\big\}, & x \to 0+ \\ \dfrac{p}{\Gamma(1-p)} T x^{-p-1}, & x \to \infty. \end{cases}$$

where

$$Q_T := \frac{(pT)^{1/(2(1-p))}}{\sqrt{2\pi(1-p)}}, \quad R_T := (1-p)(p^pT)^{1/(1-p)}, \quad T := p(q-1)\beta^p j.$$

Let us denote by X_0 the initial value of the time series $\{X_t \colon t \in \mathbb{Z}\}$, and thereafter we can formulate the following result:

THEOREM 3.3. *Let $\{\xi_t \colon t \in \mathbb{Z}\}$ be an i.i.d. sequence of random variables having distribution* (3.5). *If $q > 1$, $p \in (0,1)$, $\beta^p = \frac{1}{n}$, $n \in \mathbb{N} \setminus \{1\}$, $\kappa(p) \in \mathbb{N}$ and X_0 is from $\mathrm{AB}_{p,q}$, then relation* (3.1) *defines the time series $\{X_t \colon t \in \mathbb{Z}\}$ whose marginal distribution is $\mathrm{AB}_{p,q}$.*

REMARK 3.3. For $p = 1$ Theorem 3.3 is valid for $\mathrm{LAPLAR}_q(1)$ model.

REMARK 3.4. Since Wright hypergeometric function (3.3) is not defined for $\alpha > 1$, it is not possible to consider the case $p > 1$.

3.1. Simulation study. Histograms of beta distribution $\mathrm{B}(p, q)$, as well as histogram of $\mathrm{AB}_{p,q}$, for given values of p and q, will be presented in Example 3.1. Having in mind that $\mathrm{AB}_{p,q}$ is determined by the model, there was a problem of generating random numbers from $\mathrm{AB}_{p,q}$. The solution is to use the algorithm described in [**20**]. This algorithm is implemented, by Ridout, in software R [**25**].

In Example 3.2, we give graphical representation of PDF (3.5) together with PDF of beta distribution for certain parameter values. The normalizing constant $\|\Phi_p\|$ is calculated using adaptive Simpson quadrature. Since integral (3.6) is improper, we apply use the technique from [**22**, pp. 618–621].

The sample path of the model (3.1) will be simulated in Example 3.3. For this purpose Theorem 3.3 was used. Since the PDF (3.5) is given in the form of special function, it is impossible to build random number generator from PDF. So the initial value X_0 and random numbers from the distribution whose PDF is (3.5) is generated using the same idea as in Example 3.1.

EXAMPLE 3.1. Sample of size 5.000 was drawn from $\mathrm{B}(1/3, 4)$. Using the algorithm described in [**20**], a set of 5000 elements from $\mathrm{AB}_{1/3,\,4}$ is generated. Figure 1 represents histograms of beta distribution and its approximation.

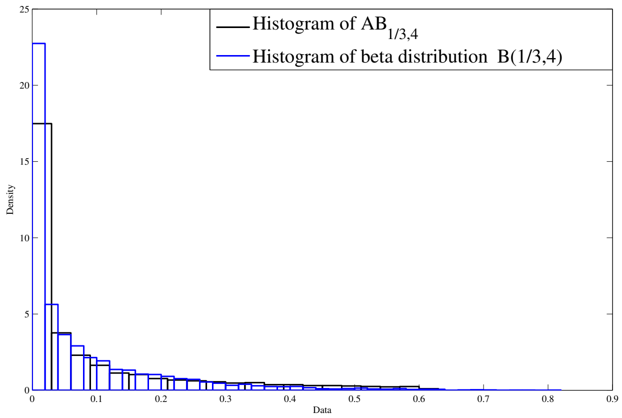

FIGURE 1. Histogram of Beta distribution and its approximation $AB_{p,q}$.

EXAMPLE 3.2. The density function of Beta distribution and PDF of its approximation (3.5), when Laplace transform argument s is large enough, will be presented in Figure 2 for two sets of different parameters.

EXAMPLE 3.3. In this example we give the sample paths of $\mathrm{LAB}_{p,q}\,\mathrm{AR}(1)$ model for different sample sizes Figure 3. Parameters values are: $p = 1/3$, $q = 4$ and $\beta = 1/8$.

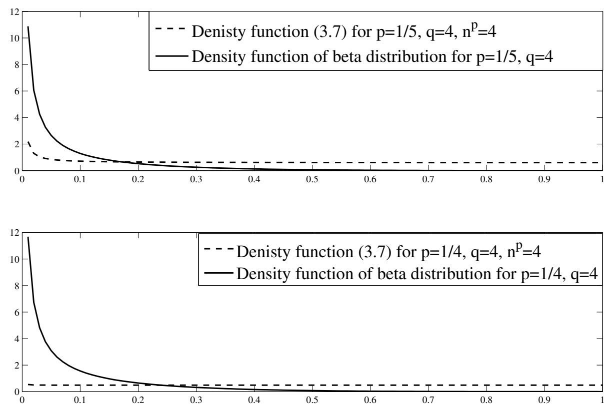

FIGURE 2. PDF of Beta distribution and of (3.5) for two different parameter arrays near the origin.

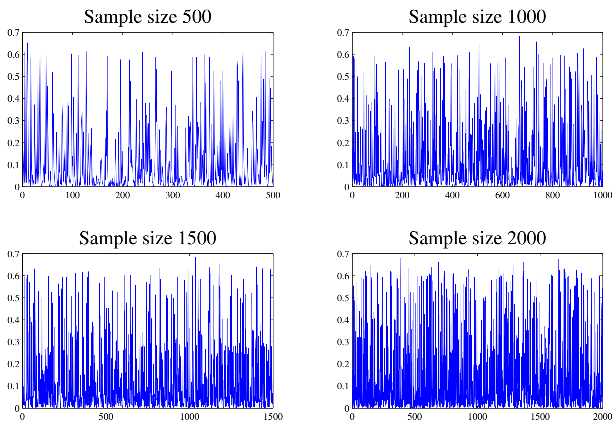

FIGURE 3. Sample paths.

4. Some properties of the LAB$_{p,q}$AR(1) model

Following the lines of the similar result for linear $AR(1)$ processes by Hamilton [**12**, pp. 53–56], it is very easy to derive the autocovariance and autocorrelation function of the model (3.1), with lag $\tau \in \mathbb{Z}$. The autocovariance and autocorrelation function are given with the following expressions, respectively:

$$\gamma(\tau) = \beta^{|\tau|} \operatorname{Var} X$$

$$\rho(\tau) = \beta^{|\tau|} \tag{4.1}$$

THEOREM 4.1. *Process X_t is not time reversible.*

PROOF. Using (3.1) and Markov properties of the process X_t, it is easy to show that the joint Laplace transform of X_t and X_{t-1}, is not symmetric with respect to λ_1 and λ_2. So the assertion of the theorem is proved. □

THEOREM 4.2. *Difference equation* (3.1) *has a unique, strictly stationary, $\mathcal{F}_t$-measurable and ergodic solution: $X_t = \sum_{j=0}^{\infty} \beta^j \xi_{t-j}$.*

PROOF. See [**17**], [**18**]. □

5. Parameter estimation in LAB$_{p,q}$AR(1) model

In this section, we estimate the parameters q and β of the LAPLAR$_q$(1) model using estimated autocorrelation function with lag 1, and estimated expectation of random variable X. Also we estimate the β appearing in the LAB$_{p,q}$AR(1), $p \in (0,1]$ model by a conditional least squares. The properties of this estimator are proved.

5.1. Parameter estimation in LAPLAR$_q$(1). The parameter β can be estimated using (4.1) for $p = 1$. Thus $\beta = \rho(1)$, where autocorrelation $\rho(1)$ with lag 1 can be estimated using the following formula:

$$\widehat{\rho}(1) = \frac{\sum_{t=2}^{n} \left(X_t - \overline{X}\right)\left(X_{t-1} - \overline{X}\right)}{\sum_{t=1}^{n} \left(X_t - \overline{X}\right)^2},$$

Since X_t is $\mathrm{B}(1,q)$, and stationarity condition is fulfilled, the expectation of random variable X is

$$EX = \frac{1}{1+q} \tag{5.1}$$

From (5.1) parameter q can be estimated using $q = \frac{\overline{X}-1}{\overline{X}}$, where the mean value of the generated sequence $\{X_t\}$ is $\overline{X}$.

5.2. Conditional least squares in LAB$_{p,q}$AR(1) model. Model (3.1) can be rewritten in the form:

$$Y_t = \beta Y_{t-1} + C_t, \qquad C_t = (\beta - 1)m + \xi_t,$$

where $Y_t = X_t - m$ and $EX_t = m$. We will assume that parameter m is known.

Let us denote by $\mathcal{F}_{t-1}$, σ-algebra defined by r.v. X_s, $s \leqslant t-1$. The conditional least squares estimator $\hat{\beta}$ of parameter β, based on the sample $(Y_1, \ldots, Y_N)$, is obtained by minimizing the following function:

$$\sum_{t=2}^{N} \big(Y_t - E(Y_t \,|\, \mathcal{F}_{t-1})\big)^2 = \sum_{t=2}^{N} \big(Y_t - \beta Y_{t-1}\big)^2, \tag{5.2}$$

with respect to β. If we equate the first derivative of (5.2) to 0, we get the estimator $\hat{\beta}$:

$$\hat{\beta} = \frac{\sum_{t=2}^{N} Y_{t-1} Y_t}{\sum_{t=2}^{N} Y_{t-1}^2} \tag{5.3}$$

THEOREM 5.1. *The estimator $\hat{\beta}$ is strongly consistent estimator of β. Furthermore, $\sqrt{N-1}(\hat{\beta}-\beta)$ has an asymptotic $\mathcal{N}(0,(1-\beta^2))$ distribution.*

PROOF. We will consider the difference:

$$\hat{\beta} - \beta = \frac{(N-1)^{-1} \sum_{t=2}^{N} u_t Y_{t-1}}{(N-1)^{-1} \sum_{t=2}^{N} Y_{t-1}^2}$$

Stationarity and ergodicity of $\{Y_t^2\}$ and $\{Y_{t-1}u_t\}$, followed by the stationarity and ergodicity of $\{X_t\}$. Expectation $V = EY_t^2$ is finite.

Since $E(R_t \,|\, \mathcal{F}_{t-1}) = 0$ and $\{Y_{t-1}\}$ is $\mathcal{F}_t$ measurable, it is valid:

$$E(u_t Y_{t-1}) = 0 \tag{5.4}$$

Ergodicity of $\{Y_t^2\}$ and $\{Y_{t-1}u_t\}$ implies:

$$\frac{1}{N-1} \sum_{t=2}^{N} Y_{t-1}^2 \xrightarrow{\text{a.s.}} V, \quad \frac{1}{N-1} \sum_{t=2}^{N} Y_{t-1} u_t \xrightarrow{\text{a.s.}} 0. \tag{5.5}$$

From (5.4) and (5.5) it can be concluded that $(\hat{\beta}-\beta) \xrightarrow{\text{a.s.}} 0$. Let us denote by $u_t = Y_t - E(Y_t \,|\, \mathcal{F}_{t-1}) = Y_t - \beta Y_{t-1}$ and $v_t = u_t^2$. It can be proved that

$$E(v_t \,|\, \mathcal{F}_{t-1}) = \big(1-\beta^2\big)\big(M - m^2\big),$$

where $EX_t^2 = M$. Let parameter α be an arbitrary constant from $\mathbb{R}$. Then, we have:

$$E(\alpha Y_{t-1} u_t)^2 = E\left\{\alpha^2 Y_{t-1}^2 E(v_t \,|\, \mathcal{F}_{t-1})\right\} = E\left\{\alpha^2 Y_{t-1}^2 \big(1-\beta^2\big)\big(M-m^2\big)\right\}$$

Since $E(\alpha Y_{t-1} u_t \,|\, \mathcal{F}_{t-1}) = 0$, from the Lindeberg–Levy central limit theorem for martingales [**3**], it follows that $(N-1)^{-1/2} \sum_{t=2}^{n} \alpha Y_{t-1} u_t$ converges to the normal distributed r.v. with mean 0 and variance $\alpha^2(1-\beta^2)(M-m^2)EY_{t-1}^2$. Bearing in mind that α is an arbitrary constant, we can conclude that $(N-1)^{-1/2} \sum_{t=2}^{N} Y_{t-1} u_t$ converges in distribution to the r.v. from normal distribution with mean 0 and variance $(1-\beta^2)(M-m^2)EY_{t-1}^2$ Now, we will consider the following expression:

$$(N-1)^{1/2}(\hat{\beta}-\beta) = \frac{(N-1)^{-1/2} \sum_{t=2}^{N} Y_{t-1} u_t}{(N-1)^{-1} \sum_{t=2}^{N} Y_{t-1}^2}. \tag{5.6}$$

From (5.6) we can conclude that $(N-1)^{1/2}(\hat{\beta}-\beta)$ converges in distribution to r.v. from normal distribution with mean 0 and variance $(1-\beta^2)$. So the assertion of the theorem is proved. $\square$

Since $X_t = \beta X_{t-1} + \xi_t > \beta X_{t-1}$, parameter β can be estimated using the following estimator:

$$\tilde{\beta} = \min_{2 \leqslant t \leqslant N} \left\{ \frac{X_t}{X_{t-1}} \right\} \tag{5.7}$$

Estimator (5.7) is a consistent estimator and it was analyzed in [**4**].

5.2.1. *Numerical Example.* Finally, let us present a numerical simulation of the parameter estimation based on previously given formulae (5.3) and (5.7). It will be assumed that parameters $p = 1/3$, $q = 4$ and $\beta = 0.125$ are known. Bearing in mind all earlier considerations, 100 samples of sizes 100, 1 000, 5 000, 10 000 and 20 000 were drawn using (3.1), and parameter β was estimated using (5.3) and (5.7). Since we had 100 samples for each size, the mean value of all 100 estimates per each sample size is reported in table [1]. The mean value of estimates which are the result of applying estimators (5.3) and (5.7) are denoted by β_{CLS} and β_{CON} respectively, and their standards deviations are STDEV(β_{CLS}) and STDEV(β_{CON}). More details about application of numerical simulations to different stochastic models, can be found in [**5**].

TABLE 1. Values of estimated parameter β

Sample size	β_{CLS}	STDEV(β_{CLS})	β_{CON}	STDEV(β_{CON})
100	0.1138	0.0923	0.1229	0.0027
1 000	0.1223	0.0309	0.125	0
5 000	0.1245	0.0137	0.125	0
10 000	0.1241	0.0099	0.125	0
20 000	0.1259	0.0066	0.125	0

The simulation shows that the estimator β_{CON} is much better than the estimator β_{CLS}. The estimator β_{CLS} converges very slowly to the true value. It is needed to generate huge samples for better accuracy of β_{CON}.

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