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COMPLEX POWERS OF NONDENSELY DEFINED OPERATORS

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ABSTRACT. The power $(-A)^b$, $b \in \mathbb{C}$ is defined for a closed linear operator A whose resolvent is polynomially bounded on the region which is, in general, strictly contained in an acute angle. It is proved that all structural properties of complex powers of densely defined operators with polynomially bounded resolvent remain true in the newly arisen situation. The fractional powers are considered as generators of analytic semigroups of growth order $r > 0$ and applied in the study of corresponding incomplete abstract Cauchy problems. In the last section, the constructed powers are incorporated in the analysis of the existence and growth of mild solutions of operators generating fractionally integrated semigroups and cosine functions.

1. Introduction and preliminaries

Throughout this paper, E denotes a nontrivial complex Banach space and A denotes a closed linear operator in E. By $L(E)$ is denoted the space which consists of all bounded linear operators from E into E and by $[D(A)]$ is denoted the Banach space $D(A)$ equipped with the norm $||x||_{[D(A)]} := \|x\|+\|Ax\|$, $x \in D(A)$. The range and the resolvent set of A are denoted by $R(A)$ and $\rho(A)$, respectively.

Given $\beta \geqslant -1$, $\varepsilon \in (0,1]$, $d \in (0,1]$, $C \in (0,1)$, $s \in \mathbb{R}$ and $\theta \in (0,\pi]$, put $B_d := \{z \in \mathbb{C} : |z| \leqslant d\}$, $\Sigma_\theta := \{z \in \mathbb{C} : z \neq 0,\ \arg(z) \in (-\theta,\theta)\}$, $P_{\beta,\varepsilon,C} := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta \in \mathbb{R},\ |\eta| \leqslant C(1+\xi)^{-\beta}\}$, $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : k \leqslant s\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : k \geqslant s\}$.

Assume that $\alpha \geqslant -1$ and that a closed linear operator A satisfies:

($\Diamond$) $(0,\infty) \subseteq \rho(A)$ and
($\Diamond\Diamond$) $\sup_{\lambda>0}(1+|\lambda|)^{-\alpha}\|R(\lambda : A)\| < \infty$.

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The central theme of this paper is the construction of the power $(-A)^b$, $b \in \mathbb{C}$.

By the usual series argument, we have that, under our standing hypotheses $(\lozenge)$ and $(\lozenge\lozenge)$, there exist $d \in (0,1]$, $C \in (0,1)$, $\varepsilon \in (0,1]$ and $M > 0$ such that:

(§) $P_{\alpha,\varepsilon,C} \cup B_d \subseteq \rho(A)$, $(\varepsilon, C(1+\varepsilon)^{-\alpha}) \in \partial B_d$ and
(§§) $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\alpha}$, $\lambda \in P_{\alpha,\varepsilon,C} \cup B_d$.

Put $\Gamma_1(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta = -C(1+\xi)^{-\alpha}\}$, $\Gamma_2(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi^2 + \eta^2 = d^2,\ \xi \leqslant \varepsilon\}$ and $\Gamma_3(\alpha,\varepsilon,C) := \{\xi + i\eta : \xi \geqslant \varepsilon,\ \eta = C(1+\xi)^{-\alpha}\}$. The curve $\Gamma(\alpha,\varepsilon,C) := \Gamma_1(\alpha,\varepsilon,C) \cup \Gamma_2(\alpha,\varepsilon,C) \cup \Gamma_3(\alpha,\varepsilon,C)$ is oriented so that $\mathrm{Im}(\lambda)$ increases along $\Gamma_2(\alpha,\varepsilon,C)$ and that $\mathrm{Im}(\lambda)$ decreases along $\Gamma_1(\alpha,\varepsilon,C)$ and $\Gamma_3(\alpha,\varepsilon,C)$. Since there is no risk for confusion, we also write Γ for $\Gamma(\alpha,\varepsilon,C)$.

The method established by Straub in [**32**], the idea of Martínez and Sanz [**22**] in their construction of complex powers of nonnegative operators and the notion of stationary dense operators introduced by Kunstmann in [**17**] are essentially utilized in our analysis. We remove density assumptions from the definition of an (analytic) semigroup of growth order $r > 0$ and consider the negatives of constructed powers as the *integral generators* [**18**] of such semigroups. We also refer the reader to the constructions of powers obtained by deLaubenfels, Yao, Wang [**7**] and deLaubenfels, Pastor [**8**] in the framework of the theory of C-regularized semigroups.

Suppose, for the time being, that A is densely defined and $\alpha \geqslant 0$. Then we introduce the complex powers of the operator $-A$ as follows [**32, 15**]. Using the arguments given in the proof of [**15**, Proposition 3.1], we have that, for every $b \in \mathbb{C}$ with $\mathrm{Re}(b) < -(\alpha+1)$, the integral $I(b) := \frac{1}{2\pi i}\int_{\Gamma(\alpha,\varepsilon,C)}(-\lambda)^b R(\lambda : A)\,d\lambda$ exists and defines a bounded linear operator. Then, for every $b \in \mathbb{C}$, we define the operator J^b by $D(J^b) := D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2})$ and

$$J^b x := \begin{cases} I(b)x, & -(\alpha+2) \leqslant \mathrm{Re}(b) < -(\alpha+1),\\ I(b - \lfloor \mathrm{Re}(b)+\alpha\rfloor - 2)(-A)^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}x, & \text{otherwise.}\end{cases}$$

Arguing as in [**15**, Proposition 3.2], we have

$$J^b x = \begin{cases} \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)x\,d\lambda, & \mathrm{Re}(b) < 0,\\ \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \mathrm{Re}(b)\rfloor-1} R(\lambda : A)(-A)^{\lfloor \mathrm{Re}(b)\rfloor+1}x\,d\lambda, & \mathrm{Re}(b) \geqslant 0,\end{cases} \tag{1.1}$$

for all $x \in D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}) = D(J^b)$. Put $C^b := (-A)^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2} J^{b-\lfloor \mathrm{Re}(b)+\alpha\rfloor-2}$. Then, for every $b \in \mathbb{C}$, C^b is a closed linear operator which contains J^b and one can prove that $C^b = \overline{J_b}$ if $|\mathrm{Re}(b)| > \alpha+1$ or $b \in \mathbb{Z}$. The complex power $(-A)^b$, $b \in \mathbb{C}$ is defined by $(-A)^b := \overline{J^b}$ and coincides with the usual power of the operator A if $b \in \mathbb{Z}$. It is worthwhile to notice that [**15**, Propositions 3.1–3.5; Lemmas 3.1–3.2; Remark 3.1 and Theorem 4.1] still hold in the case of operators satisfying (§) and (§§); this fact will be used repeatedly throughout the paper. Let $\mathrm{Re}(b) \in (-1,0)$ and $x \in D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2})$. Then there exists $y \in E$ such that $x = (-A)^{-\lfloor \mathrm{Re}(b)+\alpha\rfloor-2}y$, and (1.1) implies

$$J^b x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)(-A)^{-\lfloor \mathrm{Re}(b)+\alpha\rfloor-2}y\,d\lambda.$$

Assume $\varepsilon' > 0$, $\varepsilon'' > 0$, $\varepsilon''' > 0$, $(\varepsilon'')^2+(\varepsilon''')^2(1+\varepsilon'')^{-2\alpha} = (\varepsilon')^2$ and $P_{\alpha,\varepsilon'',\varepsilon'''}\cup B_{\varepsilon'} \subseteq P_{\alpha,\varepsilon,C} \cup B_d$. By the resolvent equation, we get that:

$$(1.2)\quad \begin{aligned} &R(\lambda : A)(-A)^{-\lfloor \mathrm{Re}(b)+\alpha\rfloor -2}y \\ &= \sum_{j=1}^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2} (-1)^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2-j}\lambda^{j-\lfloor \mathrm{Re}(b)+\alpha\rfloor-3}(-A)^{-j}y \\ &\qquad + \frac{(-1)^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}}{\lambda^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}} R(\lambda : A)y, \quad \lambda \in \rho(A) \smallsetminus \{0\}. \end{aligned}$$

Combined with the inequality $|(-\lambda)^b| \leqslant |\lambda|^{\mathrm{Re}(b)}e^{\pi|\mathrm{Im}(b)|}$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$ and the residue theorem, (1.2) indicates that one can deform the path of integration Γ, appearing in the definition of J^b, into the boundary of the region $P_{\alpha,\varepsilon'',\varepsilon'''} \cup B_{\varepsilon'}$. Letting $\varepsilon' \to 0+$, $\varepsilon'' \to 0+$, $\varepsilon''' \to 0+$ and applying the dominated convergence theorem, we obtain that $J^b x = -\frac{\sin b\pi}{\pi}\int_0^\infty t^b R(t : A)x\,dt$. By [**15**, Lemma 3.2], one gets that, for every $b \in \mathbb{C}$ such that $\mathrm{Re}(b) \notin \mathbb{Z}$ and $x \in D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2})$, the following holds:

$$(1.3)\quad (-A)^b x = \frac{\sin(\lfloor \mathrm{Re}(b)\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \mathrm{Re}(b)\rfloor-1}R(t : A)(-A)^{\lfloor \mathrm{Re}(b)\rfloor+1}x\,dt.$$

Notice that equality (1.3) extends assertion (P2) given on page 158 of [**26**].

Suppose now that a closed, densely defined operator A satisfies (§) and (§§) with $\alpha \in [-1, 0)$, or

(§$_1$) $\Sigma(\gamma, d) := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leqslant \gamma\} \cup B_d \subseteq \rho(A)$, for some $\gamma \in (0, \frac{\pi}{2})$ and

(§§$_1$) $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^\alpha$, $\lambda \in \Sigma(\gamma, d)$, for some $M > 0$ and $\alpha \in [-1, 0)$.

Then it is clear that $\|R(\cdot : A)\|$ is bounded on the region $P_{\alpha,\varepsilon,C} \cup B_d$, resp. $\Sigma(\gamma, d)$. We define the complex powers of $-A$ as in the preceding paragraph with $\alpha = 0$. Then the formula (1.3) holds for every $b \in \mathbb{C}$ such that $\mathrm{Re}(b) \notin \mathbb{Z}$ and $x \in D(A^{\lfloor \mathrm{Re}(b)\rfloor+2})$. It can be easily seen that the above construction coincides with the construction given on pages 157 and 158 of [**26**] for real values of exponents. The former conclusion remains true if (§§$_1$) holds for some $\alpha \geqslant 0$; in any case, $(-A)^b$ is a closed, densely defined linear operator and $(-A)^b \in L(E)$ provided $\mathrm{Re}(b) < -(\alpha + 1)$. Fix temporarily a number $\alpha \geqslant -1$ satisfying (◊◊). Then the construction of powers of densely defined operators does not depend on the choice of numbers $d \in (0, 1]$, $C \in (0, 1)$, $\varepsilon \in (0, 1]$ and $M > 0$ satisfying (§) and (§§). Furthermore, $\sup_{\lambda>0}(1 + |\lambda|)^{-\beta}\|R(\lambda : A)\| < \infty$ for all $\beta \in [\alpha, \infty)$, and the construction of powers of densely defined operators does not depend on the choice of such a number β.

Let us recall [**17**] that a closed linear operator A with nonempty resolvent set is stationary dense iff $n(A) = \inf\{k \in \mathbb{N}_0 : D(A^k) \subseteq \overline{D(A^{k+1})}\} < \infty$. We need the following useful assertion whose proof follows from the corresponding one of [**17**, Lemma 1.5].

Lemma 1.1. *Suppose* $\alpha \geqslant -1$ *and* A *is a closed linear operator. If there exist a constant* $M > 0$ *and a sequence* (λ_n) *in* $\rho(A)$ *such that* $\lim_{n\to\infty} |\lambda_n| = \infty$ *and* $\|R(\lambda_n : A)\| \leqslant M(1+|\lambda_n|)^{\alpha}$, $n \in \mathbb{N}$, *then* A *is stationary dense and* $n(A) \leqslant \lfloor\alpha\rfloor+2$.

2. Complex powers of nondensely defined operators

Assume that A is a closed, nondensely defined linear operator such that (◊) and (◊◊) hold. Let (§) and (§§) hold with $d \in (0,1]$, $C \in (0,1)$, $\varepsilon \in (0,1]$ and $M > 0$. By Lemma 1.1, we have that $n(A) \leqslant \lfloor\alpha\rfloor + 2$ and that the equality $\overline{D(A^{n(A)})} = \overline{D(A^{n(A)+n})}$ holds for all $n \in \mathbb{N}$. Put $F := \overline{D(A^{n(A)})}$ and denote by A_F the part of A in F, i.e., $D(A_F) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_F x := Ax$, $x \in D(A_F)$. By [**17**, Proposition 2.1], one gets that A_F is densely defined in F as well as that $\rho(A : E) = \rho(A_F : F)$ and that $\|R(\lambda : A_F)\|_F \leqslant \|R(\lambda : A)\|_E$ for all $\lambda \in \rho(A)$. This implies:

$$P_{\alpha,\varepsilon,C} \cup B_d \subseteq \rho(A_F : F) \text{ and } \big\|R(\lambda : A_F)\big\|_F \leqslant M(1+|\lambda|)^{\alpha},\ \lambda \in P_{\alpha,\varepsilon,C} \cup B_d. \tag{2.1}$$

By the foregoing, one can construct the complex powers of the operator $(-A_F)^b = \overline{J_F^b}$ in the Banach space F. Following the approach of Martínez and Sanz [**22**] for nonnegative operators, we introduce the complex powers of the operator $-A$ as follows (cf. also the construction of complex powers of almost nonnegative operators [**24**]).

Definition 2.1. Suppose $b \in \mathbb{C}$. The complex power $(-A)^b$ is defined by

$$(-A)^b := (-A)^{n(A)}(-A_F)^b(-A)^{-n(A)}.$$

Remark 2.1. It is straightforwardly checked that, for every $\lambda \in \rho(A)$ and $b \in \mathbb{C}$, we have $(-A)^b = (\lambda - A)^{n(A)}(-A_F)^b(\lambda - A)^{-n(A)}$. The definition of power $(-A)^b$ coincides with the usual definitions given in [**32**] and [**15**] when A is densely defined, and does not depend on the choice of a number $\alpha \geqslant -1$ satisfying (◊◊). Furthermore, by Lemma 1.1, $(-A_F)^b \subseteq (-A)^b \subseteq (-A)^{\lfloor\alpha\rfloor+2}(-A_F)^b(-A)^{-\lfloor\alpha\rfloor-2}$ and it is not clear whether, in general, $(-A)^{\lfloor\alpha\rfloor+2}(-A_F)^b(-A)^{-\lfloor\alpha\rfloor-2} \subseteq (-A)^b$.

Theorem 2.1. *Suppose* $b,\ c \in \mathbb{C}$ *and* $n \in \mathbb{N}$. *Then the complex powers of the operator* $-A$ *satisfy the following properties:*

(i) $(-A)^b$ *is a closed linear operator.*
(ii) $(-A)^b$ *is injective.*
(iii) $(-A)^b \in L(E)$, $\mathrm{Re}(b) < -(\alpha+1)$, $D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}) \subseteq D((-A)^b)$ *if* $b \in \mathbb{C}$ *and* $\alpha \geqslant 0$, *and* $D(A^{\lfloor \mathrm{Re}(b)\rfloor+2}) \subseteq D((-A)^b)$ *if* $b \in \mathbb{C}$ *and* $\alpha \in [-1,0)$.
(iv) $(-A)^{-b}(-A)^b x = x$, $x \in D((-A)^b)$, $(-A)^{-b} = ((-A)^b)^{-1}$ *and* $I_F \subseteq \overline{(-A)^{-b}(-A)^b} \subseteq I$.
(v) $(-A)^n = (-1)^n A \cdots A$ *n-times*, $(-A)^{-n} = R(0 : A)^n$ *and* $(-A)^0 = I$.
(vi) *Let* $x \in D((-A)^{b+c})$. *Then there exists a sequence* (x_k) *in* $D((-A)^b(-A)^c)$ *such that*

$$\lim_{k\to\infty} x_k = (-A)^{-n(A)}x \text{ and } \lim_{k\to\infty} (-A)^b(-A)^c x_k = (-A)^{-n(A)}(-A)^{b+c}x.$$

(vii) $\overline{(-A)^b(-A)^c} \subseteq (-A)^{b+c}$ *if* $(-A_F)^{b+c} = C_F^{b+c}$. *In particular, the above inclusion holds provided* $|\operatorname{Re}(b+c)| > \alpha+1$ *or* $b+c \in \mathbb{Z}$.

(viii) *Suppose* $b \in \mathbb{C}$, $\operatorname{Re}(b) \notin \mathbb{Z}$ *and* $\alpha \geqslant 0$, *resp.* $\alpha \in [-1,0)$. *Then the equality (1.3) holds for every* $x \in D(A^{\lfloor \operatorname{Re}(b)+\alpha\rfloor+2})$, *resp. for every* $x \in D(A^{\lfloor \operatorname{Re}(b)\rfloor+2})$.

(ix)(ix.1) *Let* $(-A)^{c-b} \in L(E)$. *Then* $D((-A)^b) \subseteq D((-A)^c)$ *and* $(-A)^c x = (-A)^{c-b}(-A)^b x$, $x \in D((-A)^b)$.

(ix.2) *Let* $(-A)^{-b} \in L(E)$ *and* $D((-A)^b) \subseteq D((-A)^c)$. *Then* $(-A)^{c-b} \in L(E)$.

(x) *Assume* $b \in \mathbb{C}$ *and,* $x \in D(A^{\lfloor \operatorname{Re}(b)+\alpha\rfloor+2})$ *if* $\alpha \geqslant 0$, *resp.* $x \in D(A^{\lfloor \operatorname{Re}(b)\rfloor+2})$, *if* $\alpha \in [-1,0)$. *Then*

$$(2.2)\quad (-A)^b x = \begin{cases} \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)x\,d\lambda, & \operatorname{Re}(b) < 0, \\ \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \operatorname{Re}(b)\rfloor-1} R(\lambda : A)(-A)^{\lfloor \operatorname{Re}(b)\rfloor+1}x\,d\lambda, & \operatorname{Re}(b) \geqslant 0 \end{cases}$$

and

$$(2.3)\qquad (-A)^b x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \operatorname{Re}(b)+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re}(b)+\alpha\rfloor+2}x\,d\lambda,$$

if $\alpha \geqslant 0$, *resp.*

$$(-A)^b x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \operatorname{Re}(b)\rfloor-2} R(\lambda : A)(-A)^{\lfloor \operatorname{Re}(b)\rfloor+2}x\,d\lambda,$$

if $\alpha \in [-1,0)$.

(xi) *Let* $c \in \mathbb{C}$ *and* $x \in D(A^{\lfloor \operatorname{Re}(c)+\alpha\rfloor+2})$, *if* $\alpha \geqslant 0$, *resp.* $x \in D(A^{\lfloor \operatorname{Re}(c)\rfloor+2})$, *if* $\alpha \in [-1,0)$. *Then:*

$$(2.4)\qquad \lim_{b\to c}(-A)^b x = (-A)^c x.$$

Proof. By [**15**, Theorem 4.1], we know that the properties (i)–(iv) hold for the complex powers $(-A_F)^b$ in F as well as that the powers $(-A_F)^b$, $b \in \mathbb{Z}$ coincide with the usual powers of the operator $-A_F$. Furthermore, $(-A_F)^{b+c} \subseteq \overline{(-A_F)^b(-A_F)^c}$, with the equality if $(-A_F)^{b+c} = C_F^{b+c}$, and $\overline{(-A_F)^{-b}(-A_F)^b} = I_F$. The proofs of assertions (i), (ii), (iv), (v) and (vi) follow from the corresponding properties of powers $(-A_F)^b$ and elementary definitions. We will prove the first assertion in (iii) only in the case $\alpha \in [-1,0)$ since the consideration is similar if $\alpha \geqslant 0$. Suppose $x \in E$ and $\operatorname{Re}(b) < -(\alpha+1)$. Then $n(A) = 1$ and one sees directly that $(-A_F)^b y = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A_F)y\,d\lambda$, $y \in F$. Arguing as in the proof of [**15**, Proposition 3.1], one gets that the integral $\frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)x\,d\lambda$ converges. Hence,

$$\begin{aligned}(-A_F)^b(-A)^{-1}x &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A_F)(-A)^{-1}x\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)(-A)^{-1}x\,d\lambda \\ &= \frac{1}{2\pi i}(-A)^{-1}\int_\Gamma (-\lambda)^b R(\lambda : A)x\,d\lambda \in D(A).\end{aligned}$$

Hence, $x \in D((-A)^b)$, $(-A)^b x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^b R(\lambda : A)x\,d\lambda$, $x \in E$ and the closed graph theorem implies $(-A)^b \in L(E)$. We will prove the second assertion in (iii) provided $\alpha \geqslant 0$. Notice that the first part of (iii) implies $D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}) \subseteq D((-A)^b)$ if $b \in \mathbb{C}$ and $\lfloor \mathrm{Re}(b)+\alpha\rfloor + 2 \leqslant 0$. Suppose $\lfloor \mathrm{Re}(b)+\alpha\rfloor + 2 \geqslant 1$. Then one obtains inductively $D(A^{k+n(A)}) \subseteq D(A_F^k)$, $k \in \mathbb{N}_0$, and consequently, $(-A)^{-n(A)}x \in D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2+n(A)}) \subseteq D(A_F^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2}) = D(J_F^b)$. Taking into account (1.1) and the proof of [**15**, Proposition 3.1], we get $J_F^b(-A)^{-\lfloor\alpha\rfloor-2}x \in D(A^{\lfloor\alpha\rfloor+2})$ and $x \in D((-A)^b)$. Furthermore, (vii) follows on account of:

$$\begin{aligned}
\overline{(-A)^b(-A)^c} &= \overline{[(-A)^{n(A)}(-A_F)^b(-A)^{-n(A)}][(-A)^{n(A)}(-A_F)^c(-A)^{-n(A)}]} \\
&\subseteq \overline{(-A)^{n(A)}(-A_F)^b(-A_F)^c(-A)^{-n(A)}} \subseteq \overline{(-A)^{n(A)}\overline{(-A_F)^b(-A_F)^c}(-A)^{-n(A)}} \\
&= \overline{(-A)^{n(A)}(-A_F)^{b+c}(-A)^{-n(A)}} = (-A)^{n(A)}(-A_F)^{b+c}(-A)^{-n(A)} = (-A)^{b+c}.
\end{aligned}$$

In order to prove (viii), notice that the improper integral appearing in the formulation of (1.3) converges. Without loss of generality, we may assume that $\alpha \geqslant 0$. Suppose $x \in D(A^{\lfloor \mathrm{Re}(b)+\alpha\rfloor+2})$. Owing to assertion (iii) and its proof, one gets $x \in D((-A)^b)$ and

$$\begin{aligned}
&(-A)^{-n(A)}(-A)^b x = (-A_F^b)(-A)^{-n(A)}x = J_F^b(-A)^{-n(A)}x \\
&= \frac{\sin(\lfloor \mathrm{Re}(b)\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \mathrm{Re}(b)\rfloor-1}R(t : A_F)(-A_F)^{\lfloor \mathrm{Re}(b)\rfloor+1}(-A)^{-n(A)}x\,dt \\
&= \frac{\sin(\lfloor \mathrm{Re}(b)\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \mathrm{Re}(b)\rfloor-1}R(t : A)(-A)^{\lfloor \mathrm{Re}(b)\rfloor+1}(-A)^{-n(A)}x\,dt.
\end{aligned}$$

By the closedness of $(-A)^{n(A)}$, one yields:

$$\begin{aligned}
(-A)^b x &= \frac{\sin(\lfloor \mathrm{Re}(b)\rfloor + 1 - b)\pi}{\pi} \\
&\quad\times (-A)^{n(A)}\int_0^\infty t^{b-\lfloor \mathrm{Re}(b)\rfloor-1}R(t : A)(-A)^{\lfloor \mathrm{Re}(b)\rfloor+1}(-A)^{-n(A)}x\,dt \\
&= \frac{\sin(\lfloor \mathrm{Re}(b)\rfloor + 1 - b)\pi}{\pi}\int_0^\infty t^{b-\lfloor \mathrm{Re}(b)\rfloor-1}R(t : A)(-A)^{\lfloor \mathrm{Re}(b)\rfloor+1}x\,dt,
\end{aligned}$$

as required. The proof of (x) follows immediately from that of (iii) and [**15**, Proposition 3.2]. In the case that A is densely defined, the property (ix.1) follows directly from [**15**, Proposition 3.3, Theorem 4.1(a)] and the boundedness of $(-A)^{c-b}$. Assume now that $x \in D((-A)^b)$ and that A is not densely defined. Using [**15**, Theorem 4.1(a)] and (2.2), one can simply prove that $(-A)^{-1}(-A_F)^{c-b} \subseteq (-A_F)^{c-b}(-A)^{-1}$. This implies $(-A)^{c-b}(-A)^{-k} = (-A)^{-k}(-A)^{c-b}$, $k \in \mathbb{N}_0$ and $(-A)^{n(A)}(-A)^{c-b}y = (-A)^{c-b}(-A)^{n(A)}y$, $y \in D(A^{n(A)})$. Since $(-A)^{-n(A)}y \in D((-A_F)^{c-b})$, $y \in E$, we obtain that, for every $y \in E$, $\|(-A_F)^{c-b}(-A)^{-n(A)}y\| = \|(-A)^{n(A)}(-A_F)^{c-b}(-A)^{-n(A)}(-A)^{-n(A)}y\| \leqslant \|(-A)^{c-b}\|\|(-A)^{-n(A)}y\|$. By the closedness of $(-A_F)^{c-b}$ and the previous inequality, we get that $(-A_F)^{c-b} \in L(F)$. Hence, $(-A_F)^c(-A)^{-n(A)}x = (-A_F)^{c-b}(-A_F)^b(-A)^{-n(A)}x \in (-A)^{c-b}(D(A^{n(A)})) \subseteq D(A^{n(A)})$, and

$$
\begin{aligned}
(-A)^c x &= (-A)^{n(A)}(-A_F)^{c-b}(-A_F)^b(-A)^{-n(A)}x \\
&= (-A)^{n(A)}(-A)^{c-b}(-A_F)^b(-A)^{-n(A)}x \\
&= (-A)^{c-b}(-A)^{n(A)}(-A_F)^b(-A)^{-n(A)}x = (-A)^{c-b}(-A)^b x,
\end{aligned}
$$

finishing the proof of (ix.1).

In order to prove (ix.2), notice that the closed graph theorem combined with (iv) and the prescribed assumptions implies $(-A)^c(-A)^{-b} \in L(E)$, and that the proof of (ix.1) implies $(-A_F)^{-b} \in L(F)$. Therefore,

$$
\|(-A)^{n(A)}(-A_F)^c(-A_F)^{-b}(-A)^{-n(A)}x\| \leqslant \|(-A)^c(-A)^{-b}\|\|x\|
$$

for any $x \in E$, as well as:

$$
\begin{aligned}
&\|(-A_F)^c(-A_F)^{-b}(-A)^{-n(A)}x\| \\
&\quad = \|(-A)^{n(A)}(-A_F)^c(-A_F)^{-b}(-A)^{-n(A)}(-A)^{-n(A)}x\| \\
&\quad \leqslant \|(-A)^c(-A)^{-b}\|\|(-A)^{n(A)}x\|, \quad x \in E
\end{aligned}
$$

and $\|(-A_F)^c(-A_F)^{-b}y\| \leqslant \|(-A)^c(-A)^{-b}\|\|y\|$, $y \in D(A^{n(A)})$. The closedness of $(-A_F)^c(-A_F)^{-b}$ together with the previous inequality imply $(-A_F)^c(-A_F)^{-b} \in L(F)$. By [**15**, Proposition 3.3], we have that there exists $k(b,c) \in \mathbb{N}$ such that $(-A_F)^{c-b}y = (-A_F)^c(-A_F)^{-b}y$, $y \in D(A_F^{k(b,c)})$. This yields $(-A_F)^{c-b} \in L(F)$ and $(-A_F)^{c-b} = (-A_F)^c(-A_F)^{-b}$. Let $x \in E$. Then

$$
\begin{gathered}
(-A)^{-n(A)}x \in D((-A_F)^c(-A_F)^{-b}) = D((-A_F)^{c-b}), \\
(-A_F)^{c-b}(-A)^{-n(A)}x = (-A_F)^c(-A_F)^{-b}(-A)^{-n(A)}x \in D(A^{n(A)}).
\end{gathered}
$$

Hence, $x \in D((-A)^{c-b})$ and the proof of (ix.2) follows by the closed graph theorem. We will prove (xi) only in the case $\alpha \geqslant 0$. It is clear that there exists $\sigma > 0$ such that $\lfloor \mathrm{Re}(b)+\alpha \rfloor \leqslant \lfloor \mathrm{Re}(c)+\alpha \rfloor$ if $|b-c| \leqslant \sigma$, and that $(-A)^b x$ is given by the formulae (2.2)–(2.3) in a neighborhood of the point c. In the case $\mathrm{Re}(c) \notin \mathbb{Z}$, the required continuity property follows from the formula (1.3) and the dominated convergence theorem, while in the case $\mathrm{Re}(c) \in \mathbb{Z}$ and $\alpha \notin \mathbb{N}_0$, (2.4) can be proved by means of (2.3) and the dominated convergence theorem. Let $\mathrm{Re}(c) \in \mathbb{Z}$ and $\alpha \in \mathbb{N}_0$. Then (2.3) implies that $\lim_{b \to c,\ \mathrm{Re}(b) \geqslant \mathrm{Re}(c)}(-A)^b x = (-A)^c x$. Since

$$
\begin{aligned}
(-A)^b x &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \mathrm{Re}(b)+\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \mathrm{Re}(b) \rfloor + \alpha + 2} x \, d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor \mathrm{Re}(b)+\alpha \rfloor -2} R(\lambda : A)(-A)^{-1}(-A)^{\lfloor \mathrm{Re}(c) \rfloor + \alpha + 2} x \, d\lambda \\
&= \int_\Gamma (-\lambda)^{b-\lfloor \mathrm{Re}(b)+\alpha \rfloor -3} R(\lambda : A)(-A)^{\lfloor \mathrm{Re}(c) \rfloor + \alpha + 2} x \frac{d\lambda}{2\pi i} \\
&= \int_\Gamma (-\lambda)^{b-\lfloor \mathrm{Re}(c)+\alpha \rfloor -2} R(\lambda : A)(-A)^{\lfloor \mathrm{Re}(c) \rfloor + \alpha + 2} x \frac{d\lambda}{2\pi i},
\end{aligned}
$$

for $\mathrm{Re}(b) \in (\mathrm{Re}(c)-1, \mathrm{Re}(c))$, one gets $\lim_{b \to c,\ \mathrm{Re}(b) < \mathrm{Re}(c)}(-A)^b x = (-A)^c x$. $\square$

Remark 2.2. (i) It is clear that the inclusion $(-A)^{b+c} \subseteq \overline{(-A)^b(-A)^c}$, $b, c \in \mathbb{C}$ clarified in [**32**, Theorem 1.12(iv)] and [**15**, Theorem 4.1(b)] cannot be expected if the domain of the operator A is not dense in E. The assertion (vi) quoted in the formulation of Theorem 2.1 is an interpretation of this property in the case of nondensely defined operators.

(ii) Put $(-A)^b_\alpha := (-A)^{\lfloor\alpha\rfloor+2}(-A_F)^b(-A)^{-\lfloor\alpha\rfloor-2}$, $b \in \mathbb{C}$. Then the assertions (i)–(xi) of Theorem 2.1 still hold with $n(A)$ and $(-A)^b$, replaced by $\lfloor\alpha\rfloor + 2$ and $(-A)^b_\alpha$, respectively.

(iii) Let $\omega \geqslant 0$ and $\operatorname{Re}(b) > \alpha + 1$. Then ($\diamondsuit\diamondsuit$) holds with A replaced by $A - \omega$. One can simply prove that $D((\omega - A)^b) = D((-A)^b)$ and $D((\omega - A)^b_\alpha) = D((-A)^b_\alpha)$.

(iv) Suppose that a closed linear operator A satisfies ($\S_1$) and ($\S\S_1$). Then $-A$ falls under the scope of operators considered by Periago and Straub in [**30**] and one can construct the complex powers of $-A$ by the use of an extension of McIntosh functional calculus given in [**30**, Section 2]. As the referee of the former version of the paper has noticed, the constructed complex power $(-A)^b$ coincides with that of [**30**], and the concept of stationary denseness can probably be omitted, at least at the definition of the fractional powers. As a matter of fact, one can use the recent approach of Haase [**11**] on general functional calculus for sectorial operators and McIntosh's method of multiplicative regularization to obtain a slightly different definition of fractional powers of operators satisfying ($\diamondsuit$) and ($\diamondsuit\diamondsuit$). Considering everything, we would rather use an elementary approach which enables one to simply prove Balakrishnan's representation formula as well as many other properties of powers (see, for example, the proof of [**30**, Proposition 3.4]).

Example 2.1. The operator $A = \xi\Delta^2 - i\varrho\Delta + \varsigma$ ($\xi > 0$, $\varrho \in \mathbb{R} \smallsetminus \{0\}$, $\varsigma < 0$), acting in $L^2(\mathbb{R}^n)$ with its maximal distributional domain, satisfies ($\diamondsuit$) and ($\diamondsuit\diamondsuit$) with $\alpha = (-1)/2$, and A does not satisfy ($\S_1$).

Suppose that ($\diamondsuit$) and ($\diamondsuit\diamondsuit$) hold with $\alpha > 0$. Set $T(t) := (-A)^{it}(-A)^{-\lfloor\alpha\rfloor-2}$, $t \in \mathbb{R}$. Then the closed graph theorem implies $T(t) \in L(E)$, $t \in \mathbb{R}$, and by Theorem 2.1(xi), we obtain that the mapping $t \mapsto T(t)x$, $t \in \mathbb{R}$ is continuous for every fixed $x \in E$. One can simply prove that $(T(t))_{t\in\mathbb{R}}$ is a global $(-A)^{-\lfloor\alpha\rfloor-2}$-regularized group. Denote by B the integral generator of $(T(t))_{t\in\mathbb{R}}$. The logarithm of $-A$, denoted by $\log(-A)$, is defined by $\log(-A) := -iB$. Clearly, the definition of $\log(-A)$ is independent of the choice of a number $\alpha > 0$ satisfying ($\diamondsuit\diamondsuit$), and $\pm i\log(-A)$ are the integral generators of global $(-A)^{-\lfloor\alpha\rfloor-2}$-regularized semigroups. It would take too long to develop the theory of introduced logarithms. For further information, we refer the interested reader to [**3**]–[**4**], [**6**], [**9**]–[**10**], [**19**], [**27**], [**29**] and [**34**].

3. Incomplete abstract Cauchy problems

The following definition of an (analytic) semigroup of growth order $r > 0$ is motivated by the analysis given in [**30**]–[**31**].

Definition 3.1. An operator family $(T(t))_{t>0}$ in $L(E)$ is said to be a semigroup of growth order $r > 0$ if the following conditions hold:

(i) $T(t+s) = T(t)T(s)$, $t, s > 0$,
(ii) the mapping $t \mapsto T(t)x$, $t > 0$ is continuous for every fixed $x \in E$,
(iii) $\|t^r T(t)\| = O(1)$, $t \to 0+$ and
(iv) $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

Suppose $\gamma \in (0, \frac{\pi}{2}]$ and $(T(t))_{t>0}$ has an analytic extension to the sector Σ_γ, denoted by the same symbol. If, additionally, there exists an $\omega \in \mathbb{R}$ such that, for every $\delta \in (0, \gamma)$, there exists $M_\delta > 0$ with $\|z^r T(z)\| \leqslant M_\delta e^{\omega \operatorname{Re}(z)}$, $z \in \Sigma_\delta$, then the family $(T(z))_{z \in \Sigma_\gamma}$ is called an analytic semigroup of growth order r.

Since we do not require that the set $E_0 := \bigcup_{t>0} T(t)E$ is dense in E, the introduced notion is slightly different from the former one given by Da Prato [**5**] in 1966. Furthermore, if the set E_0 is dense in E, then the definition of an analytic semigroup of growth order $r > 0$ is equivalent to the corresponding one introduced by Tanaka in [**33**]. The infinitesimal generator of $(T(t))_{t>0}$ is defined by

$$G := \Big\{(x, y) \in E \times E : \lim_{t \to 0+} \frac{T(t)x - x}{t} = y\Big\}.$$

By [**28**, Lemma 3.1], G is a closable linear operator. The closure of G, denoted by $\overline{G}$, is said to be the complete infinitesimal generator, in short, the c.i.g. of $(T(t))_{t>0}$. The notion of the integral generator $\hat{G}$ of $(T(t))_{t>0}$, introduced by Kunstmann in [**18**], is also meaningful in the study of semigroups of growth order $r > 0$:

$$\hat{G} := \Big\{(x, y) \in E \times E : T(t)x - T(s)x = \int_s^t T(r)y\,dr \text{ for all } t, s > 0 \text{ with } t \geqslant s\Big\}.$$

The integral generator $\hat{G}$ is a closed linear operator which contains the c.i.g. $\overline{G}$ and satisfies $\hat{G} = \{(x, y) \in E \times E : (T(s)x, T(s)y) \in G \text{ for all } s > 0\}$. The integral generator, resp. the c.i.g., of an analytic semigroup $(T(z))_{z \in \Sigma_\gamma}$ of growth order $r > 0$ is defined to be the integral generator, resp. the c.i.g., of $(T(t))_{t>0}$. The set $\{x \in E : \lim_{t \to 0+} T(t)x = x\}$, resp. $\{x \in E : \lim_{z \to 0,\, z \in \Sigma_{\gamma'}} T_b(z)x = x \text{ for all } \gamma' \in (0, \gamma)\}$ is said to be the continuity set of $(T(t))_{t>0}$, resp. $(T(z))_{z \in \Sigma_\gamma}$.

Suppose that $\overline{G}$ ($\hat{G}$) is the c.i.g. (the integral generator) of a semigroup $(T(t))_{t>0}$, resp. an analytic semigroup $(T(z))_{z \in \Sigma_\gamma}$, of growth order $r > 0$. Repeating literally the arguments given in [**28**] and [**33**] (cf. also [**25**, Section 5]), one gets that conditions (I), (II) and (IV) quoted in the formulation of [**28**, Theorem 1.2], resp. (b2), (b3) and (b4) quoted in the formulation of [**33**, Theorem 3], remain true if the denseness of E_0 in E is disregarded. It is an open problem to state sufficient conditions for the generation of nondense (analytic) semigroups of growth order $r > 0$. Assume now that $(T(z))_{z \in \Sigma_\gamma}$ is an analytic semigroup of growth order r. Then it is clear that, for every $\theta \in (0, \gamma)$, $(T(te^{i\theta}))_{t>0}$ is a semigroup of growth order r. With the help of C-regularized semigroups, one can prove that the integral generator of $(T(te^{i\theta}))_{t>0}$ is always $e^{i\theta}\hat{G}$, and that the c.i.g. of $(T(te^{i\theta}))_{t>0}$ is $e^{i\theta}\overline{G}$, whenever E_0 is dense in E or $r \in (0, 1)$ (cf. also [**35**, Theorem 1]). Unfortunately, it is quite questionable whether the last assertion remains true if $\overline{E_0} \neq E$ and $r \geqslant 1$.

Theorem 3.1. *Suppose that $b \in (0, \frac{1}{2})$ and that a closed linear operator A satisfies* ($\diamondsuit$) *and* ($\diamondsuit\diamondsuit$) *with $\alpha > -1$. Set* $\gamma := \arctan(\cos(\pi b))$.

(a) *Then the operator* $-(-A_F)^b$ *is the c.i.g. of an analytic semigroup* $(T_b(z))_{z\in\Sigma_\gamma}$ *of growth order* $\frac{\alpha+1}{b}$, *where*

$$T_b(z) := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)\,d\lambda, \quad z \in \Sigma_\gamma.$$

Denote by $\Omega_b(A)$, *resp.* $\Omega_{b,\theta}(A)$, *the continuity set of* $(T_b(z))_{z\in\Sigma_\gamma}$, *resp.* $(T_b(te^{i\theta}))_{t>0}$. *Then the following holds:*

(i) *For every* $\delta \in (0,\gamma)$, $\|z^{\frac{\alpha+1}{b}} T_b(z)\| = O(1)$, $z \in \Sigma_\delta$.

(ii) *The mapping* $z \mapsto T_b(z)$, $z \in \Sigma_\gamma$ *is analytic*, $\bigcup_{z\in\Sigma_\gamma} R(T_b(z)) \subseteq D_\infty(A)$,

$$\frac{d^n}{dz^n}T_b(z) = \frac{(-1)^n}{2\pi i}\int_\Gamma (-\lambda)^{nb} e^{-z(-\lambda)^b} R(\lambda : A)\,d\lambda, \quad n \in \mathbb{N}, \quad z \in \Sigma_\gamma \text{ and}$$

$$A^n T_b(z) = \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} \lambda^n R(\lambda : A)\,d\lambda, \quad z \in \Sigma_\gamma, \quad n \in \mathbb{N}. \tag{3.1}$$

(iii) *We have* $D(A^{\lfloor b+\alpha\rfloor+1}) \subseteq \Omega_b(A)$ *provided* $\lfloor b+\alpha\rfloor \geqslant 0$.

(iv) *If* $\lfloor b+\alpha\rfloor \geqslant 0$, $x \in D(A^{\lfloor b+\alpha\rfloor+2})$ *and* $\gamma' \in (0,\gamma)$, *then*

$$\lim_{z\to 0,\ z\in\Sigma_{\gamma'}} \frac{T_b(z)x - x}{z} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-1} R(\lambda : A)Ax\,d\lambda.$$

(v) *For every* $z \in \Sigma_\gamma$, $T_b(z)$ *is an injective operator.*

(vi) *The integral generator of* $(T_b(z))_{z\in\Sigma_\gamma}$, *denoted by* $\hat{G}$, *is the operator* $-(-A)^{\lfloor\alpha\rfloor+2}(-A_F)^b(-A)^{-\lfloor\alpha\rfloor-2}$; *in particular*, $-(-A)^b \subseteq \hat{G}$, *and* $-(-A)^b = \hat{G}$ *provided that* $D(A)$ *is not dense in* E *and that* $\alpha \in (-1,0)$.

(b) *Suppose* $n \in \mathbb{N}$, $n \geqslant 3$, $\theta \in [0, \arctan(\cos\frac{\pi}{n}))$ *and* $x \in \Omega_{\frac{1}{n},\theta}(A)$. *Then the function* $u : (0,\infty) \to E$, *given by* $u(t) := T_{\frac{1}{n}}(te^{i\theta})x$, $t > 0$ *is a solution of the abstract Cauchy problem*

$$(P_n): \begin{cases} u \in C((0,\infty) : [D(A)]) \cap C^\infty((0,\infty) : E), \\ \frac{d^n}{dt^n}u(t) = (-1)^{n+1}e^{in\theta}Au(t), \ t > 0, \\ \lim_{t\to 0+} u(t) = x, \ \sup_{t>0}\|u(t)\| < \infty. \end{cases}$$

Moreover, $u(\cdot)$ *can be analytically extended to the sector* $\Sigma_{\arctan(\cos\frac{\pi}{n})-|\theta|}$ *and, for every* $\delta \in (0, \arctan(\cos\frac{\pi}{n}) - |\theta|)$ *and* $i \in \mathbb{N}_0$, *we have*

$$\sup_{z\in\Sigma_\delta}\left\| z^{i+n\alpha+n}\frac{d^i}{dz^i}u(z)\right\| < \infty.$$

The previous conclusions hold in the case $\frac{1}{n} + \alpha \geqslant 0$ *and* $x \in D(A^{\lfloor\frac{1}{n}+\alpha\rfloor+1})$.

Proof. Arguing as in [**32**, Section 2] and the proof of [**15**, Theorem 4.2], one obtains that (i)–(v) hold and that $(T_b(z))_{z\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{\alpha+1}{b}$. Denote by G the infinitesimal generator of $(T_b(z))_{z\in\Sigma_\gamma}$ and put $S_b(z)x := T_b(z)x$, $z \in \Sigma_\gamma$, $x \in F$. Since $T_b(z)x \in D_\infty(A)$, $z \in \Sigma_\gamma$, $x \in E$, we obtain that $S_b(z) \in L(F)$, $z \in \Sigma_\gamma$. Furthermore, for every $\lambda \in \rho(A)$, we have $R(\lambda :$

$A)D(A^{n(A)}) \subseteq D(A^{n(A)+1})$ and $R(\lambda : A)F \subseteq F$. Hence, $R(\lambda : A)x = R(\lambda : A_F)x$, $x \in F$, $\lambda \in \rho(A : E) = \rho(A_F : F)$ and

$$\begin{aligned} S_b(z)x = T_b(z)x &= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)x\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A_F)x\,d\lambda, \quad x \in F, \quad z \in \Sigma_\gamma. \end{aligned}$$

Since A_F is densely defined in F and satisfies (2.1), one can apply the argumentation given in the proof of [**15**, Theorem 4.2] in order to see that $-(-A_F)^b$ is the c.i.g. of an analytic semigroup $(S_b(z))_{z\in\Sigma_\gamma}$ of growth order $\frac{\alpha+1}{b}$ in F. If $x \in D(G)$, then $\lim_{t\to 0+} T_b(t)x = x \in F$, and consequently, $Gx \in F$. With this in view, we get:

$$G = \Big\{(x, y) \in F \times F : \lim_{t\to 0+} \frac{S_b(t)x - x}{t} = y\Big\}.$$

This immediately implies $\overline{G} = -(-A_F)^b$. We will prove (vi) only in the case of nondensely defined operators. Since, by (ii), $\bigcup_{z\in\Sigma_\gamma} R(T_b(z)) \subseteq D_\infty(A)$, the following equality is obvious:

$$(x, y) \in \hat{G} \text{ iff } T_b(s)y = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-1} R(\lambda : A)AT_b(s)x\,d\lambda \text{ for all } s > 0. \tag{3.2}$$

Put $n := \lfloor\alpha\rfloor + 2$ and assume $(x, y) \in D((-A)^n(-A_F)^b(-A)^{-n})$. Then $(-A)^{-n}y = (-A_F)^b(-A)^{-n}x$ and one gets the existence of a sequence $(x_n, y_n) \in J_F^b$ such that $\lim_{n\to\infty} x_n = (-A)^{-n}x$ and $\lim_{n\to\infty} J_F^b x_n = (-A_F)^b(-A)^n x$. Keeping in mind (3.1), we reveal that, for every $s > 0$:

$$\begin{aligned} (-A)^{-n}T_b(s)y &= \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} T_b(s)R(\lambda : A_F)(-A_F)^{\lfloor b+\alpha\rfloor+2}x_n\,d\lambda \\ &= \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)x_n\,d\lambda \\ &= \frac{(-1)^{\lfloor b+\alpha\rfloor+2}}{2\pi i}\lim_{n\to\infty}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A) \\ &\qquad \times \bigg[\frac{1}{2\pi i}\int_\Gamma e^{-s(-\xi)^b}\xi^{\lfloor b+\alpha\rfloor+2}R(\xi : A)T_b(s)x_n d\xi\bigg] d\lambda. \end{aligned}$$

Using the dominated convergence theorem, one can verify that the last term equals

$$\begin{aligned} &= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)(-A)^{-n}x\,d\lambda \\ &= \frac{1}{2\pi i}(-A)^{-n}\int_\Gamma (-\lambda)^{b-\lfloor b+\alpha\rfloor-2} R(\lambda : A)(-A)^{\lfloor b+\alpha\rfloor+2}T_b(s)x\,d\lambda. \end{aligned}$$

The injectiveness of $(-A)^{-n}$ yields (3.2) and $-(-A)^b \subseteq \hat{G}$. Next, we will show that $D(A^{\lfloor\alpha\rfloor+2}) \subseteq \Omega_b(A)$. If $b + \alpha \geqslant 0$, the proof is obvious; suppose $b + \alpha < 0$,

$\gamma' \in (0, \gamma)$ and $\lambda_0 \in \rho(A) \smallsetminus H(a, C, d)$. Then $\lfloor \alpha \rfloor + 2 = 1$ and

$$T_b(z)(-A)^{-1}x - (-A)^{-1}x = \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)(-A)^{-1}x\, d\lambda - (-A)^{-1}x$$
$$= \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}\Big[R(\lambda : A)(-A)^{-1}x - \frac{(-A)^{-1}x}{\lambda-\lambda_0}\Big]d\lambda + \big(e^{-z(-\lambda_0)^b} - 1\big)(-A)^{-1}x$$
$$= \frac{(-1)}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}\Big[\frac{R(\lambda : A)x}{\lambda} + \frac{\lambda_0(-A)^{-1}x}{\lambda(\lambda-\lambda_0)}\Big]d\lambda + \big(e^{-z(-\lambda_0)^b} - 1\big)(-A)^{-1}x,$$

for all $z \in \Sigma_{\gamma'}$ and $x \in E$. The preceding equality, combined with the residue theorem, the inequality $|e^{-z(-\lambda)^b}| \leqslant e^{-(\operatorname{Re}(z)\cos(b\pi) - |\operatorname{Im}(z)|)|\lambda|^b}$, $z \in \Sigma_\gamma$, $\lambda \in \mathbb{C}$ and the dominated convergence theorem, implies $\lim_{z\to 0,\, z\in\Sigma_{\gamma'}} T_b(z)(-A)^{-1}x = (-A)^{-1}x$, $x \in E$ and $D(A) \subseteq \Omega_b(A)$. Hence, $\hat{G}$ is the integral generator of an exponentially bounded analytic $(-A)^{-n}$-regularized semigroup $(S_b(z) := T_b(z)(-A)^{-n})_{z\in\Sigma_\gamma}$ in the sense of [**6**, Definition 21.3, Definition 21.4]. The assumption $(x, y) \in \hat{G}$ implies $\lim_{t\to 0+} \frac{T_b(t)(-A)^{-n}x - (-A)^{-n}x}{t} = (-A)^{-n}y$, $(-A)^{-n}x \in D((-A_F)^b)$ and$(-A_F)^b(-A)^{-n}x = (-A)^{-n}y \in D(A^n)$. Thereby, $x \in D((-A)^b)$, $(-A)^b x = y = \hat{G}x$ and the proof of (a) is completed.

The proof of (b) in the case $\theta = 0$ follows from (a) and the argumentation given in the proof of [**15**, Theorem 4.3], while the proof of (b) in the case $\theta \in (0, \arctan(\cos\frac{\pi}{n}))$ is quite similar. $\square$

Remark 3.1. (i) Notice that, in general, $D(A^{\lfloor \frac{1}{n}+\alpha\rfloor+1})$ is strictly contained in $\Omega_{\frac{1}{n},\theta}(A)$ [**31**]. Suppose, further, that the number $\alpha > -1$ is minimal with respect to the property $(\Diamond\Diamond)$. Then the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$ is $-(-A)^b$ provided $\rho((-A_F)^b) \neq \emptyset$ or $n(A) = \lfloor\alpha\rfloor + 2$.

(ii) If $D(A)$ is not dense in E, then the c.i.g. of $(T_b(z))_{z\in\Sigma_\gamma}$ can be strictly contained in the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$ for all $b \in (0, \frac{1}{2})$. Indeed, suppose that $-A$ is a nondensely defined positive operator and denote by $\widehat{(-A)^b}$ the complex power of $-A$ in the sense of [**21**, Section 5]. Obviously, $(\widehat{-A_{\overline{D(A)}}})^b = (-A_{\overline{D(A)}})^b$ and $\widehat{(-A)^b} = (-A)^b$, $b \in (0, \frac{1}{2})$. On the other hand, it is clear that A satisfies $(\S_1)$ and $(\S\S_1)$ with appropriate number $\alpha \in (-1, 0)$. Now the claimed assertion follows by making use of [**21**, Corollary 5.1.12(ii)] which asserts that $(-A_{\overline{D(A)}})^b \neq (-A)^b$, $b \in (0, \frac{1}{2})$. In the present situation, the author does not know whether the denseness of $D(A)$ in E implies that the c.i.g. of $(T_b(z))_{z\in\Sigma_\gamma}$ coincides with the integral generator of $(T_b(z))_{z\in\Sigma_\gamma}$.

The proof of the following extension of [**31**, Theorem 3] is omitted.

Theorem 3.2. *Suppose $d \in (0, 1]$, $\gamma \in (0, \frac{\pi}{2})$, $\alpha \geqslant -1$, $M > 0$ and $b \in (0, \frac{\pi}{2(\pi-\gamma)})$. Set $\beta := \arctan(\cos(b(\pi - \gamma)))$ and assume that $\Sigma(\gamma, d) \subseteq \rho(A)$ and that $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^\alpha$, $\lambda \in \Sigma(\gamma, d)$.*

(a) *Denote by Γ the upwards oriented frontier of the region $\Sigma(\gamma, d)$. Then the operator $-(-A_F)^b$ is the c.i.g. of an analytic semigroup $(T_b(z))_{z\in\Sigma_\beta}$ of growth order*

$\frac{\alpha+1}{b}$, *where*

$$T_b(z) := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda, \ \ z \in \Sigma_\beta.$$

Denote by $\Omega_b(A)$, *resp.* $\Omega_{b,\theta}(A)$, *the continuity set of* $(T_b(z))_{z\in\Sigma_\beta}$, *resp.* $(T_b(te^{i\theta}))_{t>0}$. *Then the following holds:*

(i) *For every* $\delta \in (0,\beta)$, $\|z^{\frac{\alpha+1}{b}} T_b(z)\| = O(1)$, $z \in \Sigma_\delta$.

(ii) *The mapping* $z \mapsto T_b(z)$, $z \in \Sigma_\beta$ *is analytic*, $\bigcup_{z\in\Sigma_\beta} R(T_b(z)) \subseteq D_\infty(A)$,

$$\frac{d^n}{dz^n} T_b(z) = \frac{(-1)^n}{2\pi i}\int_\Gamma (-\lambda)^{nb} e^{-z(-\lambda)^b} R(\lambda : A)\, d\lambda, \ \ n \in \mathbb{N}, \ \ z \in \Sigma_\beta \ \text{and}$$

$$A^n T_b(z) = \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b} \lambda^n R(\lambda : A)\, d\lambda, \ \ z \in \Sigma_\beta, \ \ n \in \mathbb{N}.$$

(iii) *We have* $D(A^{\lfloor b+\alpha\rfloor+1}) \subseteq \Omega_b(A)$ *provided* $\lfloor b+\alpha\rfloor \geqslant 0$.

(iv) *If* $\lfloor b+\alpha\rfloor \geqslant 0$, $x \in D(A^{\lfloor b+\alpha\rfloor+2})$ *and* $\gamma' \in (0,\beta)$, *then*

$$\lim_{z\to 0,\ z\in\Sigma_{\gamma'}} \frac{T_b(z)x - x}{z} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-1} R(\lambda : A) Ax\, d\lambda.$$

(v) *For every* $z \in \Sigma_\beta$, $T_b(z)$ *is an injective operator.*

(vi) *The integral generator of* $(T_b(z))_{z\in\Sigma_\beta}$, *denoted by* $\hat{G}$, *is the operator* $-(-A)^{\lfloor\alpha\rfloor+2}(-A_F)^b(-A)^{-\lfloor\alpha\rfloor-2}$; *in particular*, $-(-A)^b \subseteq \hat{G}$, *and* $-(-A)^b = \hat{G}$ *provided that* $D(A)$ *is not dense in* E *and that* $\alpha \in (-1,0)$.

(b) *Suppose* $n \in \mathbb{N}\setminus\{1\}$, $\theta \in [0, \arctan(\cos(\frac{1}{n}(\pi-\gamma))))$ *and* $x \in \Omega_{\frac{1}{n},\theta}(A)$. *Then the function* $u : (0,\infty) \to E$, *given by* $u(t) := T_{\frac{1}{n}}(te^{i\theta})x$, $t > 0$ *is a solution of the abstract Cauchy problem* (P_n). *Furthermore, the solution of* (P_n) *is unique provided* $n = 2$ *and* $n(A) \leqslant 1$. *Put* $a_{n,\theta} := \arctan(\cos(\frac{1}{n}(\pi - a))) - |\theta|$. *Then the solution* $u(\cdot)$ *can be analytically extended to the sector* $\Sigma_{a_{n,\theta}}$ *and, for every* $\delta \in (0, a_{n,\theta})$ *and* $i \in \mathbb{N}_0$, *we have*

$$\sup_{z\in\Sigma_\delta} \Big\| z^{i+n\alpha+n} \frac{d^i}{dz^i} u(z) \Big\| < \infty.$$

The previous conclusions hold in the case $\frac{1}{n} + \alpha \geqslant 0$ *and* $x \in D(A^{\lfloor\frac{1}{n}+\alpha\rfloor+1})$.

Example 3.1. (i) In what follows, we use the notion and notation given in [**1**, Chapter 8]. Let $s > 1$, $k > 0$, $p \in [1,\infty)$, $m > 0$, $\rho \in [0,1]$, $r > 0$, $a \in S^m_{\rho,0}$ satisfies (H_r), the inequality

$$n\Big|\frac{1}{2} - \frac{1}{p}\Big|\Big(\frac{m-r-\rho+1}{r}\Big) < 1 \tag{3.3}$$

hold, $E = L^p(\mathbb{R}^n)$ or $E = C_0(\mathbb{R}^n)$ (in the last case, we assume $p = \infty$), and $A = \mathrm{Op}_E(a)$. If $a(\cdot)$ is an elliptic polynomial of order m, then $m = r$, $\rho = 1$ and (3.3) is valid. Assume $\mathrm{dist}(a(\mathbb{R}^n), [0,\infty)) > 0$. Using [**1**, Lemma 8.2.1, Proposition 8.2.6, the proof of Lemma 8.2.8], it follows that Theorem 3.1 can be applied with a convenable chosen constant $\alpha > -1$.

(ii) Assume that A generates an exponential distribution semigroup [**16**]. Then there exist $\omega > 0$ and $\alpha \geqslant -1$ such that $(\Diamond)$ and $(\Diamond\Diamond)$ hold with A replaced by $\pm i(A-\omega)$.

4. The existence and growth of mild solutions of operators generating fractionally integrated semigroups and cosine functions

Recall that the function $u(\cdot)$ is a mild solution of the abstract Cauchy problem

$$(ACP_1): \ u'(t) = Au(t),\ t \geqslant 0,\ u(0) = x,\ \text{resp.,}$$
$$(ACP_2): \ u''(t) = Au(t),\ t \geqslant 0,\ u(0) = x,\ u'(0) = y,$$

iff the mapping $t \mapsto u(t)$, $t \geqslant 0$ is continuous, $\int_0^t u(s)\,ds \in D(A)$ and $A\int_0^t u(s)\,ds = u(t) - x$, $t \geqslant 0$, resp., the mapping $t \mapsto u(t)$, $t \geqslant 0$ is continuous, $\int_0^t (t-s)u(s)\,ds \in D(A)$ and $A\int_0^t (t-s)u(s)\,ds = u(t) - x - ty$, $t \geqslant 0$.

Suppose $\alpha \geqslant 0$ and A generates an exponentially bounded α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Then, for every $\gamma \in (0, \frac{\pi}{2})$ and $\sigma > 0$, there exists $d \in (0,1]$ such that $\Sigma(\gamma, d) \subseteq \rho(A - \omega - \sigma)$ and that $\|R(\lambda : A - \omega - \sigma)\| = O((1 + |\lambda|)^{\alpha-1})$, $\lambda \in \Sigma(\gamma, d)$. Therefore, one can construct the powers of $\omega + \sigma - A \equiv -A_{\omega+\sigma}$.

Albeit the method chosen to define the fractional powers is not embedded into a functional calculus, it enables one to simply prove results concerning the existence and growth of mild solutions of operators generating fractionally integrated semigroups and cosine functions. This is why we deeply believe that our approach has some advantages.

The main objective in the following theorem is to transfer the assertions of [**26**, Theorem 1.1, Theorem 1.2] to nondensely defined generators of fractionally integrated semigroups.

THEOREM 4.1. (i) *Let $\alpha > 0$ and let A be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Then, for every $\epsilon > 0$, $\sigma > 0$ and $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\epsilon})$, the abstract Cauchy problem (ACP_1) has a unique mild solution. Moreover, this solution is exponentially bounded and its exponential type is at most ω. If $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\epsilon})$, the solution is classical.*

(ii) *Let $\alpha > 0$ and let A be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$ satisfying $\|S_\alpha(t)\| \leqslant M(1 + t^\gamma)$, $t \geqslant 0$, for appropriate constants $M \geqslant 1$ and $\gamma \geqslant 0$. Then, for every $\epsilon > 0$, $\sigma > 0$ and $x_0 \in D((-A_\sigma)^{\alpha+\epsilon})$, the abstract Cauchy problem (ACP_1) has a unique mild solution. Moreover, this solution is polynomially bounded and its polynomial type is at most* $\max(\alpha - 1 + \epsilon, \gamma + \epsilon, 2\gamma - \alpha + \epsilon)$. *If $x_0 \in D((-A_\sigma)^{1+\alpha+\epsilon})$, the solution is classical.*

PROOF. We basically follow the notation given in [**26**] and consider only the nontrivial case $\alpha + \varepsilon \notin \mathbb{N}$ (cf. the proof of [**26**, Theorem 1.1]). First of all, notice that all structural results proved in [**26**, Section 3, Section 4] still hold for nondensely defined generators of fractionally integrated semigroups. In particular, the singular integral

$$v_{\omega+\sigma}(t,x) := \Gamma_{\alpha,\varepsilon}\int_0^\infty \frac{ds}{s-1}\Big(s^{\lfloor\alpha+\varepsilon\rfloor-\alpha-\varepsilon}S_{\omega+\sigma}^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}(t)-\frac{1}{s}S_{\omega+\sigma}^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}\Big(\frac{t}{s}\Big)\Big)x,$$

where $\Gamma_{\alpha,\varepsilon} := \frac{\sin(\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor)\pi}{\pi}$, is absolutely convergent for all $x \in D(A^{\lfloor\alpha+\varepsilon\rfloor})$. We will prove that the mild solution of (ACP_1) is given by

$$u(t,x_0) := e^{(\omega+\sigma)t}v_{\omega+\sigma}\big(t,(-A_{\omega+\sigma})^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}x_0\big),\ \ t \geqslant 0.$$

Since $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\epsilon}) = D(((-A_{\omega+\sigma})^{-\alpha-\varepsilon})^{-1}) = R((-A_{\omega+\sigma})^{-\alpha-\varepsilon})$, we have that there exists $y_0 \in E$ such that $x_0 = (-A_{\omega+\sigma})^{-\alpha-\varepsilon}y_0$. By Theorem 2.1(ix) and Theorem 2.1(vii), $D((-A_{\omega+\sigma})^{\alpha+\varepsilon}) \subseteq D((-A_{\omega+\sigma})^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor})$ and $(-A_{\omega+\sigma})^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}x_0 = (-A_{\omega+\sigma})^{-\alpha-\varepsilon}y_0 \in D(A^{\lfloor\alpha+\varepsilon\rfloor})$. Hence, the integral

$$\int_0^\infty \frac{ds}{s-1}\Big(s^{\lfloor\alpha+\varepsilon\rfloor-\alpha-\varepsilon}S_{\omega+\sigma}^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}(t)-\frac{1}{s}S_{\omega+\sigma}^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}\Big(\frac{t}{s}\Big)\Big)(-A_{\omega+\sigma})^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}x_0 \tag{4.1}$$

is absolutely convergent. Put

$$f(t) := e^{(\omega+\sigma)t}v_{\omega+\sigma}\big(t,(-A_{\omega+\sigma}^F)^{\alpha+\varepsilon-\lfloor\alpha+\varepsilon\rfloor}(-A_{\omega+\sigma})^{-n(A)}x_0\big),\ \ t \geqslant 0.$$

Then it follows from [**26**] that the mapping $t \mapsto f(t) \in F$, $t \geqslant 0$ is continuous and

$$A_F\int_0^t f(s)ds = f(t) - (-A_{\omega+\sigma})^{-n(A)}x_0,\ \ t \geqslant 0.$$

By the closedness of $(-A_{\omega+\sigma})^{n(A)}$ and the absolute convergence of (4.1), one gets that $f(t) \in D(A^{n(A)})$, $t \geqslant 0$ and $(-A_{\omega+\sigma})^{n(A)}f(t) = u(t,x_0)$, $t \geqslant 0$. This, in turn, implies that the function $t \mapsto u(t,x_0)$, $t \geqslant 0$ is a mild solution of (ACP_1). The uniqueness is a consequence of the Ljubich theorem. Thanks to Theorem 2.1(vii) and Theorem 2.1(ix), we get that the assertion of [**26**, Lemma 6.3] still holds in the case of nondensely defined generators. Now one can repeat literally the final part of the proof of [**26**, Lemma 6.4] in order to see that the solution is classical provided $x_0 \in D((-A_{\omega+\sigma})^{1+\alpha+\epsilon})$. This completes the proof of (i) while the proof of (ii) follows analogically. □

REMARK 4.1. The proof of Theorem 4.1 combined with Remark 2.2(ii) implies that the mild solution of (ACP_1) in (i), resp. (ii), exists for all

$$x_0 \in \bigcup_{\beta>0} D\Big((-A_{\omega+\sigma})^{\lfloor\beta\rfloor+2}(-A_{\omega+\sigma}^F)^{\alpha+\varepsilon}(-A_{\omega+\sigma})^{-\lfloor\beta\rfloor-2}\Big),\ \text{resp.}$$

$$x_0 \in \bigcup_{\beta>0} D\Big((-A_{\sigma})^{\lfloor\beta\rfloor+2}(-A_{\sigma}^F)^{\alpha+\varepsilon}(-A_{\sigma})^{-\lfloor\beta\rfloor-2}\Big).$$

Notice that [**26**, Theorem 1.3, Remark, p. 164, Corollary 7.3, Theorem 7.4 and Theorem 7.5] still hold in the case of nondensely defined generators of fractionally integrated semigroups and that the above comment can be applied again.

The following generalization of [**20**, Theorem 3.1-Theorem 3.2] follows immediately from Theorem 4.1.

Theorem 4.2. (i) *Let* $\alpha > 0$ *and let* A *be the generator of an* α*-times integrated semigroup* $(S_\alpha(t))_{t\geqslant 0}$ *satisfying* $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, *for appropriate constants* $M \geqslant 1$ *and* $\omega \geqslant 0$. *Then, for every* $\epsilon > 0$ *and* $\sigma > 0$, A *is the integral generator of an exponentially bounded* $(-A_{\omega+\sigma})^{-(\alpha+\epsilon)}$*-regularized semigroup* $(T(t))_{t\geqslant 0}$ *which satisfies that, for every* $\sigma' > \sigma$, *there exists* $M' \geqslant 1$ *such that* $\|T(t)\| \leqslant M'e^{(\omega+\sigma')t}$, $t \geqslant 0$.

(ii) *Assume that there exist constants* $M \geqslant 1$, $\omega \geqslant 0$, $\beta > 0$ *and* $\gamma \in (0, \frac{\pi}{2})$ *such that* $\omega + \Sigma_\gamma \subseteq \rho(A)$ *and that* $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^{\beta-1}$, $\lambda \in \omega + \Sigma_\gamma$. *If* A *generates an exponentially bounded* $(-A_{\omega+\sigma})^{-\alpha}$*-regularized semigroup for some* $\alpha > \beta$ *and* $\sigma > 0$, *then, for every* $\epsilon > 0$, A *generates an exponentially bounded* $(\alpha+\epsilon)$*-times integrated semigroup.*

Before proceeding further, we would like to observe that Theorem 4.1, Remark 4.1 and Theorem 4.2 can be applied to nondensely defined convolution operators considered by Hieber in [**12**, Section 4]. In such a way, one can prove an extension of [**20**, Theorem 3.7] for the operators acting in $L^\infty(\mathbb{R}^n)$ and $C_b(\mathbb{R}^n)$.

Using [**16**, Theorem 2.1.11] and Theorem 4.1, we state without proof the following theorem.

Theorem 4.3. (i) *Let* $\alpha > 0$ *and let* A *be the generator of an* α*-times integrated cosine function* $(C_\alpha(t))_{t\geqslant 0}$ *satisfying* $\|C_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, *for appropriate constants* $M \geqslant 1$ *and* $\omega \geqslant 0$. *Put* $\mathcal{A} := \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$. *Then, for every* $\epsilon > 0$, $\sigma > 0$ *and* $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\epsilon+1})$, *the abstract Cauchy problem* (ACP_2) *has a unique mild solution. Moreover, this solution is exponentially bounded and its exponential type is at most* ω. *If* $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\epsilon+2})$, *the solution is classical.*

(ii) *Let* $\alpha > 0$ *and let* A *be the generator of an* α*-times integrated cosine function* $(C_\alpha(t))_{t\geqslant 0}$ *satisfying* $\|C_\alpha(t)\| \leqslant M(1+t^\gamma)$, $t \geqslant 0$, *for appropriate constants* $M \geqslant 1$ *and* $\gamma \geqslant 0$. *Then, for every* $\epsilon > 0$, $\sigma > 0$ *and* $(x_0, y_0) \in D((-\mathcal{A}_\sigma)^{\alpha+\epsilon+1})$, *the abstract Cauchy problem* (ACP_2) *has a unique mild solution. Moreover, this solution is polynomially bounded and its polynomial type is at most* $\max(\alpha+\epsilon, \max(\alpha, \gamma+2)+\epsilon, 2\max(\alpha, \gamma+2)-(\alpha+1)+\epsilon)$. *If* $(x_0, y_0) \in D((-\mathcal{A}_\sigma)^{\alpha+\epsilon+2})$, *the solution is classical.*

Remark 4.2. Let $\alpha \in (2n, 2n+1)$ for some $n \in \mathbb{N}_0$, resp. $\alpha \in (2n-1, 2n)$ for some $n \in \mathbb{N}$. Then it is well known (cf. for example [**16**, Section 2.3]) that the classical solution of (ACP_2) exists for all $(x_0, y_0) \in D(A^{n+2}) \times D(A^{n+1}) = D(\mathcal{A}^{2n+3})$, resp. for all $(x_0, y_0) \in D(A^{n+1}) \times D(A^{n+1}) = D(\mathcal{A}^{2n+2})$. Notice that the set $\bigcup_{\varepsilon\in(0,\lfloor\alpha\rfloor+1-\alpha]} D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+2})$ strictly contains $D(\mathcal{A}^{2n+3})$, resp. the set $\bigcup_{\varepsilon\in(0,\lfloor\alpha\rfloor+1-\alpha]} D((-\mathcal{A}_{\omega+\sigma})^{\alpha+\varepsilon+2})$ strictly contains $D(\mathcal{A}^{2n+2})$ (cf. Remark 4.1 and [**26**, Remark, p. 164]). The same conclusion holds in the case of mild solutions.

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