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HEREDITARILY INDECOMPOSABLE HAUSDORFF CONTINUA HAVE UNIQUE HYPERSPACES 2^X AND $C_n(X)$

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Abstract. Let X be a Hausdorff continuum (a compact connected Hausdorff space). Let 2^X (respectively, $C_n(X)$) denote the hyperspace of nonempty closed subsets of X (respectively, nonempty closed subsets of X with at most n components), with the Vietoris topology. We prove that if X is hereditarily indecomposable, Y is a Hausdorff continuum and 2^X (respectively $C_n(X)$) is homeomorphic to 2^Y (respectively, $C_n(Y)$), then X is homeomorphic to Y.

1. Introduction

A *Hausdorff continuum* is a compact connected Hausdorff space X with more than one point. A *subcontinuum* A of X is a closed connected subset of X.

For a Hausdorff continuum X and a positive integer n, define the hyperspaces

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$C(X) = C_1(X),$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

The hyperspace 2^X is endowed with the Vietoris topology. That is, the basis for the topology of 2^X is the family $\mathcal{B} = \{\langle U_1, \dots, U_n\rangle : n \text{ is a positive integer and } U_1, \dots, U_n \text{ are open subsets of } X\}$, where $\langle U_1, \dots, U_n\rangle = \{A \in 2^X : A \subset U_1 \cup \dots \cup U_n \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, \dots, n\}\}$.

The Hausdorff continuum X is *hereditarily indecomposable* provided that given $A, B \in C(X)$, either $A \cap B = \emptyset$ or $A \subset B$ or $B \subset A$ and X is said to *have unique hyperspace* 2^X (resp., $C_n(X)$), provided that, if Y is a Hausdorff continuum and

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2^X is homeomorphic to 2^Y (resp., $C_n(X)$ is homeomorphic to $C_n(Y)$), then X is homeomorphic to Y.

Uniqueness of hyperspaces has been widely studied (see [**2**], [**3**, p. 285] and [**4**]). Macías proved in [**7**] and [**8**, Theorem 6.1] that if X is a metric hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n). Lončar [**6**, Theorem 2.4] proved that rim metrizable hereditarily indecomposable continua have unique hyperspace $C(X)$. Using techniques of inverse limits, Peláez generalized these results by proving that if X is a rim metrizable (X has a basis of neighborhoods with metrizable boundary) hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n).

In this paper we generalize these results by proving:

THEOREM 1.1. *If X is a hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n).*

2. Generalized arcs

The proof of Theorem 1.1 is based in an analysis of the generalized arcs (see below) in the hyperspace 2^X.

A *generalized arc joining p and q* in a topological space Z is a subcontinuum α of Z such that $p, q \in \alpha$ and each point $z \in \alpha - \{p, q\}$, separates p and q in α. Given $z \in \alpha - \{p, q\}$, let $\alpha - \{z\} = U_z \cup V_z$, where U_z and V_z are disjoint open subsets of α such that $p \in U_z$ and $p \in V_z$. Let $U_p = \emptyset = V_q$, $V_p = \alpha - \{p\}$ and $U_q = \alpha - \{q\}$. If $z, w \in \alpha$ and $z \neq w$, define $z < w$ if and only if $z \in U_w$.

The following lemma summarizes the basic facts about generalized arcs. For the proof of (b), see Theorem 6.16 of [**10**]. The rest of Lemma 2.1 is easy to prove.

LEMMA 2.1. *Let α be a generalized arc joining points p and q in a topological space Z. Then:*

(a) *the relation $<$ is a well defined linear order,*
(b) *α has the topology induced by $<$,*
(c) *given $z, w \in \alpha$, the intervals $[z, w), (z, w), (z, w]$ and $[z, w]$, defined in the natural way, are connected subsets of α,*
(d) *$(\alpha, <)$ has the property of the supremum (every nonempty subset of α has a supremum in α),*
(e) *the relation $\backsim$ defined in Z by $z \backsim w$ if an only if there exists a generalized arc joining z and w in Z, is an equivalence relation; the equivalence classes are called* g-arcwise components.

Let X be a Hausdorff continuum. Given $A, B \in 2^X$, with $A \subsetneq B$, an *order arc in 2^X, from A to B* is a subcontinuum α of 2^X such that $A \subset C \subset B$ for each $C \in \alpha$ and, for every $C, D \in \alpha$, either $C \subset D$ or $D \subset C$. The fundamental theorem for order arcs in hyperspaces is the following.

THEOREM 2.2. [**5**, Theorem 15.3] *Let X be a Hausdorff continuum and let $A, B \in 2^X$ be such that $A \subsetneq B$. Then there exists an order arc in 2^X, from A to B if and only if each component of B intersects A.*

LEMMA 2.3. *Let X be a Hausdorff continuum, $A, B \in 2^X$ and let α be and order arc in 2^X, from A to B. Then α is a generalized arc joining A and B in 2^X.*

PROOF. It is enough to show that for each $C \in \alpha - \{A, B\}$, C separates A and B in α. Let $\mathcal{A} = \{D \in \alpha : D \subset C\}$ and $\mathcal{B} = \{D \in \alpha : C \subset D\}$. It is easy to show that $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of α. By the definition of order arc, we have $\alpha = \mathcal{A} \cup \mathcal{B}$, $A \in \mathcal{A} - \{C\}$, $B \in \mathcal{B} - \{C\}$ and $\mathcal{A} \cap \mathcal{B} = \{C\}$. This proves that C separates A and B in α. $\square$

3. Property of Kelley

A Hausdorff continuum X is said to have the *Property of Kelley* (see [**1**, p. 115]), provided that for every $p \in X$, $A \in C(X)$ and open subset $\mathcal{U}$ of $C(X)$ such that $p \in A \in \mathcal{U}$, there exists an open subset T of X such that $p \in T$ and, for each $q \in T$, there exists $B \in C(X)$ such that $q \in B \in \mathcal{U}$.

For metric Hausdorff continua, property of Kelley has been widely studied, see ([**5**, pp. 167 and 405] and [**9**, Chapter XIV]). In [**9**, Theorem 16.27] it is proved that hereditarily indecomposable Hausdorff metric continua have property of Kelley. Next, we extend this result to Hausdorff continua.

THEOREM 3.1. *Let X be a hereditarily indecomposable Hausdorff continuum. Then X has property of Kelley.*

PROOF. Suppose that X does not have the property of Kelley. Then there exist $p \in X$, $A \in C(X)$ and $\mathcal{V} = \langle V_1, \ldots, V_m \rangle \cap C(X)$, a basic set in $C(X)$, such that $p \in A \in \mathcal{V}$ and for each open subset T of X such that $p \in T$ there exists a point $q \in T$ for which there is no element $B \in C(X)$ with the property that $q \in B \in \mathcal{V}$. Note that $A \neq X$.

Let W be an open subset of X such that $A \subset W \subset \mathrm{cl}_X(W) \subset V_1 \cup \cdots \cup V_m$ and $W \neq X$. Let $\mathcal{T} = \{T \subset X : T$ is an open subset of X such that $p \in T \subset W\}$. For each $T \in \mathcal{T}$, choose a point $q_T \in T$ for which there is no element $B \in C(X)$ with the property that $q_T \in B \in \mathcal{V}$ and let D_T be the component of $\mathrm{cl}_X(W)$ such that $q_T \in D_T$. Since $D_T \subset V_1 \cup \cdots \cup V_m$ and $D_T \notin \mathcal{V}$, there exists $i \in \{1, \ldots, m\}$ such that $D_T \cap V_i = \emptyset$. Let $E = \{q_T \in X : T \in \mathcal{T}\}$. Then $p \in \mathrm{cl}_X(E)$. For each $i \in \{1, \ldots, m\}$, let $\mathcal{T}_i = \{T \in \mathcal{T} : D_T \cap V_i = \emptyset\}$ and $E_i = \{q_T \in X : T \in \mathcal{T}_i\}$. Then $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_m$ and $E = E_1 \cup \cdots \cup E_m$. Thus, there exists $i \in \{1, \ldots, m\}$ such that $p \in \mathrm{cl}_X(E_i)$. We may assume that $p \in \mathrm{cl}_X(E_1)$.

For each $Q \in \mathcal{T}$, let $F(Q) = \mathrm{cl}_X\left(\bigcup\{D_T : T \in \mathcal{T}_1 \text{ and } q_T \in Q\}\right)$. Since $p \in \mathrm{cl}_X(E_1)$, it follows that $p \in F(Q)$. Thus, $F(Q)$ is a nonempty compact subset of X. Let $F = \bigcap\{F(Q) : Q \in \mathcal{T}\}$ and let G be the component of F such that $p \in G$.

We need to show that $\mathrm{Fr}_X(W) \cap G \neq \emptyset$. Suppose to the contrary that $\mathrm{Fr}_X(W) \cap G = \emptyset$. By [**5**, Theorem 12.9], there exist disjoint compact subsets K and L of X such that $F = K \cup L$, $G \subset K$ and $K \cap \mathrm{Fr}_X(W) = \emptyset$. Let R and S be disjoint open subsets of X such that $K \subset R$ and $\mathrm{Fr}_X(W) \cup L \subset S$. Since $F \subset R \cup S$, there exist $k \geqslant 1$ and elements $Q_1, \ldots, Q_k \in \mathcal{T}$ of X such that $F \subset F(Q_1) \cap \cdots \cap F(Q_k) \subset R \cup S$. Let $Q_0 = Q_1 \cap \cdots \cap Q_k$. Then Q_0 is an open subset of X such that $p \in Q_0 \subset W$ and

$F(Q_0) \subset R \cup S$. Since $p \in R \cap Q_0$ and $p \in \mathrm{cl}_X(E_1)$, there exists $T \in \mathcal{T}_1$ such that $q_T \in R \cap Q_0 \cap D_T$. Then $D_T \subset F(Q_0) \subset R \cup S$. Since D_T is connected, $D_T \subset R$, so $\mathrm{Fr}_X(W) \cap D_T = \emptyset$, this contradicts [**5**, Theorem 12.10]. We have shown that $\mathrm{Fr}_X(W) \cap G \neq \emptyset$.

Recall that $G \in C(X)$ and $p \in G \cap A$. Since X is hereditarily indecomposable, $G \subset A$ or $A \subset G$. Since $\mathrm{Fr}_X(W) \cap G \neq \emptyset$ and $\mathrm{Fr}_X(W) \cap A = \emptyset$, we obtain that $A \subset G$. Since $A \cap V_1 \neq \emptyset$, we can choose a point $x \in A \cap V_1 \subset F \cap V_1$. Then $x \in V_1 \cap F(W)$. Thus, there exists $T \in \mathcal{T}_1$ such that $V_1 \cap D_T \neq \emptyset$. This contradicts the definition of $\mathcal{T}_1$ and ends the proof of the theorem. □

4. Main results

Lemma 4.1. *Let X be a hereditarily indecomposable Hausdorff continuum. Suppose that $A, B \in 2^X$, $C \in C(X)$, $A \subset C$, $B \nsubseteq C$ and α is a generalized arc joining A and B in 2^X. Then $C \in \alpha$.*

Proof. Let $\mathcal{D} = \{D \in \alpha : D \subset C\}$. Then $\mathcal{D}$ is a nonempty proper closed subset of α. Since α is connected, there exists $D_0 \in \mathrm{Fr}_\alpha(\mathcal{D})$. Since $\mathcal{D}$ is closed in α, $D_0 \subset C$. Let $<$ be the order defined, as in the paragraph previous to Lemma 2.1, for α satisfying $A < B$. We claim that $D_0 = C$. Suppose to the contrary that $D_0 \neq C$. Fix a point $x_0 \in C - D_0$. Since $\langle X - \{x_0\}\rangle \cap \alpha$ is an open subset of α containing D_0, by Lemma 2.1(b), we may assume that there exists an element $D_1 \in \alpha$, such that $D_0 < D_1$ such that $[D_0, D_1] \subset \langle X - \{x_0\}\rangle$ and $[D_0, D_1] \nsubseteq \mathcal{D}$. Let $E = \bigcup\{F : F \in [D_0, D_1]\}$. Notice that $D_0 \subset E$, $x_0 \notin E$ and $E \nsubseteq C$. We check that E is closed in X. Let $x \in X - E$. Then $\{\langle X - \mathrm{cl}_X(U)\rangle : x \in U$ and U is an open subset of $X\}$ is an open cover of the compact set $[D_0, D_1]$. Thus, there exists an open subset U_0 of X such that $x \in U_0$ and $[D_0, D_1] \subset \langle X - \mathrm{cl}_X(U_0)\rangle$. Hence, $U_0 \cap E = \emptyset$. This proves that E is closed in X. Fix a point $y_0 \in E - C$.

We claim that the component G of E that contains y_0 intersects C. Suppose to the contrary that $G \cap C = \emptyset$. By [**5**, Theorem 12.9], there exists an open and closed subset K of E such that $y_0 \in K$ and $K \cap C = \emptyset$. Let $L = E - K$. Then L is a compact subset of X and $D_0 \subset L$. Let $\mathcal{E} = \{F \in [D_0, D_1] : F \subset L\}$ and $\mathcal{F} = \{F \in [D_0, D_1] : F \cap K \neq \emptyset\}$. Clearly, $\mathcal{E}$ and $\mathcal{F}$ are closed disjoint subsets of $[D_0, D_1]$, $D_0 \in \mathcal{E}$, $\mathcal{F} \neq \emptyset$ and $[D_0, D_1] = \mathcal{E} \cup \mathcal{F}$. This contradicts the connectedness of $[D_0, D_1]$ and completes the proof that $G \cap C \neq \emptyset$.

Since X is hereditarily indecomposable, G and C are subcontinua of X and $G \cap C \neq \emptyset$, we have $G \subset C$ or $C \subset G$. Since $y_0 \in G - C$ and $G \subset E \subset X - \{x_0\}$, we obtain a contradiction. This ends the proof that $C = D_0 \in \alpha$. □

The following lemma can be proved imitating the proof of Theorem 11.3 of [**9**] and using Theorem 2.2.

Lemma 4.2. *Let Y be a Hausdorff continuum, $n \geqslant 1$ and $A \in 2^Y - C(Y)$. Then $2^Y - \{A\}$ and $C_n(Y) - \{A\}$ are arcwise connected.*

Lemma 4.3. *Let X be a hereditarily indecomposable continuum and $n \geqslant 1$. Let $A \in C(X) - (\{X\} \cup F_1(X))$. Let $\mathcal{E}$ be the g-arcwise component of $C_n(X) - \{A\}$ that contains X. Then $\mathcal{E}$ is the only dense g-arcwise component of $C_n(X) - \{A\}$.*

Proof. Let $\mathcal{F}$ be a g-arcwise component of $C_n(X)-\{A\}$ such that $\mathcal{F}\neq\mathcal{E}$. Given $B\in C_n(X)-\{A\}$ such that $B\nsubseteq A$, taking an order arc from B to X, we obtain that $B\in\mathcal{E}$. This implies that $\mathcal{F}\subset C_n(A)$. Since $C_n(A)$ is compact and $C_n(A)\neq C_n(X)$, we conclude that $\mathcal{F}$ is not dense in $C_n(X)-\{A\}$.

Now we check that $\mathcal{E}$ is dense in $C_n(X)-\{A\}$. Let $B\in C_n(X)-\{A\}$ and $\mathcal{U}=\langle U_1,\ldots,U_m\rangle\cap C_n(X)$ be a basic set in $C_n(X)$ such that $B\in\mathcal{U}$, where each U_i is a nonempty open subset of X. We may assume that $B\notin\mathcal{E}$. Then $B\subsetneq A$. Let $B_1,\ldots,B_k$ be the components of B. Then $k\leqslant n$. Fix points $p_1\in B_1,\ldots,p_k\in B_k$. For each $i\in\{1,\ldots,k\}$, let $F_i=\{j\in\{1,\ldots,m\}:B_i\cap U_j\neq\emptyset\}$ and $\mathcal{U}_i=\{C\in C(X):C\subset\bigcup\{U_j:j\in F_i\}$ and $C\cap U_j\neq\emptyset$ for each $j\in F_i\}$. Then $\mathcal{U}_i$ is open in $C(X)$ and $B_i\in\mathcal{U}_i$. By Theorem 3.1, there exists an open subset V_i of X such that $p_i\in V_i$ and, if $u\in V_i$, then there exists $Q\in C(X)$ such that $u\in Q\in\mathcal{U}_i$.

Fix a point $x\in X-A$. Let α be an order arc in $C(X)$ joining $\{x\}$ to X. Notice that $\alpha\subset\mathcal{E}$. Let $E_0\in\alpha$ be the first element of α containing A, that is $E_0\in\alpha$, $A\subset E_0$ and, if $E\subsetneq E_0$ and $E\in\alpha$, then $A\nsubseteq E$. Since the set $\langle V_1,\ldots,V_k,X\rangle\cap C(X)$ is an open subset of $C(X)$ containing E_0, there exists $E\in\alpha$ such that $E\subsetneq E_0$ and $E\in\langle V_1,\ldots,V_k,X\rangle$. For each $i\in\{1,\ldots,k\}$, choose a point $u_i\in E\cap V_i$. By the choice of V_i, there exists $Q_i\in C(X)$ such that $u_i\in Q_i\in\mathcal{U}_i$. Let $D=\{u_1,\ldots,u_k\}$.

Since $x\in E$, $E\nsubseteq A$. Since X is hereditarily indecomposable and $E\in C(X)$, $E\cap A=\emptyset$. Thus, $D\cap A=\emptyset$. Let $Q=Q_1\cup\cdots\cup Q_k$, since $D\subset Q$, we have that $Q\in\mathcal{E}$. It is easy to show that $Q\in\mathcal{U}$. This completes the proof of the lemma. □

Theorem 4.4. *If X is a hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$, for each n.*

Proof. Let Y be a Hausdorff continuum. Let $\mathcal{K}(X)$ denote one of the hyperspaces 2^X or $C_n(X)$ of X and let $\mathcal{K}(Y)$ be the correspondig hyperspace of Y. Suppose that $\mathcal{K}(X)$ is homeomorphic to $\mathcal{K}(Y)$. Let $h:\mathcal{K}(X)\to\mathcal{K}(Y)$ be a homeomorphism. Since $F_1(X)$ (respectively $F_1(Y)$) is homeomorphic to X (respectively Y) it is enough to show that $h(F_1(X))=F_1(Y)$.

Using Proposition 6.3 of [**9**], it is easy to show that every proper subcontinuum of X has empty interior.

Claim 1. Let $A\in C(X)-(\{X\}\cup F_1(X))$. Let $\mathcal{E}$ be the g-arcwise component of $\mathcal{K}(X)-\{A\}$ that contains X. Then $\mathcal{E}$ is the only dense g-arcwise component of $\mathcal{K}(X)-\{A\}$.

We prove Claim 1. By Lemma 4.3, Claim 1 holds for the case that $\mathcal{K}(X)=C_n(X)$. Thus, in the proof of Claim 1, we assume that $\mathcal{K}(X)=2^X$. Let $\mathcal{F}$ be a g-arcwise component of $2^X-\{A\}$ such that $\mathcal{F}\neq\mathcal{E}$. Given $B\in 2^X-\{A\}$ such that $B\nsubseteq A$, taking an order arc from B to X, we obtain that $B\in\mathcal{E}$. This implies that $\mathcal{F}\subset 2^A$. Since 2^A is compact and $2^A\neq 2^X$, we conclude that $\mathcal{F}$ is not dense in $2^X-\{A\}$. Now we check that $\mathcal{E}$ is dense in $2^X-\{A\}$. Let $B\in 2^X-\{A\}$ and let $\mathcal{U}=\langle U_1,\ldots,U_m\rangle$ be a basic set in 2^X such that $B\in\mathcal{U}$, where each U_i is a nonempty open subset of X. Since $\operatorname{int}_X(A)=\emptyset$, for each $i\in\{1,\ldots,m\}$, $U_i\nsubseteq A$.

Given $i\in\{1,\ldots,m\}$, fix a point $p_i\in U_i-A$. Let $G=\{p_1,\ldots,p_m\}$. Then $G\in\mathcal{U}\cap\mathcal{E}$. This completes the proof of Claim 1.

Claim 2. $h(C(X)) \subset C(Y)$.

In order to prove Claim 2, take an element $A \in C(X) - (\{X\} \cup F_1(X))$. By Lemma 4.1, $\mathcal{K}(X) - \{A\}$ is not g-arcwise connected. By Lemma 4.2, if $h(A)$ is not connected, then $\mathcal{K}(Y) - \{h(A)\}$ is g-arcwise connected. Since h is a homeomorphism, we conclude that $h(A)$ is connected. That is, $h(A) \in C(Y)$. Therefore, $h(C(X) - (\{X\} \cup F_1(X))) \subset C(Y)$. Since $\mathrm{cl}_{2^X}(C(X) - (\{X\} \cup F_1(X))) = C(X)$ and $C(Y)$ is compact, we conclude that $h(C(X)) \subset C(Y)$.

Claim 3. Let $A \in C(X) - \{X\}$ and let $B \in \mathcal{K}(X)$ be such that $B \subset A$. Then $h(B) \subset h(A)$.

We prove Claim 3. In the case that $A \in F_1(X)$, $B = A$ and $h(B) = h(A)$. Thus, suppose that $A \notin F_1(X)$. By Claim 2, $h(A)$ is connected. We can suppose that $h(A) \neq Y$. Let $\mathcal{E}$ (respectively, $\mathcal{F}$) be the g-arcwise component of $\mathcal{K}(X) - \{A\}$ (respectively, $\mathcal{K}(Y) - \{h(A)\}$) that contains X (respectively, Y). By Lemma 4.1, $B \notin \mathcal{E}$. If $h(B) \not\subset h(A)$, then taking an order arc from $h(B)$ to Y in $\mathcal{K}(Y)$ we can prove that $h(B) \in \mathcal{F}$. Since h is a homeomorphism, $h(\mathcal{E}) \neq \mathcal{F}$, so $h(\mathcal{E}) \cap \mathcal{F} = \emptyset$. Since $\mathcal{K}(Y) - (2^{h(A)} \cap \mathcal{K}(Y)) \subset \mathcal{F}$, $h(\mathcal{E}) \subset 2^{h(A)} \cap \mathcal{K}(Y)$. Since $h(A) \neq Y$, $2^{h(A)} \cap \mathcal{K}(Y)$ is a proper compact subset of $\mathcal{K}(Y)$. Thus, $h(\mathcal{E})$ is not dense in $\mathcal{K}(Y) - \{h(A)\}$. This implies that $\mathcal{E}$ is not dense in $\mathcal{K}(X) - \{A\}$, contrary to Claim 1. This proves that $h(B) \subset h(A)$.

Claim 4. $h(F_1(X)) \subset F_1(Y)$.

Suppose, contrary to Claim 4, that there exists a point $p \in X$ such that $h(\{p\}) \notin F_1(Y)$. Since $F_1(Y)$ is compact, there exists an open subset U of X such that $p \in U$ and $\langle U \rangle \subset \mathcal{K}(X) - h^{-1}(F_1(Y))$. Taking an order arc from $\{p\}$ to X in $C(X)$, it is possible to find two nondegenerate proper subcontinua A and B of X such that $p \in A \subsetneq B \subset U$. Let $\mathcal{E}$ be the g-arcwise connected component of $\mathcal{K}(X) - \{B\}$ such that $\{p\}, A \in \mathcal{E}$. By Lemma 4.1, $\mathcal{E} \subset \mathcal{K}(X) \cap 2^B \subset \langle U \rangle$. Thus $h(\mathcal{E}) \subset \mathcal{K}(Y) - F_1(Y)$. Since h is a homeomorphism, $h(\mathcal{E})$ is a g-arcwise component of $\mathcal{K}(Y) - \{h(B)\}$. By Claims 2 and 3, $h(A) \in C(Y)$ and $h(A) \subsetneq h(B)$. Fix a point $y \in h(A)$ and take an order arc β from $\{y\}$ to $h(A)$ in $C(Y)$. Notice that $\beta \subset \mathcal{K}(Y) - \{h(B)\}$. Since $h(A) \in h(\mathcal{E})$, $\beta \subset h(\mathcal{E})$. Hence, $\{y\} \in h(\mathcal{E}) \cap F_1(Y)$, a contradiction. Claim 4 is proved.

Claim 5. Let $A \in C(X)$. Then $h(A) = \bigcup\{h(\{p\}) \in \mathcal{K}(Y) : p \in A\}$.

Let $B = \bigcup\{h(\{p\}) \in \mathcal{K}(Y) : p \in A\}$. By [**1**, Lemma 2.1], B is a subcontinuum of Y. In the case that $A \neq X$, by Claim 3, $B \subset h(A)$. Now, we see that, in the case that $A = X$, we also have that $B \subset h(A)$. It is enough to show that, if $p \in X$, then $h(\{p\}) \subset h(X)$. Suppose to the contrary that there exists a point $y \in h(\{p\}) - h(X)$. Let $\mathcal{W} = h^{-1}(\langle Y - \{y\} \rangle)$. Then $\mathcal{W}$ is an open subset of $\mathcal{K}(X)$ such that $X \in \mathcal{W}$. Let λ be an order arc from $\{p\}$ to X in $C(X)$. Then there exists $E \in \lambda$ such that $E \neq X$ and $E \in \mathcal{W}$. By Claim 3, $h(\{p\}) \subset h(E)$. Thus, $y \in h(E) \in \langle Y - \{y\} \rangle$. This contradiction proves that, in every case, $B \subset h(A)$.

Suppose that $B \neq h(A)$. Let $\mathcal{V} = \mathcal{K}(Y) - (2^B \cap \mathcal{K}(Y))$ and $\mathcal{U} = h^{-1}(\mathcal{V})$. Then $\mathcal{U}$ is an open subset of $\mathcal{K}(X)$ such that $A \in \mathcal{U}$. Fix a point $p_0 \in A$ and let α be an order arc from $\{p_0\}$ to A in $C(X)$. Then there exists $C \in \alpha - \{A, \{p_0\}\}$ such that $C \in \mathcal{U}$. Then $\{p_0\} \subsetneq C \subsetneq A$. Fix a point $q_0 \in A - C$. Since $h(\{p_0\}), h(\{q_0\}) \subset B$, taking order arcs in $C(Y)$, it is possible to construct a generalized arc β joining $h(\{p_0\})$ and $h(\{q_0\})$ in $2^B \cap \mathcal{K}(Y)$. Thus, $h^{-1}(\beta)$ is a generalized arc in $\mathcal{K}(X)$ joining $\{p_0\}$ and $\{q_0\}$. By Lemma 4.1, $C \in h^{-1}(\beta)$. Thus, $h(C) \in \beta \subset 2^B \cap \mathcal{K}(Y)$. Hence, $h(C) \in 2^B \cap \mathcal{K}(Y) \cap \mathcal{V}$, a contradiction. We have shown that $B = h(A)$. This ends the proof of Claim 5.

Claim 6. Let $A \in \mathcal{K}(X)$. Then $h(A) \subset h(X)$.

In order to prove claim 6, we may assume that $A \neq X$. By Claim 2, $h(X)$ is connected. In order to prove Claim 6, since $2^{h(X)} \cap \mathcal{K}(Y)$ is compact, it is enough to show that $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y))$. Let $\mathcal{V}$ be an open subset of $\mathcal{K}(Y)$ such that $h(A) \in \mathcal{V}$. Let $\mathcal{U} = \langle U_1, \dots, U_m\rangle \cap \mathcal{K}(X)$ be a basic open subset of $\mathcal{K}(X)$ such that $A \in \mathcal{U} \subset h^{-1}(\mathcal{V})$ and $X \notin \mathcal{U}$, where each U_i is open in X. Note that $X \nsubseteq U_1 \cup \dots \cup U_m$. Fix a point $p \in X$ and let α be an order arc from $\{p\}$ to X in $C(X)$. We analyze two cases:

Case 1. $\mathcal{K}(X) = 2^X$. Since $X \in \langle U_1, \dots, U_m, X\rangle$, there exists $B \in \alpha \cap \langle U_1, \dots, U_m, X\rangle$ such that $B \neq X$. For each $i \in \{1, \dots, m\}$, choose a point $p_i \in B \cap U_i$. Let $C = \{p_1, \dots, p_m\}$. Then $C \subset B$, $C \in \mathcal{U}$ and $h(C) \in \mathcal{V}$. By Claim 3, $h(C) \subset h(B)$. By Claim 5, $h(B) = \bigcup\{h(\{b\}) \in 2^Y : b \in B\} \subset \bigcup\{h(\{x\}) \in 2^Y : x \in X\} = h(X)$. Hence, $h(C) \in 2^{h(X)}$. Hence, $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y)) = 2^{h(X)} \cap \mathcal{K}(Y)$.

Case 2. $\mathcal{K}(X) = C_n(X)$. Let $A_1, \dots, A_k$ be the components of A. Then $k \leqslant n$. For each $i \in \{1, \dots, k\}$, choose a point $a_i \in A_i$, let $F_i = \{j \in \{1, \dots, m\} : A_i \cap U_j \neq \emptyset\}$ and $\mathcal{U}_i = \{C \in C(X) : C \subset \bigcup\{U_j : j \in F_i\}$ and $C \cap U_j \neq \emptyset$ for each $j \in F_i\}$. Then $\mathcal{U}_i$ is open in $C(X)$ and $A_i \in \mathcal{U}_i$. By Theorem 3.1, there exists an open subset V_i of X such that $a_i \in V_i$ and, if $u \in V_i$, then there exists $Q \in C(X)$ such that $u \in Q \in \mathcal{U}_i$. Since $X \in \langle V_1, \dots, V_k, X\rangle \cap \mathcal{K}(X)$, there exists $B \in (\alpha - \{X\}) \cap \langle V_1, \dots, V_k, X\rangle$. For each $i \in \{1, \dots, k\}$, fix a point $b_i \in B \cap V_i$ and let $Q_i \in C(X)$ be such that $b_i \in Q_i \in \mathcal{U}_i$. Let $Q = Q_1 \cup \dots \cup Q_k$. Then $Q \in C_n(X)$ and $Q \in \mathcal{U}$. Since $Q \subset U_1 \cup \dots \cup U_m$, $Q \neq X$. Since X is hereditarily indecomposable and $B \neq X$, we obtain that $B \cup Q \neq X$. Notice that $B \cup Q$ is a subcontinuum of X. By Claim 3, $h(Q) \subset h(B \cup Q)$ and, by Claim 5, $h(B \cup Q) = \bigcup\{h(\{b\}) \in 2^Y : b \in B \cup Q\} \subset \bigcup\{h(\{x\}) \in 2^Y : x \in X\} = h(X)$. Thus, $h(Q) \subset h(X)$ and $h(Q) \in \mathcal{V}$. Therefore, $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y)) = 2^{h(X)} \cap \mathcal{K}(Y)$. Claim 6 is proved.

We are ready to show that $h(F_1(X)) = F_1(Y)$. Let $y \in Y$ and $B = h^{-1}(\{y\})$. By Claim 6, $\{y\} = h(B) \subset h(X)$. By Claim 5, $h(X) = \bigcup\{h(\{x\}) \in 2^Y : x \in X\}$. Thus, there exists $x \in X$ such that $y \in h(\{x\})$. By Claim 4, $h(\{x\})$ is a one-point set. Hence, $\{y\} = h(\{x\}) \in h(F_1(X))$. We have shown that $F_1(Y) \subset h(F_1(X))$. Thus, Claim 4 implies that $h(F_1(X)) = F_1(Y)$. This ends the proof of the theorem. □

Question 1. (see [**3**, Problem 37]) Let X be a hereditarily indecomposable Hausdorff continuum. Is it true that X has a unique hyperspace $F_n(X)$, for each

$n \geqslant 2$? That is, suppose that Y is a Hausdorff continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$, does it follows that X is homeomorphic to Y?

The answer to this question is not known even for the case that X is a metric continuum and $n = 2$.

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