PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 89(103) (2011), 89–93 DOI: 10.2298/PIM1103089Z

ON ROOTS OF POLYNOMIALS WITH POSITIVE COEFFICIENTS

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Communicated by Žarko Mijajlović

Abstract. Let α be an algebraic number with no nonnegative conjugates over the field of the rationals. Settling a recent conjecture of Kuba, Dubickas proved that the number α is a root of a polynomial, say P, with positive rational coefficients. We give in this note an upper bound for the degree of P in terms of the discriminant, the degree and the Mahler measure of α; this answers a question of Dubickas.

1. Introduction

An element α of the set $\mathbb{C}$ of complex numbers is called an algebraic number if it is a root of a nonzero polynomial with coefficients in the field of the rationals $\mathbb{Q}$. Among nonzero elements P of the ring $\mathbb{Q}[x]$ and satisfying the condition $P(\alpha) = 0$, there is only one monic polynomial having the smallest possible degree; this polynomial is called the minimal polynomial of α and is usually noted Min_α. The roots of Min_α are the conjugates of α, and the degree of α is the degree of Min_α. In these pages, the notions of minimal polynomial, conjugates and degree of an algebraic number are considered over $\mathbb{Q}$.

In his study of some classes of algebraic numbers on the unit circle, Kuba [3] considered the roots of polynomials with positive rational coefficients. A complex number is said to be *positively algebraic* if it is a root of a polynomial, say P, with positive rational coefficients [3]. In fact (as it was signaled in [2] and [3]) we may replace in this last definition the word *positive* by the sentence *nonnegative and such that* $P(0) \neq 0$, because the coefficients of the polynomial $P(x)(1 + x + \cdots + x^{\deg(P)})$, where $\deg(P)$ is the degree of P, are positive when the coefficients of P are nonnegative and $P(0) > 0$. Clearly, a positively algebraic number is an algebraic number, and none of its conjugates is a nonnegative real number. Kuba conjectured that the converse of the last proposition is true, and verified this conjecture for some particular cases, especially when α is quadratic or when

2010 *Mathematics Subject Classification:* 11R04; 12D10; 11R06.

the Galois group of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is isomorphic to the symmetric group $S_{\deg(\mathrm{Min}_\alpha)}$ [3]. The question of Kuba did not remain open for a long time, since Dubickas has shown that "*an algebraic number with no nonnegative conjugates is a root of a polynomial, say again P, with positive rational coefficients*" [2]. At the end of his paper, Dubickas has remarked that the proof of the last mentioned proposition does not provide any estimation for the degree of P. In fact, replacing the arguments of the distribution modulo 1, by a simple geometrical argument we obtain the following result.

Theorem 1.1. *Let α be an algebraic number with no nonnegative conjugates. Then, there is a polynomial with positive rational coefficients, vanishing at α and with degree less than*

$$\frac{2d\pi}{\arcsin\left(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\right)},$$

where d, Δ and M are the degree, the discriminant, and the Mahler measure of α, respectively.

Recall that if

$$\mathrm{Min}_\alpha(x) = \prod_{1\leqslant j\leqslant d}(x-\alpha_j) = x^d + \frac{a_{d-1}}{b_{d-1}}x^{d-1} + \cdots + \frac{a_0}{b_0},$$

where the rational integers $a_0, \ldots, a_{d-1}$, and the positive rational integers $b_0, \ldots, b_{d-1}$ are so that the fractions $\frac{a_{d-1}}{b_{d-1}}, \ldots, \frac{a_0}{b_0}$ are irreducible, then

$$\Delta = \mathrm{lcm}(b_0, \ldots, b_{d-1})^{2d-2} \prod_{1\leqslant j<k\leqslant d}(\alpha_j-\alpha_k)^2$$

and

$$M = \mathrm{lcm}(b_0, \ldots, b_{d-1}) \prod_{1\leqslant j\leqslant d} \max\{1, |\alpha_j|\}.$$

The proof of Theorem 1.1 appears in the last section and is based on two auxiliary results, due essentially to Dubickas, and explained in the next section.

2. Two lemmas

The following result is an improvement of Lemma 2 of [2].

Lemma 2.1. *Let $\omega = |\omega|e^{i\theta} \in \mathbb{C} - \{0\}$, where $i^2=-1$, $\theta \in \left[\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}\right[$ and n is a nonnegative rational integer. Then, there is $T \in \mathbb{Q}[x]$, with degree $2^{n+2}-3$ and such that the coefficients of the polynomial $(x-\omega)(x-\bar{\omega})T(x)$, where $\bar{\omega}$ is the complex conjugate of ω, are positive.*

Proof. The scheme of the proof is identical to the one of Lemma 2 of [2] with minor modifications. We prefer to give some details of this proof. To simplify the notation set $m=2^n$. Then, $\omega^m = |\omega|^m e^{im\theta}$, $\frac{\pi}{2} \leqslant m\theta < \pi$, $|\omega|^{2m} > 0$, $\cos m\theta \leqslant 0$ and the coefficients of the polynomial

$$(x^m-\omega^m)(x^m-\bar{\omega}^m) = x^{2m} - 2|\omega|^m(\cos m\theta)x^m + |\omega|^{2m},$$

are nonnegative real numbers. A simple calculation shows that the coefficients of

$$(x^m-\omega^m)(x^m-\bar{\omega}^m)\sum_{k=0}^{2m-1}x^k$$

are positive. For $z\in\mathbb{C}$ let

$$T_z(x):=\frac{(x^m-z^m)(x^m-\bar{z}^m)}{(x-z)(x-\bar{z})}\sum_{k=0}^{2m-1}x^k.$$

Then, the coefficients of the polynomial T_z are real,

$$T_z(x)=(x^{m-1}+zx^{m-2}+\cdots+z^{m-1})(x^{m-1}+\bar{z}x^{m-2}+\cdots+\bar{z}^{m-1})\sum_{k=0}^{2m-1}x^k$$

and $\deg(T_z)=2^{n+2}-3$. For each $k\in\{0,\dots,2^{n+2}-3\}$ let $c_k(z)$ be the "coefficient" function defined by the identity

$$T_z(x)=\sum_{0\leqslant k\leqslant 2^{n+2}-3}c_k(z)x^k.$$

Since the complex conjugation is a continuous map on $\mathbb{C}$, then so is each function c_k; in particular we have $\lim_{z\to\omega}c_k(z)=c_k(\omega)$, and the coefficients of

$$(x-\omega)(x-\bar{\omega})T_z(x)$$

are positive when z is close to ω. Finally, as the set $\mathbb{Q}(i)=\{a+ib,(a,b)\in\mathbb{Q}^2\}$ is dense in $\mathbb{C}$, and $T_z(x)\in\mathbb{Q}[x]$ when $z\in\mathbb{Q}(i)$, it is enough to choose z in an appropriate neighborhood of ω in $\mathbb{C}$ which meets $\mathbb{Q}(i)$, and T the corresponding polynomial T_z. □

The following lemma is a corollary of a theorem of [1]. In [4], Mignotte obtained a slight improvement of this result.

LEMMA 2.2. *Let ω be a nonreal algebraic number. Then*

$$|\omega-\bar{\omega}|\geqslant\frac{2|\omega|\,|\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}}M^{d-1}},$$

where d, Δ and M are the degree, the discriminant and the Mahler measure of ω, respectively.

PROOF. See [1]. □

3. Proof of Theorem 1.1

Let $\{\alpha_1,\dots,\alpha_r\}$ and $\{\alpha_{r+1},\overline{\alpha_{r+1}},\dots,\alpha_{r+s},\overline{\alpha_{r+s}}\}$ be a partition of the set of the conjugates of α, where the first subset is real (if it is not empty) and the second one does not meet the real line. It is clear that $r\geqslant0$, $s\geqslant0$, $r+2s=d$ and the numbers $\alpha_1,\dots,\alpha_r$ are negative. Let

$$\mathrm{Min}_\alpha(x)=(x-\alpha_1)\dots(x-\alpha_r)\prod_{j=1}^{s}(x-\alpha_{j+r})(x-\overline{\alpha_{j+r}}),$$

where $\alpha_{j+r} = |\alpha_{j+r}|\, e^{i\theta_j}$ for $j \in \{1, \dots, s\}$, and $0 < \theta := \theta_1 \leqslant \dots \leqslant \theta_s < \pi$ when $s \geqslant 1$. We want to show that there is a multiple, say Q, of Min_α with positive rational coefficients and degree at most $C_\alpha d$, where

$$C_\alpha = \frac{2\pi}{\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr)} - \frac{1}{2} < \frac{2\pi}{\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr)}.$$

As a finite product of polynomials with nonnegative coefficients is also a polynomial with nonnegative coefficients, we obtain immediately that the coefficients of Min_α are nonnegative when $\theta \geqslant \frac{\pi}{2}$, because

$$(x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}) = x^2 - 2|\alpha_{j+r}|(\cos\theta_{j+r})x + |\alpha_{j+r}|^2$$

and $\cos\theta_{j+r} \leqslant 0$ for each $j \in \{1, \dots, s\}$. It follows that the polynomial

$$Q(x) := \mathrm{Min}_\alpha(x)(1 + x + \dots + x^{d-1}),$$

has positive rational coefficients, and satisfy $Q(\alpha) = 0$. From the trivial inequality $\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr) \leqslant \frac{\pi}{2}$, we have $C_\alpha d \geqslant \frac{7}{2} d > \deg(Q) = 2d - 1$, and so Theorem 1.1 is true. Now, suppose $\theta < \frac{\pi}{2}$, and let t be the largest rational integer satisfying $\theta_t < \frac{\pi}{2}$. For each $j \in \{1, \dots, t\}$ let n_j be the largest rational integer satisfying

$$\theta_j < \frac{\pi}{2^{n_j}}. \tag{3.1}$$

Then

$$n := n_1 \geqslant \dots \geqslant n_t \tag{3.2}$$

and $\theta_j \geqslant \frac{\pi}{2^{n_j+1}}$, for each $j \in \{1, \dots, t\}$. Lemma 2.1 asserts that there is $T_j(x) \in \mathbb{Q}[x]$ with degree $2^{n_j+2} - 3$ and such that the coefficients of the polynomial $(x - \alpha_{j+r}) \times (x - \overline{\alpha_{j+r}}) T_j(x)$ are positive. Set

$$Q(x) := (x - \alpha_1) \cdots (x - \alpha_r) \biggl(\prod_{j=1}^{t} (x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}) T_j(x) \biggr) \prod_{j=t+1}^{s} (x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}).$$

Then, the coefficients of Q are positive, $Q(x) = \mathrm{Min}_\alpha(x) \prod_{j=1}^{t} T_j(x) \in \mathbb{Q}[x]$, and $\deg(Q) = d + \sum_{j=1}^{t} (2^{n_j+2} - 3)$. It follows by the relation (3.2) that

$$\deg(Q) \leqslant d + \sum_{j=1}^{t} (2^{n+2} - 3) \leqslant d + \sum_{j=1}^{s} (2^{n+2} - 3) \leqslant 2^{n+1} d - \frac{d}{2}, \tag{3.3}$$

since $t \leqslant s \leqslant \frac{d}{2}$. By Lemma 2.2 we have

$$|\alpha_{1+r} - \overline{\alpha_{1+r}}| = 2|\alpha_{1+r}| \sin\theta \geqslant \frac{2|\alpha_{1+r}|\, |\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}} M^{d-1}},$$

and so

$$\theta \geqslant \arcsin\biggl(\frac{|\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}} M^{d-1}} \biggr).$$

The last inequality together with the relation (3.1) (with $j = 1$) yield

$$2^{n+1} < \frac{2\pi}{\arcsin\left(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\right)},$$

and the result follows immediately by (3.3).

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(Received 06 03 2010)