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CHAOS EXPANSION METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS INVOLVING THE MALLIAVIN DERIVATIVE–PART I

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Abstract. We consider Gaussian, Poissonian, fractional Gaussian and fractional Poissonian white noise spaces, all represented through the corresponding orthogonal basis of the Hilbert space of random variables with finite second moments, given by the Hermite and the Charlier polynomials. There exist unitary mappings between the Gaussian and Poissonian white noise spaces. We investigate the relationship of the Malliavin derivative, the Skorokhod integral, the Ornstein–Uhlenbeck operator and their fractional counterparts on a general white noise space.

1. Introduction

Generalized stochastic processes on white noise spaces have a series expansion form given by the Hilbert space basis of square integrable processes (processes with finite second moments), and depending on the stochastic measure this basis can be represented as a family of orthogonal polynomials defined on an infinite dimensional space. The classical Hida approach (see [**7, 8**]) suggests to start with a nuclear space E and its dual E', such that $E \subset L^2(\mathbb{R}) \subset E'$, and then take the basic probability space to be $\Omega = E'$ endowed with the Borel σ–algebra of the weak topology and an appropriate probability measure P. Since Gaussian processes and Poissonian processes represent the two most important classes of Lévy processes, we will focus on these two types of measures.

In case of a Gaussian measure, the orthogonal basis of $L^2(\Omega, P)$ can be constructed from any orthogonal basis of $L^2(\mathbb{R})$ that belongs to E and from the Hermite polynomials, while in the case of a Poissonian measure the orthogonal basis of

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$L^2(\Omega, P)$ is constructed using the Charlier polynomials together with the orthogonal basis of $L^2(\mathbb{R})$. We will focus on the case when E and E' are the Schwartz spaces of rapidly decreasing test functions $\mathcal{S}(\mathbb{R})$ and tempered distributions $\mathcal{S}'(\mathbb{R})$. In this case the orthogonal family of $L^2(\mathbb{R})$ can be represented by the Hermite functions.

Following the idea of the construction of $\mathcal{S}'(\mathbb{R})$ as an inductive limit space over $L^2(\mathbb{R})$ with appropriate weights, one can define stochastic generalized random variable spaces over $L^2(\Omega, P)$ by adding certain weights in the convergence condition of the series expansion (also known as the Wiener-Itô chaos expansion) and thus weakening the topology of the L^2 norm. We will define several spaces of this type, weighted by a sequence q and denote them by $(Q)_{-1}^P$, thus obtaining a Gel'fand triplet $(Q)_1^P \subset L^2(\Omega, P) \subset (Q)_{-1}^P$.

Recently, there have been made improvements in economics and financial modelling by replacing the Brownian motion with the fractional Brownian motion, and replacing white noise by fractional white noise (see [**2, 3, 9**]). We will define the fractional Poissonian process in a framework that will make it easy to link it to its regular version.

In [**8**] it was proved that there exists a unitary mapping between the Gaussian and the Poissonian white noise space, by mapping the Hermite polynomial basis into the Charlier polynomial basis. In [**6**] and [**10**] a unitary mapping was introduced between the Gaussian and the fractional Gaussian white noise space. We extend these ideas to define the fractional Poissonian white noise space itself and to link it to the classical Poissonian white noise space. As a result we obtain four types of white noise spaces: Gaussian, Poissonian, fractional Gaussian and fractional Poissonian, where any two of them can be identified through a unitary mapping.

The Skorokhod integral δ represents an extension of the Itô integral to nonanticipating processes. Its adjoint operator $\mathbb{D}$ is known as the Malliavin derivative. Both operators, having an interpretation also in the Fock space sense as the annihilation and the creation operator, are widely used in solving stochastic differential equations (see [**4, 13, 14, 16, 17**]). Their composition $\delta\mathbb{D}$ is known as the Ornstein–Uhlenbeck operator, and it is a selfadjoint operator on $L^2(\Omega, P)$ that has the elements of the orthogonal basis (Hermite or Charlier polynomials) as its eigenvalues. In this paper we continue our work from [**11**] and [**12**] in providing examples of stochastic differential equations involving the Malliavin derivative and the Ornstein–Uhlenbeck operator. The Malliavin derivative and its related operators are all defined on either of the four white noise spaces we are working on, and their domains are characterized in the terms of convergence in a stochastic distribution space $(Q)_{-1}^P$ with special q-weights.

This paper consists of two separately published parts. In the present paper (Part I) we focus on the structural properties of the aforementioned four types of white noise spaces and operators defined on them. Examples of stochastic differential equations and their solutions on all four spaces will be included in Part II of the paper.

The paper is organized as follows: In Section 2 we provide the basic notation used throughout the paper, followed by the construction of the Gaussian and Poissonian white noise spaces in Section 3. In Section 4 their fractional counterparts are introduced and the unitary mappings between the four white noise spaces are established. In Section 5 the chaos expansion theorem for generalized stochastic processes and $\mathcal{S}'(\mathbb{R})$-valued generalized stochastic processes is reviewed, and the Malliavin derivative, the Skorokhod integral and the Ornstein–Uhlenbeck operator are defined together with their fractional versions.

2. Notations

Throughout the paper we will use the following notations. Let

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n e^{x^2/2}}{dx^n}, \quad \xi_n(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{(n-1)!}} e^{-x^2/2} h_{n-1}\big(\sqrt{2}x\big), \quad n \in \mathbb{N},$$

be the families of Hermite polynomials and Hermite functions, respectively. The latter one forms a complete orthonormal system of $L^2(\mathbb{R})$. It is well known that the Schwartz space of rapidly decreasing functions can be constructed as the projective limit $S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$, where

$$S_l(\mathbb{R}) = \Big\{ \varphi = \sum_{k=1}^{\infty} a_k\, \xi_k \in L^2(\mathbb{R}) : \ \|\varphi\|_l^2 = \sum_{k=1}^{\infty} a_k^2 (2k)^l < \infty \Big\},\ l \in \mathbb{N}_0,$$

and the Schwartz space of tempered distributions is its dual $S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$, where

$$S_{-l}(\mathbb{R}) = \Big\{ f = \sum_{k=1}^{\infty} b_k\, \xi_k : \ \|f\|_{-l}^2 = \sum_{k=1}^{\infty} b_k^2 (2k)^{-l} < \infty \Big\},\ l \in \mathbb{N}_0.$$

We also consider the test space of deterministic functions of exponential growth introduced in [**19**] $\exp S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} \exp S_l(\mathbb{R})$, where

$$\exp S_l(\mathbb{R}) = \Big\{ \varphi = \sum_{k=1}^{\infty} a_k\, \xi_k \in L^2(\mathbb{R}) : \ \|\varphi\|_{\exp,l}^2 = \sum_{k=1}^{\infty} a_k^2 e^{2kl} < \infty \Big\},\ l \in \mathbb{N}_0,$$

and the corresponding space of deterministic distributions of exponential growth $\exp S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} \exp S_{-l}(\mathbb{R})$, where

$$\exp S_{-l}(\mathbb{R}) = \Big\{ f = \sum_{k=1}^{\infty} b_k\, \xi_k : \ \|f\|_{\exp,-l}^2 = \sum_{k=1}^{\infty} b_k^2 e^{-2kl} < \infty \Big\},\ l \in \mathbb{N}_0.$$

These spaces satisfy $\exp S(\mathbb{R}) \subseteq S(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq S'(\mathbb{R}) \subseteq \exp S'(\mathbb{R})$, where each inclusion mapping is compact.

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of sequences of nonnegative integers which have finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m, 0, 0, \dots)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, \dots, m$, $m \in \mathbb{N}$. The k-th unit vector $\varepsilon^{(k)} = (0, \dots, 0, 1, 0, \dots)$, $k \in \mathbb{N}$ is the sequence of zeros with the number 1 as the kth component. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$ and $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$. Let $(2\mathbb{N})^\alpha = \prod_{k=1}^{\infty} (2k)^{\alpha_k}$. Note that $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ for $p > 1$ and $\sum_{\alpha\in\mathcal{I}} e^{-p(2\mathbb{N})^\alpha} < \infty$ if $p > 0$.

Denote by $\chi[0,t]$ the characteristic function of $[0,t]$, $t \in \mathbb{R}$ and by $\circ$ the function composition $F \circ G(x) = F(G(x))$.

3. White noise spaces

Let the basic probability space $(\Omega, \mathcal{F}, P)$ be $(S'(\mathbb{R}), \mathcal{B}, P)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel σ-algebra generated by the weak topology on $S'(\mathbb{R})$ and P denotes the unique probability measure on $(S'(\mathbb{R}), \mathcal{B})$ corresponding to a given characteristic function. Recall, a mapping $C : S(\mathbb{R}) \to \mathbb{C}$ given on a nuclear space $S(\mathbb{R})$ is called a characteristic function if it is continuous, positive definite, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_i \overline{z}_j \, C(\varphi_i - \varphi_j) \geqslant 0, \quad \varphi_1, \ldots, \varphi_n \in S(\mathbb{R}),\ z_1, \ldots, z_n \in \mathbb{C},$$

for all $\varphi_1, \ldots, \varphi_n \in S(\mathbb{R})$ and $z_1, \ldots, z_n \in \mathbb{C}$, and if it satisfies $C(0) = 1$. Then by the Bochner–Minlos theorem (see [**7**], [**8**]) there exists a unique probability measure P on $(S'(\mathbb{R}), \mathcal{B})$ such that $E_P\big(e^{i\langle \omega, \varphi \rangle}\big) = C(\varphi)$, for all $\varphi \in S(\mathbb{R})$, where E_P denotes the expectation with respect to the measure P and $\langle \omega, \varphi \rangle$ denotes the usual dual paring between a tempered distribution $\omega \in \mathcal{S}'(\mathbb{R})$ and a rapidly decreasing function $\varphi \in \mathcal{S}(\mathbb{R})$. Thus,

$$\int_{S'(\mathbb{R})} e^{i\langle \omega, \varphi \rangle} dP(\omega) = C(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{3.1}$$

Let $L^2(P) = L^2(S'(\mathbb{R}), \mathcal{B}, P)$ be the Hilbert space of square integrable functions on $S'(\mathbb{R})$ with respect to the measure P and with norm induced by the inner product $(f, g)_{L^2(P)} = E_P(fg) = \int_{S'(\mathbb{R})} f(\omega)\, g(\omega)\, dP(\omega)$. Let K_α, $\alpha \in \mathcal{I}$ be the orthogonal basis of $L^2(P)$. The Wiener–Itô chaos expansion theorem states that every element $F \in L^2(P)$ has a unique representation of the form

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha K_\alpha(\omega), \quad c_\alpha \in \mathbb{R}, \tag{3.2}$$

such that

$$\|F\|^2_{L^2(P)} = \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \|K_\alpha\|^2_{L^2(P)} < \infty. \tag{3.3}$$

In Subsections 3.1 and 3.2 we will consider two special cases, when the measure P is a Gaussian measure and a Poissonian measure. In these cases K_α can be taken as a family of Hermite and Charlier polynomials respectively, defined on an infinite-dimensional space. Later in Subsection 3.3 we introduce q-weighted spaces with respect to the probability measure P, which represent the stochastic analogue of the deterministic spaces $S_l(\mathbb{R})$, $S_{-l}(\mathbb{R})$, $\exp S_l(\mathbb{R})$ and $\exp S_{-l}(\mathbb{R})$ for $l \in \mathbb{N}_0$.

3.1. Gaussian white noise space. If we choose in (3.1) the characteristic function of a Gaussian random variable

$$C(\varphi) = \exp\left[-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}\right], \quad \varphi \in \mathcal{S}(\mathbb{R}), \tag{3.4}$$

then the corresponding unique measure P from the Bochner–Minlos theorem is called the *Gaussian white noise measure* μ and the triplet $(S'(\mathbb{R}), \mathcal{B}, \mu)$ is called the *Gaussian white noise probability space*.

Note that from (3.1) and (3.4) follows $E_\mu(\langle\omega, f\rangle) = 0$ and $E_\mu(\langle\omega, f\rangle^2) = \|f\|^2_{L^2(\mathbb{R})}$, for $f \in S(\mathbb{R})$. Also the polarization formula $E_\mu\big(\langle\omega, f\rangle\,\langle\omega, g\rangle\big) = (f, g)_{L^2(\mathbb{R})}$ holds for all $f, g \in S(\mathbb{R})$.

By extending the action of a distribution $\omega \in \mathcal{S}'(\mathbb{R})$ not only onto test functions from $S(\mathbb{R})$ but also onto elements of $L^2(\mathbb{R})$ we obtain Brownian motion in the form $B_t(\omega) := \langle\omega, \chi[0, t]\rangle$, $\omega \in S'(\mathbb{R})$. It is a Gaussian process with zero expectation and covariance function $E_\mu(B_t(\omega)\, B_s(\omega)) = \min\{t, s\}$. The Itô integral of $f \in L^2(\mathbb{R})$ is given by $I(f) = \langle\omega, f\rangle = \int_\mathbb{R} f(t)\, dB_t(\omega)$. Then $E_\mu(I(f)) = 0$ and the Itô isometry $\|I(f)\|_{L^2(\mu)} = \|f\|_{L^2(\mathbb{R})}$ holds for $f \in L^2(\mathbb{R})$.

The family of Fourier–Hermite polynomials (cf. [**8**])

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle\omega, \xi_k\rangle), \quad \alpha \in \mathcal{I}, \tag{3.5}$$

forms an orthogonal basis of $L^2(\mu)$, where $\|H_\alpha\|^2_{L^2(\mu)} = \alpha!$. In particular, for the kth unit vector $\varepsilon^{(k)}$ we have $H_{\varepsilon^{(k)}}(\omega) = \langle\omega, \xi_k\rangle = \int_\mathbb{R} \xi_k(t)\, dB_t(\omega) = I(\xi_k)$, $k \in \mathbb{N}$. From the Wiener–Itô chaos expansion theorem (3.2) each element $F \in L^2(\mu)$ has a unique chaos expansion representation of the form $F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega)$, where $\|F\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!$ and $c_\alpha = \frac{1}{\alpha!} E_\mu(F H_\alpha)$ and $\|F\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha! < \infty$ (see (3.3)).

3.2. Poissonian white noise space. If we choose in (3.1) the characteristic function of a compound Poisson random variable

$$C(\varphi) = \exp\left[\int_\mathbb{R} (e^{i\varphi(x)} - 1)\, dx\right], \quad \varphi \in \mathcal{S}(\mathbb{R}), \tag{3.6}$$

then the corresponding unique measure P from the Bochner–Minlos theorem is called the *Poisonian white noise measure* ν and the triplet $(S'(\mathbb{R}), \mathcal{B}, \nu)$ is called the *Poissonian white noise probability space*.

From (3.1) and (3.6) follows that $\langle\omega, \varphi\rangle$ has a nonzero expectation $E_\nu(\langle\omega, \varphi\rangle) = \int_\mathbb{R} \varphi(x)\, dx$ and $E_\nu(\langle\omega, \varphi\rangle^2) = \|\varphi\|^2_{L^2(\mathbb{R})} + (\int_\mathbb{R} \varphi(x)\, dx)^2$ i.e., its variance is $\mathrm{Var}(\langle\omega, \varphi\rangle) = \|\varphi\|^2_{L^2(\mathbb{R})}$, for all $\varphi \in S(\mathbb{R})$.

Hence the map $J : \varphi \mapsto \langle\omega, \varphi\rangle - \int_\mathbb{R} \varphi(x)\, dx$, $\varphi \in S(\mathbb{R})$ can be extended to an isometry from $L^2(\mathbb{R})$ into $L^2(\nu)$. Then $E_\nu(J(\varphi)) = 0$ and $\|J(\varphi)\|^2_{L^2(\nu)} = \|\varphi\|^2_{L^2(\mathbb{R})}$, for all $\varphi \in L^2(\mathbb{R})$. The polarization formula $E_\nu(J(\phi)J(\varphi)) = (\phi, \varphi)_{L^2(\mathbb{R})}$ holds for all $\phi, \varphi \in L^2(\mathbb{R})$. A right continuous integer valued version of the process

$$P_t(\omega) = J(\chi[0, t]) = \langle\omega, \chi[0, t]\rangle - t, \quad \omega \in S'(\mathbb{R}),$$

belongs to $L^2(\nu)$ and is called the compensated Poisson process.

The family of Charlier polynomial functionals $C_\alpha(\omega)$, defined by

$$C_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{\xi_1, \ldots, \xi_1}_{\alpha_1}, \ldots, \underbrace{\xi_m, \ldots, \xi_m}_{\alpha_m}), \quad \alpha = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots) \in \mathcal{I}, \tag{3.7}$$

where

$$C_k(\omega;\varphi_1,\dots,\varphi_k) = \frac{\partial^k}{\partial u_1\dots\partial u_k}\exp\left[\left\langle \omega, \log\left(1+\sum_{j=1}^k u_j\varphi_j\right) - \sum_{j=1}^k u_j\int_{\mathbb{R}}\varphi_j(y)\,dy\right\rangle\right]\Big|_{u_1=\dots=u_k=0},$$

for $k\in\mathbb{N}$ and $\varphi_j\in S(\mathbb{R})$, forms an orthogonal basis of the space of Poissonian square integrable random variables $L^2(\nu)$ and $\|C_\alpha\|^2_{L^2(\nu)}=\alpha!$. In particular, $C_{\varepsilon^{(k)}}(\omega)=C_1(\omega,\xi_k)=\langle\omega,\xi_k\rangle-\int_{\mathbb{R}}\xi_k(x)\,dx=J(\xi_k)$, $\omega\in S'(\mathbb{R})$, $k\in\mathbb{N}$. From the Wiener–Itô chaos expansion theorem follows that every element $G\in L^2(\nu)$ is given in the form $G(\omega)=\sum_{\alpha\in\mathcal{I}}b_\alpha C_\alpha(\omega)$, $b_\alpha\in\mathbb{R}$, where $\|G\|^2_{L^2(\nu)}=\sum_{\alpha\in\mathcal{I}}\alpha!b_\alpha^2$ is finite.

The following important theorem, proved by Benth and Gjerde in [**5**], states the existence of a unitary correspondence between the Gaussian and the Poissonian spaces of random variables.

THEOREM 3.1. [**5**] *The map* $\mathcal{U}:L^2(\mu)\to L^2(\nu)$ *defined by*

$$\mathcal{U}\left(\sum_{\alpha\in\mathcal{I}}b_\alpha H_\alpha(\omega)\right)=\sum_{\alpha\in\mathcal{I}}b_\alpha C_\alpha(\omega),\quad b_\alpha\in\mathbb{R},\ \ \alpha\in\mathcal{I},$$

is unitary i.e., it is surjective and the isometry $\|\mathcal{U}(F)\|_{L^2(\nu)}=\|F\|_{L^2(\mu)}$ *holds.*

Using the isometry $\mathcal{U}$ all results obtained in the Gaussian case can be carried over to the Poissonian case. The Fourier–Hermite orthogonal basis $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ of the space of Gaussian random variables just has to be replaced with the corresponding elements of the Charlier polynomials orthogonal basis $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of the space of Poissonian random variables. In the next section we will use this isometry to interpret stochastic differential equations with the Malliavin derivative and their solutions obtained in the Gaussian versions of q-weighted spaces with their corresponding Poissonian versions.

For more details on Gaussian white noise spaces, Poissonian white noise spaces, Hermite and Charlier polynomials we refer to [**1, 3, 7, 8**].

3.3. q-weighted stochastic spaces. In this subsection we define q-weighted stochastic spaces of test functions $(Q)_1^P$ and stochastic generalized functions $(Q)_{-1}^P$, with respect to the measure P. Let $q_\alpha>1$, $\alpha\in\mathcal{I}$. The space of q-weighted P-stochastic test functions $(Q)_1^P=\bigcap_{p\in\mathbb{N}_0}(Q)_{1,p}^P$ is the projective limit of the spaces

$$(Q)_{1,p}^P=\left\{f=\sum_{\alpha\in\mathcal{I}}a_\alpha K_\alpha\in L^2(P):\ \|f\|^2_{(Q)_{1,p}^P}=\sum_{\alpha\in\mathcal{I}}(\alpha!)^2\,a_\alpha^2\,q_\alpha^p<\infty\right\},\ \ p\in\mathbb{N}_0.$$

The space of q-weighted P-stochastic generalized functions $(Q)_{-1}^P=\bigcup_{p\in\mathbb{N}_0}(Q)_{-1,-p}^P$ is the inductive limit of the spaces

$$(Q)_{-1,-p}^P=\left\{F=\sum_{\alpha\in\mathcal{I}}b_\alpha K_\alpha:\ \|F\|^2_{(Q)_{-1,-p}^P}=\sum_{\alpha\in\mathcal{I}}b_\alpha^2\,q_\alpha^{-p}<\infty\right\},\ \ p\in\mathbb{N}_0.$$

Two important special cases will be given by weights of the form $q_\alpha = (2\mathbb{N})^\alpha$ and $q_\alpha = e^{(2\mathbb{N})^\alpha}$. For weights of the form $q_\alpha = (2\mathbb{N})^\alpha$ we obtain the Kondratiev spaces of P-stochastic test functions and P-stochastic generalized functions, denoted by $(S)_1^P$ and $(S)_{-1}^P$ respectively. For $q_\alpha = e^{(2\mathbb{N})^\alpha}$ we obtain the exponential growth spaces of P-stochastic test functions and P-stochastic generalized functions, denoted by $\exp(S)_1^P$ and $\exp(S)_{-1}^P$ respectively. It holds that

$$\exp(S)_1^P \subseteq (S)_1^P \subseteq L^2(P) \subseteq (S)_{-1}^P \subseteq \exp(S)_{-1}^P. \tag{3.8}$$

In particular, for $P = \mu$ the spaces in (3.8) become Gaussian q-weighted spaces and relation (3.8) was proven in [**20**]. For $P = \nu$ we obtain the Poissonian q-weighted spaces. For more details on the Kondratiev spaces we refer to [**8**] and on spaces of exponential growth to [**20**].

We can extend the unitary mapping $\mathcal{U}$ given in the Theorem 3.1. into a linear and isometric mapping on q-weighted spaces by defining $\mathcal{U} : (Q)_{-1}^\mu \to (Q)_{-1}^\nu$ such that

$$\mathcal{U}\Big[\sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega)\Big] = \sum_{\alpha\in\mathcal{I}} a_\alpha C_\alpha(\omega), \quad a_\alpha \in \mathbb{R}, \tag{3.9}$$

for elements $F = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega) \in (Q)_{-1,-p_0}^\mu$ and the isometry $\|\mathcal{U}(F)\|_{(Q)_{-1,-p}^\nu} = \|F\|_{(Q)_{-1,-p}^\mu}$ holds for all $p \geqslant p_0$. More details can be found in [**5**] and [**8**].

3.4. Generalized stochastic processes of type (O). Generalized stochastic processes of type (O) are measurable mappings from $\mathbb{R}$ into some q-weighted space of generalized functions i.e., measurable mappings $\mathbb{R} \to (Q)_{-1}^P$. Since generalized stochastic processes of type (O) with values in $(Q)_{-1}^P$ are defined pointwisely with respect to the parameter $t \in \mathbb{R}$, their chaos expansion follows directly from the Wiener-Itô chaos expansion theorem i.e., $F_t(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha(t) K_\alpha(\omega)$, $t \in \mathbb{R}$, where $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha \in \mathcal{I}$ are measurable functions and there exists $p \in \mathbb{N}_0$ such that $\sum_{\alpha\in\mathcal{I}} |f_\alpha(t)|^2 q_\alpha^{-p} < \infty$ for all $t \in \mathbb{R}$. The unitary mapping $\mathcal{U}$ can be extended to the class of generalized stochastic processes of type (O) in the similar way as in (3.9).

Example 3.1. Brownian motion is given by the chaos expansion

$$B_t(\omega) = \sum_{k=1}^{\infty} \Big(\int_0^t \xi_k(s)\, ds\Big) H_{\epsilon^{(k)}}(\omega) \tag{3.10}$$

and it is an element of $L^2(\mu)$.

Singular white noise $W_t(\cdot)$ is defined by the chaos expansion

$$W_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega), \tag{3.11}$$

and it is an element of the space $(\mathcal{S})_{-1}^\mu$, for all $t \in \mathbb{R}$. It is integrable and the relation $\frac{d}{dt} B_t = W_t$ holds (see [**8**]) in the $(S)_{-1}^\mu$ sense.

Example 3.2. The chaos expansion of a compensated Poisson process $P_t(\omega) \in L^2(\nu)$ is given by

$$P_t(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s)\, ds \right) C_{\varepsilon^{(k)}}(\omega). \tag{3.12}$$

The Poissonian compensated white noise $V_t(\cdot)$ is defined by the chaos expansion

$$V_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\, C_{\varepsilon^{(k)}}(\omega), \tag{3.13}$$

and it is an element of the space $(S)^{\nu}_{-1}$ for all $t \in \mathbb{R}$. It is integrable and the relation $\frac{dP_t}{dt} = V_t$ holds. Note that $P_t(\omega) = \mathcal{U}(B_t(\omega))$ and $V_t(\omega) = \mathcal{U}(W_t(\omega))$, which is consistent with (3.9).

4. Fractional white noise spaces

4.1. Fractional Gaussian white noise space. A fractional Brownian motion with Hurst index $H \in (0,1)$ on the probability space $(\Omega, \mathcal{F}, P)$ is defined to be a Gaussian process $B^{(H)} = \big\{B_t^{(H)}(\cdot),\ t \in \mathbb{R}\big\}$ with $B_0^{(H)} = 0$ a.s., zero expectation $E\big[B_t^{(H)}\big] = 0$ for all $t \in \mathbb{R}$, and covariance function

$$E\big[B_s^{(H)} B_t^{(H)}\big] = \frac{1}{2}\big\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big\}, \quad s, t \in \mathbb{R}. \tag{4.1}$$

The fractional Brownian motion is a centered Gaussian process with nonindependent stationary increments and its dependence structure is modified by the Hurst parameter $H \in (0,1)$. For $H = \frac{1}{2}$ the process $B_t^{(1/2)}$ becomes a standard Brownian motion and it has independent increments. From (4.1) it follows that $E\big(B_t^{(H)} - B_s^{(H)}\big)^2 = |t-s|^{2H}$ and according to Kolmogorov's theorem, $B^{(H)}$ has a continuous modification.

The fractional Brownian motion $B^{(H)}$ is an H self-similar process, i.e., $B_{\alpha t}^{(H)} = \alpha^H B_t^{(H)}$, $\alpha > 0$. For any $n \in \mathbb{Z}$, $n \neq 0$ the autocovariance function is given by

$$r(n) := E\big[B_1^{(H)}(B_{n+1}^{(H)} - B_n^{(H)})\big] \sim H(2H-1)|n|^{2H-1}, \quad \text{when } |n| \to \infty.$$

For $H \in (\frac{1}{2}, 1)$ the fractional Brownian motion has the long-range dependence property $\sum_{n=1}^{\infty} r(n) = \infty$ and for $H \in (0, \frac{1}{2})$ the property $\sum_{n=1}^{\infty} |r(n)| < \infty$. More details on the fractional Brownian motion can be found in [**2, 3, 6, 9, 10, 15, 22**].

Further on we follow the ideas of [**6**] where the fractional white noise theory for the Hurst parameter $H \in (0,1)$ was developed. In [**6**] the fractional operator $M = M^{(H)}$ was introduced, which connects the fractional Brownian motion $B_t^{(H)}$ and the classical Brownian motion B_t on the white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$.

Definition 4.1. Let $H \in (0,1)$. Define the operator $M = M^{(H)} : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ by

$$\widehat{Mf}(y) = |y|^{\frac{1}{2} - H}\, \widehat{f}(y), \quad y \in \mathbb{R},\ f \in S(\mathbb{R}), \tag{4.2}$$

where $\widehat{f}(y) := \int_{\mathbb{R}} e^{-ixy} f(x)\, dx$ is the Fourier transformation of f.

Note that the operator $M = M^{(H)}$ has the structure of a convolution operator. From (4.2) the form of the inverse operator $M^{-1} = M^{(1-H)}$ follows, i.e., for all $H \in (0,1)$

$$M^{(H)} \circ M^{(1-H)}(f) = f, \quad f \in S(\mathbb{R}). \tag{4.3}$$

An equivalent definition of the operator M is given by

$$Mf(x) = -\frac{d}{dx}\frac{c_H}{H-\frac{1}{2}}\int_{\mathbb{R}}(t-x)\,\frac{f(t)}{|t-x|^{H-\frac{3}{2}}}\,dt, \quad f \in S(\mathbb{R}), \tag{4.4}$$

where $c_H = \left[2\Gamma(H-\frac{1}{2})\cos(\frac{\pi}{2}(H-\frac{1}{2}))\right]^{-1}\left[\Gamma(2H+1)\sin(\pi H)\right]^{\frac{1}{2}}$ and $\Gamma(\cdot)$ is the Gamma function. From (4.4) it follows that the operator M can be interpreted as the αth Riemann–Liouville fractional integral of f, where $\alpha = \frac{1}{2} - H$. For more details on the theory of deterministic fractional derivatives and integrals we refer to [**21**]. Let $L^2_H(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R};\ Mf(x) \in L^2(\mathbb{R})\}$. The space $L^2_H(\mathbb{R})$ is the closure of $S(\mathbb{R})$ with respect to the norm $\|f\|_{L^2_H(\mathbb{R})} = \|Mf\|_{L^2(\mathbb{R})}$, $f \in S(\mathbb{R})$ induced by the inner product $(f,g)_{L^2_H(\mathbb{R})} = (Mf, Mg)_{L^2(\mathbb{R})}$. The operator M is self-adjoint and for $f, g \in L^2(\mathbb{R}) \cap L^2_H(\mathbb{R})$ we have $(f, Mg)_{L^2_H(\mathbb{R})} = (Mf, g)_{L^2_H(\mathbb{R})}$.

Theorem 4.1. [**6**] *Let $M : L^2_H(\mathbb{R}) \to L^2(\mathbb{R})$ defined by* (4.2) *be the extension of the operator M from Definition* 4.1*. Then M is an isometry between the two Hilbert spaces $L^2(\mathbb{R})$ and $L^2_H(\mathbb{R})$. The functions*

$$e_n(x) = M^{-1}\xi_n(x), \quad n \in \mathbb{N}, \tag{4.5}$$

belong to $S(\mathbb{R})$ and form an orthonormal basis in $L^2_H(\mathbb{R})$.

Following [**2**] and [**6**] we extend M onto $\mathcal{S}'(\mathbb{R})$ and define $M : S'(\mathbb{R}) \to S'(\mathbb{R})$ by $\langle M\omega, f\rangle = \langle \omega, Mf\rangle$, $f \in S(\mathbb{R})$, $\omega \in S'(\mathbb{R})$. The fractional Itô integral of a deterministic function $f \in L^2_H(\mathbb{R})$ is defined by $I^{(H)}(f) = \int_{\mathbb{R}} f(t)\,dB^{(H)}_t(\omega) = \int_{\mathbb{R}} Mf(t)\,dB_t(\omega) = I(Mf)$. Then the relation $\|I(Mf)\|_{L^2(\mu)} = \|Mf\|_{L^2(\mathbb{R})} = \|f\|_{L^2_H(\mathbb{R})}$ holds. For more details on this subject we refer to [**2, 3, 6, 8**].

The t-continuous version of the process $B^{(H)}_t(\omega) := \langle \omega, M\chi[0,t](\cdot)\rangle$, $\omega \in S'(\mathbb{R})$ is an element of $L^2(\mu)$ called fractional Brownian motion (see [**2, 6, 10**]).

Now we extend the action of the operator M from $\mathcal{S}'(\mathbb{R})$ onto $L^2(\mu)$ and define the stochastic analogue of $L^2_H(\mathbb{R})$. Let

$$L^2(\mu_H) = L^2(\mu \circ M^{-1}) = \{G : \Omega \to \mathbb{R};\ G \circ M \in L^2(\mu)\}. \tag{4.6}$$

It is the space of square integrable functions on $S'(\mathbb{R})$ with respect to the *fractional Gaussian white noise measure* μ_H. Since $G \in L^2(\mu_H)$ if and only if $G \circ M \in L^2(\mu)$, it follows that G has an expansion of the form

$$\begin{aligned} G(M\omega) &= \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \xi_i\rangle) \\ &= \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, Me_i\rangle) = \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle M\omega, e_i\rangle). \end{aligned}$$

Define the family of Fourier–Hermite polynomials by

$$\widetilde{\mathcal{H}}_\alpha(\omega)=\prod_{k=1}^{\infty} h_{\alpha_k}(\langle\omega,e_k\rangle),\quad \alpha\in\mathcal{I}. \tag{4.7}$$

Now, it follows that the family $\{\widetilde{\mathcal{H}}_\alpha;\ \alpha\in\mathcal{I}\}$ forms an orthogonal basis of $L^2(\mu_H)$, $\|\widetilde{\mathcal{H}}_\alpha\|^2_{L^2(\mu_H)}=\alpha!$, $\alpha\in\mathcal{I}$, and $G\in L^2(\mu_H)$ has a chaos expansion representation of the form $G(\omega)=\sum_{\alpha\in\mathcal{I}} c_\alpha\widetilde{\mathcal{H}}_\alpha(\omega)$, $c_\alpha\in\mathbb{R}$ and $\|G\|^2_{L^2(\mu_H)}=\sum_{\alpha\in\mathcal{I}} c_\alpha^2\alpha!$. Moreover, $c_\alpha=\frac{1}{\alpha!}E_{\mu_H}(G\widetilde{\mathcal{H}}_\alpha(\omega))$ and $\|G\|_{L^2(\mu_H)}=\|G\circ M\|_{L^2(\mu)}$.

Definition 4.2. Let $\mathcal{M}: L^2(\mu_H)\to L^2(\mu)$ be defined by $\mathcal{M}(\widetilde{\mathcal{H}}_\alpha)=H_\alpha$ and extend it by linearity and continuity to

$$\mathcal{M}\bigg(\sum_{\alpha\in\mathcal{I}} c_\alpha\widetilde{\mathcal{H}}_\alpha\bigg)=\sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha \tag{4.8}$$

for $G=\sum_{\alpha\in\mathcal{I}} c_\alpha\widetilde{\mathcal{H}}_\alpha\in L^2(\mu_H)$.

Note that from (3.5), (4.7) and (4.8) follows $\mathcal{M}(\widetilde{\mathcal{H}}_\alpha(\omega))=\widetilde{\mathcal{H}}_\alpha(M\omega)=H_\alpha(\omega)$, $\omega\in\mathcal{S}'(\mathbb{R})$, $\alpha\in\mathcal{I}$. It holds that $\|\mathcal{M}(\widetilde{\mathcal{H}}_\alpha)\|_{L^2(\mu)}=\|H_\alpha\|_{L^2(\mu)}=\alpha!=\|\widetilde{\mathcal{H}}_\alpha\|_{L^2(\mu_H)}$, thus the operator $\mathcal{M}$ is an isometry between spaces of classical Gaussian and fractional Gaussian random variables and its action can be seen as a transformation of the corresponding elements of the orthogonal basis $\{\widetilde{\mathcal{H}}_\alpha\}_{\alpha\in\mathcal{I}}$ into $\{H_\alpha\}_{\alpha\in\mathcal{I}}$. The connection between the two bases is given by $H_\alpha(\omega)=\mathcal{M}\widetilde{\mathcal{H}}_\alpha(\omega)$ and $\widetilde{\mathcal{H}}_\alpha(\omega)=\mathcal{M}^{-1}H_\alpha(\omega)$, $\omega\in\mathcal{S}'(\mathbb{R})$, $\alpha\in\mathcal{I}$. Thus, every element of $L^2(\mu_H)$ can be represented as the image of a unique $f(\omega)=\sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega)\in L^2(\mu)$ such that $F=\mathcal{M}^{-1}f$. Then, F is of the form $F(\omega)=\sum_{\alpha\in\mathcal{I}} c_\alpha\,\widetilde{\mathcal{H}}(\omega)$.

For $P=\mu_H$ the spaces in (3.8) reduce to fractional q-weighted spaces of stochastic test functions and stochastic generalized functions. In [**10**] we considered the following inclusions $\exp(S)_1^{\mu_H}\subseteq(S)_1^{\mu_H}\subseteq L^2(\mu_H)\subseteq(S)_{-1}^{\mu_H}\subseteq\exp(S)_{-1}^{\mu_H}$. The action of the operator $\mathcal{M}$ can be extended to $q-$weighted spaces by defining $\mathcal{M}:(Q)_{-1}^{\mu_H}\to(Q)_{-1}^{\mu}$ given by

$$\mathcal{M}\bigg[\sum_{\alpha\in\mathcal{I}} a_\alpha\widetilde{\mathcal{H}}_\alpha(\omega)\bigg]=\sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega),\quad a_\alpha\in\mathbb{R}. \tag{4.9}$$

This extension is well defined since there exists $p\in\mathbb{N}$ such that $\sum_{\alpha\in\mathcal{I}} a_\alpha^2 q_\alpha^{-p}<\infty$.

In an analogous way the action of the operator $\mathcal{M}$ can be extended to generalized stochastic processes of type (O).

Example 4.1. The fractional Brownian motion $B_t^{(H)}(\omega)$ as an element of $L^2(\mu)$, is defined by the chaos expansion

$$
\begin{aligned}
B_t^{(H)}(\omega) &= \langle \omega, M\chi[0,t]\rangle = \langle M\omega, \chi[0,t]\rangle \\
&= \sum_{k=1}^{\infty} (\chi[0,t], e_k)_{L^2_H(\mathbb{R})} \langle M\omega, e_k\rangle = \sum_{k=1}^{\infty} (M\chi[0,t], Me_k)_{L^2(\mathbb{R})} \langle \omega, Me_k\rangle \\
&= \sum_{k=1}^{\infty} (\chi[0,t], M\xi_k)_{L^2(\mathbb{R})} \langle \omega, \xi_k\rangle = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s)\, ds \right) H_{\varepsilon^{(k)}}(\omega).
\end{aligned} \tag{4.10}
$$

Applying the map $\mathcal{M}^{-1} = \mathcal{M}^{(1-H)}$ we obtain the chaos decomposition form of the fractional Brownian motion in $L^2(\mu_H)$:

$$
B_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s)\, ds \right) \widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega). \tag{4.11}
$$

On the other hand,

$$
B_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \langle \chi[0,t], \xi_k\rangle_{L^2(\mathbb{R})} \langle \omega, M\xi_k\rangle = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s)\, ds \right) \langle \omega, M\xi_k\rangle. \tag{4.12}
$$

Note that for a fixed Hurst parameter $H \in (0,1)$ we have $M = M^{(H)}$ and due to (4.3) $M^{-1} = M^{(1-H)}$, thus $e_k^{(H)} = M^{(1-H)}\xi_k$ implies $M^{(H)}\xi_k = e_k^{(1-H)}$ and we may consider (4.12) to be the chaos decomposition of the fractional Brownian motion in $L^2(\mu_{(1-H)}) = L^2(\mu \circ M^{(H)})$ by the orthogonal basis $\widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega) = \langle \omega, e_k^{(1-H)}\rangle$. In other words, the fractional Brownian motion with the Hurst parameter $H \in (0,1)$ is the image of the classical Brownian motion under the mapping $\mathcal{M} = \mathcal{M}^{(H)}$ in the fractional white noise space $L^2(\mu_{(1-H)})$. Thus, we can consider fractional Brownian motion as an element of three different spaces, as defined in (4.10), (4.11) and (4.12).

Example 4.2. Fractional white noise $W_t^{(H)}(\cdot)$ is defined by the chaos expansions

$$
W_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\ \widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty} M\xi_k(t) H_{\varepsilon^{(k)}}(\omega), \tag{4.13}
$$

in the spaces $(S)_{-1}^{\mu_{(1-H)}}$ and $(S)_{-1}^{\mu}$ respectively. It is integrable and the relation $\frac{d}{dt}B_t^{(H)} = W_t^{(H)}$ holds (see [**8**]) in the $(S)_{-1}^{\mu}$ sense.

4.2. Fractional Poissonian white noise space. In this subsection we use the same idea as in the Gaussian case and apply the isomorphism M to the elements of the Poissonian white noise space to obtain their corresponding fractional versions. Let $H \in (0,1)$ and $J : L^2(\mathbb{R}) \to L^2(\nu)$, $J(f) = \langle \omega, f\rangle - \int_{\mathbb{R}} f(x)\, dx$ as in Subsection 3.2. Define $J^{(H)} := J \circ M$ as the mapping $L^2(\mathbb{R}) \to L^2(\nu)$, $f \mapsto \langle \omega, Mf\rangle - \int_{\mathbb{R}} Mf(x)\, dx$. Then $\|J(Mf)\|_{L^2(\nu)} = \|Mf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ holds for $f \in L^2(\mathbb{R})$.

Similarly as in (4.6) we let $L^2(\nu_H) = L^2(\nu \circ M^{-1})$ be the space of square integrable functions on $S'(\mathbb{R})$ with respect to *fractional Poissonian white noise measure* ν_H. The family of Charlier polynomials

$$\widetilde{\mathcal{C}}_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{e_1, \dots, e_1}_{\alpha_1}, \dots, \underbrace{e_m, \dots, e_m}_{\alpha_m}), \quad \alpha = (\alpha_1, \dots, \alpha_m, 0, 0, \dots) \in \mathcal{I}$$

(see (3.7) and (4.5)) forms the orthogonal basis of the Hilbert space of fractional Poissonian random variables i.e., $L^2(\nu_H)$ consists of elements $F = \sum_{\alpha\in\mathcal{I}} a_\alpha \widetilde{\mathcal{C}}_\alpha(\omega)$, $a_\alpha \in \mathbb{R}$ such that $\|F\|^2_{L^2(\nu_H)} = \sum_{\alpha\in\mathcal{I}} a_\alpha^2 \alpha! < \infty$.

The mapping $\mathcal{M}^{-1} : L^2(\nu) \to L^2(\nu_H)$ defined by $\widetilde{\mathcal{C}}_\alpha(\omega) = \mathcal{M}^{-1} C_\alpha(\omega)$, $\alpha \in \mathcal{I}$, extends by linearity and continuity to $L^2(\nu)$. Thus every element $G \in L^2(\nu_H)$ can be represented as an inverse picture of a unique $g = \sum_{\alpha\in\mathcal{I}} a_\alpha C_\alpha(\omega) \in L^2(\nu)$ such that $G(\omega) = \mathcal{M}^{-1} g(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha\, C_\alpha(\omega) \in L^2(\nu)$.

Example 4.3. There exists a right t-continuous version of the process $P_t^{(H)}(\omega) = J(M\chi[0,t])(\omega)$, $\omega \in S'(\mathbb{R})$, that belongs to $L^2(\nu_H)$ and is called the fractional compensated Poisson process. It is given by the chaos expansions

$$P_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s) ds \right) C_{\varepsilon^{(k)}}(\omega) \quad \text{in } L^2(\nu), \tag{4.14}$$

$$P_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) \widetilde{\mathcal{C}}_{\varepsilon^{(k)}}(\omega) \quad \text{in } L^2(\nu_{(1-H)}). \tag{4.15}$$

Example 4.4. The fractional compensated Poissonian noise is defined by the chaos expansions

$$V_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\, \widetilde{\mathcal{C}}_{\varepsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty} M\xi_k(t) C_{\varepsilon^{(k)}}(\omega), \tag{4.16}$$

in the spaces $(S)_{-1}^{\nu_{(1-H)}}$ and $(S)_{-1}^{\nu}$ respectively.

Remark 4.1. Since there exists an isomorphism $\mathcal{M}$ between the classical white noise spaces (Gaussian or Poissonian) and their corresponding fractional white noise spaces; and also there exists the isomorphism $\mathcal{U}$ between Gaussian and Poissonian white noise spaces (classical or fractional), all results obtained, for example, in the classical Gaussian case can be interpreted in all other spaces. In this manner, the space $L^2(\nu_H)$ can be obtained from the fractional Gaussian white noise space by $L^2(\nu_H) = \mathcal{U}[L^2(\mu_H)]$ or directly from the Gaussian white noise space $L^2(\nu_H) = \mathcal{U}[\mathcal{M}^{-1}[L^2(\mu)]]$ or $L^2(\nu_H) = \mathcal{M}^{-1}[\mathcal{U}[L^2(\mu)]]$. All connections are described in the following commutative diagram.

```tikzcd
L^2(\mu) \arrow[r, "\mathcal{M}^{-1}"] \arrow[d, "\mathcal{U}"'] \arrow[dr, "\mathcal{U}\circ\mathcal{M}^{-1}", "\mathcal{M}^{-1}\circ\mathcal{U}"'] & L^2(\mu_H) \arrow[d, "\mathcal{U}"] \\
L^2(\nu) \arrow[r, "\mathcal{M}^{-1}"'] & L^2(\nu_H)
```

Diagram 1

5. Chaos expansion of generalized random process of type (I)

Denote by $\mathfrak{e}_k, k \in \mathbb{N}$ the orthonormal basis of $L^2_H(\mathbb{R})$, i.e., $\mathfrak{e}_k$ is either the orthonormal Hermite basis ξ_k, $k \in \mathbb{N}$ (for $H = \frac{1}{2}$) or the orthonormal fractional basis $e_k = M^{-1}\xi_k$, $k \in \mathbb{N}$ (for $H \in (0,1)$). Note, $\|\mathfrak{e}_k\|^2_{-l} = (2k)^{-l}$ and $\|\mathfrak{e}_k\|^2_{\exp,-l} = e^{-2kl}$ for all $k, l \in \mathbb{N}$. Denote by K_α, $\alpha \in \mathcal{I}$ the orthogonal basis of the space of square integrable random variables $L^2(P)$ on the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, P)$.

TABLE 1.

white noise space	classical		fractional	
	Gaussian	Poissonian	Gaussian	Poissonian
measure P	μ	ν	μ_H	ν_H
basis K_α	H_α	C_α	$\widetilde{\mathcal{H}}_\alpha$	$\widetilde{\mathcal{C}}_\alpha$
basis $\mathfrak{e}_k$	ξ_k	ξ_k	e_k	e_k

We extend the chaos expansion theorem to the class of generalized random processes of type (I). Let X be a topological vector space and X' its dual.

We consider generalized stochastic processes of type (I) as linear and continuous mappings from X into the space of q-weighted generalized functions $(Q)^P_{-1}$ i.e., elements of $\mathcal{L}(X, (Q)^P_{-1})$. If at least one of the spaces X or $(Q)^P_{-1}$ is nuclear, then $\mathcal{L}(X, (Q)^P_{-1}) \cong X' \otimes (Q)^P_{-1}$.

EXAMPLE 5.1. Brownian motion $B_t(\omega)$ defined in (3.10) and fractional Brownian motion defined in (4.10), as well as the Poissonian process defined in (3.12) and fractional Poissonian process (4.14) are stochastic processes of type (I), i.e., elements of the space $X \otimes (Q)^P_{-1}$ where $X = C^\infty([0, +\infty))$. White noise (3.11), fractional white noise (4.13), Poissonian noise (3.13) and fractional Poissonian noise (4.16) are elements of $X \otimes (Q)^P_{-1}$, where $X = \mathcal{S}'(\mathbb{R})$.

THEOREM 5.1. [**20**] *Let X be a Banach space endowed with $\|\cdot\|_X$. Generalized stochastic processes as elements of $\mathcal{L}(X, (Q)^P_{-1})$ have a chaos expansion of the form*

$$u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes K_\alpha, \quad f_\alpha \in X,\ \alpha \in \mathcal{I}, \tag{5.1}$$

and there exists $p \in \mathbb{N}_0$ such that

$$\|u\|^2_{X \otimes (Q)^P_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X q_\alpha^{-p} < \infty.$$

THEOREM 5.2. [**20**] *Let $X = \bigcap_{k=0}^\infty X_k$ be a nuclear space endowed with a family of seminorms $\{\|\cdot\|_k; k \in \mathbb{N}_0\}$ and let $X' = \bigcup_{k=0}^\infty X_{-k}$ be its topological dual. Generalized stochastic processes as elements of $X' \otimes (Q)^P_{-1}$ have a chaos expansion of the form $u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes K_\alpha$, $f_\alpha \in X_{-k}$, $\alpha \in \mathcal{I}$, where $k \in \mathbb{N}_0$ does not depend on $\alpha \in \mathcal{I}$, and there exists $p \in \mathbb{N}_0$ such that*

$$\|u\|^2_{X' \otimes (Q)^P_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_{-k} q_\alpha^{-p} < \infty.$$

With the same notation as in (5.1) we will denote by $E(u) = f_{(0,0,0,\dots)}$ the generalized expectation of the process u.

We extend the action of the operator $\mathcal{U}$ given by (3.9) and also the action of the operator $\mathcal{M}$ given by (4.9) to the class of generalized stochastic processes of type (I). We define $\mathcal{U} : X \otimes (Q)_{-1}^{\mu} \to X \otimes (Q)_{-1}^{\nu}$ such that for every $\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes (Q)_{-1}^{\mu}$

$$\mathcal{U}\Big[\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha\Big] = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes C_\alpha, \quad u_\alpha \in X,\ \alpha \in \mathcal{I}.$$

For all processes in $X \otimes (Q)_{-1}^{\mu_H}$, represented in the form $\sum_{\alpha\in\mathcal{I}} v_\alpha \otimes \widetilde{\mathcal{H}}_\alpha(\omega)$ we define the operator $\mathcal{M} : X \otimes (Q)_{-1}^{\mu_H} \to X \otimes (Q)_{-1}^{\mu}$ by

$$\mathcal{M}\Big[\sum_{\alpha\in\mathcal{I}} v_\alpha \otimes \widetilde{\mathcal{H}}_\alpha\Big] = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha, \quad v_\alpha \in X,\ \alpha \in \mathcal{I}.$$

Remark 5.1. Note that $\mathcal{U} \circ \mathcal{M}^{-1} : X \otimes (Q)_{-1}^{\mu} \to X \otimes (Q)_{-1}^{\nu_H}$ such that

$$\mathcal{U} \circ \mathcal{M}^{-1}\Big[\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha\Big] = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes \widetilde{\mathcal{C}}_\alpha, \quad u_\alpha \in X,\ \alpha \in \mathcal{I}.$$

The same is obtained by action of the operator $\mathcal{M}^{-1} \circ \mathcal{U}$ i.e., $\mathcal{U} \circ \mathcal{M}^{-1} = \mathcal{M}^{-1} \circ \mathcal{U}$ (Diagram 1).

5.1. $\mathcal{S}'$-valued generalized random process. In [**23**] and [**24**] we provided a general setting of vector-valued generalized random processes. $S'(\mathbb{R})$-valued generalized random processes are elements of $\widetilde{X} \otimes (Q)_{-1}^{P}$, where $\widetilde{X} = X \otimes S'(\mathbb{R})$, and are given by chaos expansions of the form

$$f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha = \sum_{\alpha\in\mathcal{I}} b_\alpha \otimes K_\alpha = \sum_{k\in\mathbb{N}} c_k \otimes \mathfrak{e}_k, \tag{5.2}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \mathfrak{e}_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes K_\alpha \in X \otimes (Q)_{-1}^{P}$ and $a_{\alpha,k} \in X$. Thus, for some $p, l \in \mathbb{N}_0$,

$$\|f\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes (Q)^{P}_{-1-p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X (2k)^{-l} q_\alpha^{-p} < \infty.$$

In a similar manner one can also consider $\exp S'(\mathbb{R})$-valued generalized stochastic processes as elements of $X \otimes \exp S'(\mathbb{R}) \otimes (Q)_{-1}^{P}$ given by a chaos expansion of the form (5.2), with the convergency condition

$$\|f\|^2_{X\otimes \exp S_{-l}(\mathbb{R})\otimes (Q)^{P}_{-1-p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X e^{-2kl} q_\alpha^{-p} < \infty,$$

for some $p, l \in \mathbb{N}_0$.

5.2. The Malliavin derivative and the Skorokhod integral. From this section and further on we will consider only the Kondratiev-type spaces $(S)_{-1}^P$ and $\exp(S)_{-1}^P$ defined by the weights $q_\alpha = (2\mathbb{N})^\alpha$ and $q_\alpha = e^{(2\mathbb{N})^\alpha}$ respectively. We will omit writing the measure P, and denote these spaces $(S)_1$ and $\exp(S)_{-1}$, since there exist unitary mappings between all four white noise spaces (Diagram 1).

We give now the definitions of the Malliavin derivative and the Skorokhod integral which are slightly more general than in [**4, 16, 17, 18**]. Instead of setting the domain in a way that the Malliavin derivative and the Skorokhod integral take values in $L^2(P)$, we allow values in $(S)_{-1}$ and $\exp(S)_{-1}$ and thus obtain a larger domain for both operators.

Denote by ι the multi-index $\iota = \sum_{k=1}^{\infty} \varepsilon^{(k)} = (1, 1, 1, \ldots)$. Note that $\iota \notin \mathcal{I}$, but we will use the following convention: for $\alpha \in \mathcal{I}$, define $\alpha - \iota$ as the multi-index with k-th component

$$(\alpha - \iota)_k = \begin{cases} \alpha_k - 1, & \alpha_k \geqslant 2 \\ 0, & \alpha_k \in \{0, 1\} \end{cases}.$$

Thus, $\alpha - \iota \in \mathcal{I}$, for all $\alpha \in \mathcal{I}$.

Definition 5.1. Let $u \in X \otimes (S)_{-1}$ be of the form (5.1). If there exists $p \in \mathbb{N}_0$ such that

$$\sum_{\alpha \in \mathcal{I}} |\alpha|^2 \, \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty, \tag{5.3}$$

then the Malliavin derivative of u is defined by

$$\mathbb{D}u = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k f_\alpha \otimes \mathfrak{e}_k \otimes K_{\alpha - \varepsilon^{(k)}}. \tag{5.4}$$

The operator $\mathbb{D}$ is also called the stochastic gradient of a generalized stochastic process u. The set of processes u such that (5.3) is satisfied is the domain of the Malliavin derivative, which will be denoted by $\mathrm{Dom}(\mathbb{D})$. All processes which belong to $\mathrm{Dom}(\mathbb{D})$ are called differentiable in the Malliavin sense. We proved in [**11**] that the Malliavin derivative $\mathbb{D}$ is a linear and continuous mapping from $\mathrm{Dom}(\mathbb{D}) \subseteq X \otimes (S)_{-1,-p}$ to $X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}$, for some $p \in \mathbb{N}_0$ and $l > p + 1$, $l \in \mathbb{N}$.

Definition 5.2. A process $u \in X \otimes \exp(S)_{-1}$ of the form (5.1) is differentiable in the Malliavin sense, i.e., $u \in \mathrm{Dom}_{\exp}(\mathbb{D})$ if there exists $p \in \mathbb{N}_0$ such that

$$\sum_{\alpha \in \mathcal{I}} |\alpha|^2 \|f_\alpha\|_X^2 \, e^{-p(2\mathbb{N})^{\alpha - \iota}} < \infty.$$

Then the Malliavin derivative is defined by (5.4).

Theorem 5.3. *The Malliavin derivative is a linear and continuous mapping from* $\mathrm{Dom}_{\exp}(\mathbb{D}) \subseteq X \otimes \exp(S)_{-1,-p}$ *to* $X \otimes \exp S_{-l}(\mathbb{R}) \otimes \exp(S)_{-1,-p}$*, for all* $l \in \mathbb{N}_0$*.*

PROOF. Clearly,

$$\begin{aligned}\|\mathbb{D}u\|^2_{X\otimes\exp S_{-l}(\mathbb{R})\otimes\exp(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}}\Big\|\sum_{k=1}^{\infty}\alpha_k f_\alpha\otimes\mathfrak{e}_k\Big\|^2_{X\otimes\exp S_{-l}(\mathbb{R})} e^{-p(2\mathbb{N})^{(\alpha-\varepsilon^{(k)})}}\\ &\leqslant \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\alpha_k^2\, e^{-2kl}\,\|f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha-\iota}}\\ &\leqslant \sum_{\alpha\in\mathcal{I}}|\alpha|^2\|f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha-\iota}}<\infty. \qquad \square\end{aligned}$$

Note that $\mathrm{Dom}_{exp}(\mathbb{D})\supseteq\mathrm{Dom}(\mathbb{D})$.

DEFINITION 5.3. Let $F=\sum_{\alpha\in\mathcal{I}}f_\alpha\otimes v_\alpha\otimes K_\alpha\in X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-1,-p}$, $p\in\mathbb{N}_0$ be a generalized $S_{-p}(\mathbb{R})$-valued stochastic process and let $v_\alpha\in S_{-p}(\mathbb{R})$ be given by the expansion $v_\alpha=\sum_{k\in\mathbb{N}}v_{\alpha,k}\,\mathfrak{e}_k$, $v_{\alpha,k}\in\mathbb{R}$. Then the process F is integrable in the Skorokhod sense and the chaos expansion of its stochastic integral is given by

$$\delta(F)=\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}v_{\alpha,k}\,f_\alpha\otimes K_{\alpha+\varepsilon^{(k)}}. \tag{5.5}$$

We proved in [**11**] that the Skorokhod integral δ is a linear and continuous mapping $\delta: X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-1,-p}\to X\otimes(S)_{-1,-p}$.

THEOREM 5.4. *Let $F=\sum_{\alpha\in\mathcal{I}}f_\alpha\otimes v_\alpha\otimes K_\alpha\in X\otimes\exp S_{-p}(\mathbb{R})\otimes\exp(S)_{-1,-p}$, $p\in\mathbb{N}_0$ be a generalized $\exp S_{-p}(\mathbb{R})$-valued stochastic process and let $v_\alpha\in\exp S_{-p}(\mathbb{R})$ be given by the expansion $v_\alpha=\sum_{k\in\mathbb{N}}v_{\alpha,k}\,\mathfrak{e}_k$, $v_{\alpha,k}\in\mathbb{R}$. Let $\delta(F)$ be its Skorokhod integral defined by* (5.5). *Then the Skorokhod integral δ is a linear and continuous mapping $\delta: X\otimes\exp S_{-p}(\mathbb{R})\otimes\exp(S)_{-1,-p}\to X\otimes\exp(S)_{-1,-p}$.*

PROOF. This follows from the inequality $e^{-p(2\mathbb{N})^{\alpha}(2k)}\leqslant e^{-2kp}\cdot e^{-p(2\mathbb{N})^{\alpha}}$, for $\alpha\in\mathcal{I}$ and $k,p\geqslant 0$. Clearly,

$$\begin{aligned}\|\delta(F)\|^2_{X\otimes\exp(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}}\Big\|\sum_{k\in\mathbb{N}}v_{\alpha,k}\,f_\alpha\Big\|^2_X e^{-p(2\mathbb{N})^{\alpha+\varepsilon^{(k)}}}\\ &\leqslant \sum_{\alpha\in\mathcal{I}}\Big(\sum_{k\in\mathbb{N}}v_{\alpha,k}^2\,e^{-2kp}\Big)\|f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha}}\\ &= \sum_{\alpha\in\mathcal{I}}\|v_\alpha\|^2_{\exp,-p}\,\|f_\alpha\|^2_X\,e^{-p(2\mathbb{N})^{\alpha}}<\infty,\end{aligned}$$

since $F\in X\otimes\exp S_{-p}(\mathbb{R})\otimes\exp(S)_{-1,-p}$, $p\in\mathbb{N}_0$. $\square$

From $2(2\mathbb{N})^{\alpha}\leqslant(2\mathbb{N})^{2\alpha}$ we conclude that the image of the Malliavin derivative is included in the domain of the Skorokhod integral. Their composition $\mathcal{R}=\delta\circ\mathbb{D}$ is called the Ornstein–Uhlenbeck operator. The Hermite i.e., the Charlier polynomials are eigenfunctions of $\mathcal{R}$ and the corresponding eigenvalues are $|\alpha|$, $\alpha\in\mathcal{I}$, i.e., $\mathcal{R}(K_\alpha)=|\alpha|K_\alpha$. Moreover, if we apply the previous identity k times successively, we obtain $\mathcal{R}^k(K_\alpha)=|\alpha|^kK_\alpha$, $k\in\mathbb{N}$, for $\alpha\in\mathcal{I}$.

Let a generalized stochastic process $u \in \mathrm{Dom}(\mathbb{D})$ be given by the chaos expansion $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes K_\alpha$, $u_\alpha \in X$. Then

$$\mathcal{R}u = \sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes K_\alpha. \tag{5.6}$$

Denote by

$$\mathrm{Dom}(\mathcal{R}) = \Big\{ u \in X \otimes (S)_{-1} :\ \exists p \in \mathbb{N}_0,\ \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty \Big\} \quad \text{and}$$

$$\mathrm{Dom}_{\exp}(\mathcal{R}) = \Big\{ u \in X \otimes \exp(S)_{-1} :\ \exists p \in \mathbb{N}_0,\ \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|u_\alpha\|_X^2 e^{-p(2\mathbb{N})^\alpha} < \infty \Big\}.$$

The operator $\mathcal{R}$ is a linear and continuous mapping from $\mathrm{Dom}(\mathcal{R}) \subset X \otimes (S)_{-1}$ into the space $X \otimes (S)_{-1}$, and in this case the domains of $\mathbb{D}$ and $\mathcal{R}$ coincide, i.e., $\mathrm{Dom}(\mathcal{R}) = \mathrm{Dom}(\mathbb{D})$. Clearly, if $u \in \mathrm{Dom}(\mathbb{D}) \subset X \otimes (S)_{-1,-p}$ then $\mathcal{R}u \in X \otimes (S)_{-1,-p}$. This follows from (5.6) and

$$\|\mathcal{R}u\|^2_{X\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} = \|u\|^2_{\mathrm{Dom}(\mathbb{D})} < \infty.$$

For $u \in X \otimes \exp(S)_{-1,-p}$ it follows that $\mathrm{Dom}_{exp}(\mathbb{D}) \subseteq \mathrm{Dom}_{exp}(\mathcal{R})$.

5.3. The fractional Malliavin derivative and Skorokhod integral. Consider the extension of the operator M from $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ onto generalized stochastic processes: Let $\mathbf{M} = M \otimes Id : \mathcal{S}'(\mathbb{R}) \otimes (Q)^P_{-1} \to \mathcal{S}'(\mathbb{R}) \otimes (Q)^P_{-1}$ be given by

$$\mathbf{M}\Big(\sum_{\alpha\in\mathcal{I}} a_\alpha(t) \otimes K_\alpha(\omega) \Big) = \sum_{\alpha\in\mathcal{I}} M a_\alpha(t) \otimes K_\alpha(\omega). \tag{5.7}$$

Its restriction onto $L^2_H(\mathbb{R}) \otimes L^2(P)$ is an isometric mapping $L^2_H(\mathbb{R}) \otimes L^2(P) \to L^2(\mathbb{R}) \otimes L^2(P)$. In Example 4.1 and Example 4.2 we have seen that $B_t^{(H)} = \mathbf{M}B_t$ in $L^2(\mu)$, and $W_t^{(H)} = \mathbf{M}W_t$ in $(S)_{-1}$.

In [**18**] the fractional Malliavin derivative in $L^2(\mu)$ was defined as $\mathbb{D}^{(H)} = \mathbf{M}^{-1} \circ \mathbb{D}$. Thus, we extend this notion to generalized stochastic processes of type (I), e.g. on Kondratiev white noise spaces with Gaussian measure $\mathbb{D}^{(H)} : X \otimes (S)^\mu_{-1} \to X \otimes \mathcal{S}'(\mathbb{R}) \otimes (S)^\mu_{-1}$ is given by

$$\begin{aligned} \mathbb{D}^{(H)} F = \mathbf{M}^{-1} \circ \mathbb{D}\, F = \mathbf{M}^{-1} \Big(\sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \, \otimes \, \xi_k \, \otimes H_{\alpha-\varepsilon^{(k)}} \Big) \\ = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes e_k \otimes H_{\alpha-\varepsilon^{(k)}}, \end{aligned} \tag{5.8}$$

for $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$, $f_\alpha \in X$, $\alpha \in \mathcal{I}$. Note that the domain of the fractional Malliavin derivative coincides with the domain of the classical Malliavin derivative. The following definition holds on a general white noise space (Gaussian, Poissonian, fractional Gaussian or fractional Poissonian).

DEFINITION 5.4. Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha \in X \otimes (S)_{-1}$, respectively $X \otimes \exp(S)_{-1}$. If $F \in \mathrm{Dom}(\mathbb{D})$, respectively $F \in \mathrm{Dom}_{\exp}(\mathbb{D})$, then the fractional Malliavin derivative of F is defined by

$$\mathbb{D}^{(H)} F = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes M^{-1}\mathfrak{e}_k \otimes K_{\alpha-\varepsilon^{(k)}}. \tag{5.9}$$

In the following theorem, P will denote either the Gaussian or Poissonian measure, and P_H will denote their corresponding fractional measures. The notation $(Q)_{-1}$ will refer to either $(S)_{-1}$ or $\exp(S)_{-1}$ with the appropriate measure.

THEOREM 5.5. *Let $\mathbb{D}$ and $\mathbb{D}^{(H)}$ denote the Malliavin derivative, respectively the fractional Malliavin derivative on $X \otimes (Q)_{-1}^{P}$. Let $\widetilde{\mathbb{D}}$ denote the Malliavin derivative on $X \otimes (Q)_{-1}^{P_H}$. Then,*

$$\mathbb{D}^{(H)} F = \mathbf{M}^{-1} \circ \mathbb{D}\, F = \mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1} F, \tag{5.10}$$

for all $F \in \mathrm{Dom}(\mathbb{D})$.

PROOF. We will conduct the proof for the Gaussian case. Since $D^{(H)}F = \mathbf{M}^{-1} \circ \mathbb{D} F$ follows directly from (5.7) and (5.9), we need to prove that (5.8) is equal to $\mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1} F$, where $\widetilde{\mathbb{D}}$ stands for the Malliavin derivative in $L^2(\mu_H)$. Clearly,

$$\begin{aligned}
\mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1} \Big(\sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha \Big) &= \mathcal{M} \circ \widetilde{\mathbb{D}} \Big(\sum_{\alpha\in\mathcal{I}} f_\alpha \otimes \widetilde{\mathcal{H}}_\alpha \Big) \\
&= \mathcal{M} \Big(\sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes e_k \otimes \widetilde{\mathcal{H}}_{\alpha-\varepsilon^{(k)}} \Big) \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes e_k \otimes H_{\alpha-\varepsilon^{(k)}} \qquad \square
\end{aligned}$$

EXAMPLE 5.2. It is well known that in $L^2(\mu)$, the Malliavin derivative of Brownian motion is $\mathbb{D}B_t(\omega) = \chi[0,t] = \sum_{k=1}^{\infty} c_k \xi_k$. Thus

$$\mathbb{D}^{(H)} B_t(\omega) = M^{-1}\chi[0,t] = M^{(1-H)}(0,t) = \sum_{k=1}^{\infty} c_k e_k,$$

where

$$c_k = (\xi_k, \chi[0,t])_{L^2(\mathbb{R})} = (M^{-1}\xi_k, M^{-1}\chi[0,t])_{L^2_{1-H}(\mathbb{R})} = \big(e_k, M^{(1-H)}(0,t)\big)_{L^2_{1-H}(\mathbb{R})}.$$

DEFINITION 5.5. Let $\delta : X \otimes \mathcal{S}'(\mathbb{R}) \otimes (Q)_{-1}^{P} \to X \otimes (Q)_{-1}^{P}$ denote the Skorokhod integral in sense of Definition 5.3 and Theorem 5.4. The fractional Skorokhod integral $\delta^{(H)} : X \otimes \mathcal{S}'(\mathbb{R}) \otimes (Q)_{-1}^{P} \to X \otimes (Q)_{-1}^{P}$ is defined for every $F \in \mathrm{Dom}(\delta)$ by

$$\delta^{(H)} F = \delta \circ \mathbf{M}\, F. \tag{5.11}$$

Finally, for the Ornstein–Uhlenbeck operator we note that its fractional version coincides with the regular one, i.e., from (5.10) and (5.11) it follows that

$$\mathcal{R}^{(H)} = \delta^{(H)} \circ \mathbb{D}^{(H)} = \delta \circ \mathbf{M} \circ \mathbf{M}^{-1} \circ \mathbb{D} = \delta \circ \mathbb{D} = \mathcal{R}.$$

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