PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 91(105) (2012), 59–62 DOI: 10.2298/PIM1205059J

ON GAUSS–BONNET THEOREM

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ABSTRACT. A very simple proof of the Gauss–Bonnet theorem is given in invariant form, i.e., independent of the coordinate system of a surface.

The Gauss–Bonnet theorem is one of the most important theorem of the theory of surfaces. This theorem is an example of differential geometry in the large. Moreover, it is analogous to Green's theorem and can be obtained from this theorem [1].

THEOREM. *Let S be a simple connected portion of a surface for which a representation $\boldsymbol{x}(u^1, u^2)$ of class $r \geqslant 3$ exists and whose boundary C is a simple closed curve which has a representation $x(u^1(s), u^2(s))$ of class $r^* \geqslant 2$, where s is the arc length of C. Let k_g be the geodesic curvature of C and let K be the Gaussian curvature of S. Then*

$$\int\limits_C k_g ds + \iint\limits_S K da = 2\pi, \tag{1}$$

where da is the element of area of C. The integration along C has to be carried out in such direction that S stays on the left side.

The integral $\iint_S K da$ occurring in the Gauss–Bonnet theorem is called the integral curvature of a surface under consideration [2].

The proof of the theorem can be found in every good book on differential geometry. There are many different approaches to the proof. The proof is simplified by the use of special coordinate systems u^α, $\alpha = 1, 2$, on the surface S. But then the invariant approach to the proof of the theorem is lost. Therefore it is desirable to give the proof the theorem independently of the choice of coordinate system on S. This gives the theorem the right sense to its invariant property. Here we present an approach of this kind, so far unknown to me. For that purpose the standard notation of tensor calculus of the surface S is applied: $a_{\alpha\beta}$ is the metric tensor, $a = \det(a_{\alpha\beta})$, $\varepsilon^{\alpha\beta}$ is the Ricci tensor of alternation; $(\cdot)_,$ denote covariant

2010 *Mathematics Subject Classification:* Primary 53A04; Secondary 53A05.

Dedicated to the memory of Professor Antun Bilimovich, the founder of the journal Publications de L'Institut Mathematique de L'Academie Serbe des Sciences.

derivative. The Greek indices $\alpha, \beta, \dots$, indicate values from 1 to 2. The summation over repeated indices is understood.

Further we shall require the following:

I. If λ^α and λ_α are the contravariant and covariant components of a unit vector, one has $\lambda^\alpha_{,\beta}\lambda_\alpha = \lambda_{\alpha,\beta}\lambda^\alpha = 0$, from which it follows that

$$\lambda^\alpha_{,\beta} = \mu^\alpha \nu_\beta, \qquad \lambda_{\alpha,\beta} = \mu_\alpha \nu_\beta, \tag{2}$$

where μ^α is the vector perpendicular to the given vector λ_α, and ν_β is a vector, [3]. Now, it is easy to show that

$$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}\lambda^\gamma_{,\alpha}\lambda^\delta_{,\beta} = 0. \tag{3}$$

II. Next we have

$$m_{\alpha,\beta\gamma} - m_{\alpha,\gamma\beta} = m_\delta R^\delta_{\alpha\beta\gamma}, \tag{4}$$

where

$$R_{\alpha\beta\gamma\delta} = K\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} \tag{5}$$

is the Riemann–Christoffel tensor. Then from (4) and (5) we have

$$\varepsilon^{\beta\gamma}m_{\alpha,\beta\gamma} = Km^\gamma\varepsilon_{\gamma\alpha}. \tag{6}$$

III. From the Green theorem [4] we have

$$\oint A_\alpha\lambda^\alpha\,ds = \iint\limits_S \varepsilon^{\alpha\beta}A_{\beta,\alpha}\,da. \tag{7}$$

IV. Now, let $C : u^\alpha = u^\alpha(s)$ and $C_a : \varphi(u^1, u^2) = a$ be family of curves on S; s is the arc length of C, and a is an arbitrary constant. Further, $\lambda^\alpha = du^\alpha/ds$ denotes the unit tangent vector of C and $m_\alpha = \varphi_{,\alpha}/|\operatorname{grad}\varphi|$ the unit vector normal on C_a; $|\operatorname{grad}\varphi| = \sqrt{a^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}}$. We assume that C intersects with the family of curves C_a. At the point of intersection the angle between C and C_a will be denoted by θ.

We want to calculate the rate of change with respect to s along C of the angle θ. This can be done making use of $\theta = \frac{\pi}{2} - \vartheta$, where ϑ is the angle between the unit vectors λ^α and m_α at the point of intersection since then $\frac{d\theta}{dt} = -\frac{d\vartheta}{dt}$. Obviously,

$$\tan\vartheta = \frac{\varepsilon_{\alpha\beta}\lambda^\alpha m^\beta}{a_{\gamma\delta}\lambda^\gamma m^\delta}$$

so that

$$\begin{aligned}
\frac{d\vartheta}{dt} = \frac{\delta\vartheta}{\delta s} &= \varepsilon_{\alpha\beta}\frac{\delta}{\delta s}\left(\varepsilon_{\alpha\beta}\lambda^\alpha m^\beta\right)a_{\gamma\delta}\lambda^\gamma m^\delta - \varepsilon_{\alpha\beta}\lambda^\alpha m^\beta\frac{\delta}{\delta s}\left(a_{\gamma\delta}\lambda^\gamma m^\delta\right)\\
&= \varepsilon_{\alpha\beta}a_{\gamma\delta}\left(\frac{\delta\lambda^\alpha}{\delta s}\lambda^\gamma - \lambda^\alpha\frac{\delta\lambda^\gamma}{\delta s}\right)m^\beta m^\delta + \varepsilon_{\alpha\beta}a_{\gamma\delta}\left(\frac{\delta m^\beta}{\delta s}m^\delta - m^\beta\frac{\delta m^\delta}{\delta s}\right)\lambda^\alpha\lambda^\gamma\\
&= \varepsilon_{\alpha\beta}a_{\gamma\delta}\delta^{\alpha\gamma}_{\sigma\tau}\frac{\delta\lambda^\sigma}{\delta s}\lambda^\tau m^\beta m^\delta + \varepsilon_{\alpha\beta}a_{\gamma\delta}\delta^{\beta\delta}_{\sigma\tau}\frac{\delta m^\sigma}{\delta s}m^\tau\lambda^\alpha\lambda^\gamma\\
&= a_{\beta\delta}m^\beta m^\delta\varepsilon_{\sigma\tau}\frac{\delta\lambda^\sigma}{\delta s}\lambda^\tau - a_{\gamma\alpha}\lambda^\alpha\lambda^\gamma\varepsilon_{\sigma\tau}\frac{\delta m^\sigma}{\delta s}m^\tau.
\end{aligned}$$

Thus,

$$\frac{d\vartheta}{ds} = \varepsilon_{\sigma\tau}\frac{\delta\lambda^\sigma}{\delta s}\lambda^\tau - \varepsilon_{\sigma\tau}\frac{\delta m^\sigma}{\delta s}m^\tau$$

and from this

$$\frac{d\vartheta}{ds} = k_g - \varepsilon_{\sigma\tau}\frac{\delta m^\sigma}{\delta s}m^\tau$$

since $\dfrac{\delta\lambda^\sigma}{\delta s} = k_g\upsilon^\sigma$ and $\varepsilon_{\sigma\tau}\upsilon^\sigma\lambda^\tau = -1$. Hence,

$$\frac{d\theta}{ds} = k_g + \varepsilon_{\sigma\tau}\frac{\delta m^\sigma}{\delta s}m^\tau. \tag{8}$$

Now we are ready to give the proof of the theorem.

Proof. From (8) we have

$$\oint d\theta - \oint k_g\,ds = \oint \varepsilon_{\sigma\tau}\delta m^\sigma m^\tau = \oint \varepsilon_{\sigma\tau}m^\sigma_{,\alpha}m^\tau\lambda^\alpha\,ds$$

where we make use of

$$\frac{\delta m^\alpha}{\delta s} = m^\alpha_{,\varrho}\lambda^\varrho.$$

But, in view of Green's theorem (7), (2), (3) and (6),

$$\begin{aligned}\oint \varepsilon_{\sigma\tau}m^\sigma_{,\alpha}m^\tau\lambda^\alpha\,ds &= \iint_S \varepsilon^{\beta\alpha}\big(\varepsilon_{\sigma\tau}m^\sigma_{,\alpha}m^\tau\big)_{,\beta}\,da\\ &= \iint_S \varepsilon^{\beta\alpha}\varepsilon_{\sigma\tau}\big(m^\sigma_{,\alpha\beta}m^\tau + m^\sigma_{,\alpha}m^\tau_{,\beta}\big)da\\ &= -\iint_S Km_\gamma\varepsilon^{\gamma\sigma}\varepsilon_{\sigma\tau}m^\tau\,da = \iint_S K\,da.\end{aligned}$$

Thus,

$$\oint d\theta - \oint k_g ds = \iint_S K da$$

from which (1) follows. □

Of course the Gauss–Bonnet theorem can be formulated for more general cases, for example for a simply-connected portion of a surface which is bounded by piecewise regular curves, but the procedure is the same. Then instead of (1) we have

$$\oint k_g\,ds + \sum_{i=1}^{n}(\pi - \theta_i) + \iint_S K\,da = 2\pi, \tag{9}$$

where θ_i are the corresponding exterior angles of C at the points of cusps. Usually we make use of interior angles α_i, i.e. of the relation $\alpha_i = \pi - \theta_i$, so that we write (9) as

$$(n-2)\pi + \oint k_g\,ds + \iint_S K\,da = \sum_{i=1}^{n}\alpha_i. \tag{10}$$

Remark. The approach is a quite general one. For instance, in [3] it is assumed that $\varphi = u^\alpha = \text{const}$, $\alpha = \text{fixed}$ (usually $\alpha = 2$). Then $\varphi_{,\sigma} = \delta^\alpha_\sigma$, $|\text{grad}\,\varphi| = \sqrt{a^{\alpha\alpha}}$, $m_\sigma = \dfrac{1}{\sqrt{a^{\alpha\alpha}}}\delta^\alpha_\sigma = \sqrt{\dfrac{a}{a^{\beta\beta}}}\delta^\alpha_\sigma$, (no sum over $\alpha, \beta \neq \alpha$). Then, from (8) we have

$$\frac{d\theta_\alpha}{ds} = k_{g_\alpha} + \varepsilon^{\sigma\tau}\frac{\delta m_\sigma}{\delta s}m_\tau = k_{g_\alpha} + \varepsilon^{\beta\alpha}\frac{1}{\sqrt{a}}\frac{\delta m_\beta}{\delta s}m_\alpha, \quad \text{(no sum over } \alpha, \beta).$$

But

$$\frac{\delta m_\beta}{\delta s} = \frac{dm_\beta}{ds} - m_\varrho\Gamma^\varrho_{\beta\delta}\frac{du^\delta}{ds} = -m_\varrho\Gamma^\varrho_{\beta\delta}\frac{du^\delta}{ds} = -\sqrt{\frac{a}{a_{\beta\beta}}}\Gamma^\alpha_{\beta\delta}\frac{du^\delta}{ds}$$

and hence

$$\frac{d\theta_\alpha}{ds} = k_{g_\alpha} - \varepsilon^{\beta\alpha}\frac{\sqrt{a}}{a_{\beta\beta}}\Gamma^\alpha_{\beta\delta}\frac{du^\delta}{ds}.$$

Particularly, for $\varphi = u^2 = \text{const}$, from (10) we obtain

$$\frac{d\theta_2}{ds} = k_{g_2} - \frac{\sqrt{a}}{a_{11}}\Gamma^2_{1\delta}\frac{du^\delta}{ds}.$$

If $\varphi = u^2 = \text{const}$ is geodesic, then $k_{g_2} = 0$ and

$$\frac{d\theta_2}{ds} + \frac{\sqrt{a}}{a_{11}}\Gamma^2_{1\delta}\frac{du^\delta}{ds} = 0,$$

see [**3**, (32.14)].

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(Received 20 01 2011)
(Revised 29 12 2011)