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FORMULATION AND ANALYSIS OF A PARABOLIC TRANSMISSION PROBLEM ON DISJOINT INTERVALS

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ABSTRACT. We investigate an initial-boundary-value problem for one dimensional parabolic equations in disjoint intervals. Under some natural assumptions on the input data we proved the well-posedness of the problem. Nonnegativity and energy stability of its weak solutions are also studied.

1. Introduction

It is well known that the transfer of energy or mass is fundamental for many biological, chemical, environmental and industrial processes. The basic transport mechanisms of such processes are diffusion (or dispersion) and bulkflow. Therefore, the corresponding flux has two components: a diffusive one and a convective one. Here we pay attention to diffusion in one-dimensional domain with layers. Layers with material properties which significantly differ from those of surrounding medium appear in variety of applications [**1, 2, 3, 4, 7, 8, 12, 14, 17**]. The layer may have structural role (as in the case of glue), a thermal role (as in the case of thin thermal insulator), an electromagnetic or optical role, etc., depending on the application. Traditionally there are two ways of handling such layers in the numerical modelling: either they are fully modelled or, they are totally ignored. We use a method proposed in [**7, 8**] of modelling of a thin layer as an *interface*. The effect of the layer on the solution is modelled by means of nonlocal *jump conditions* across the interface. In order to explain the method proposed in this paper for mathematical modelling of layer phenomena we consider in Section 2 a physical model. As a result the following initial-boundary-value problem (IBVP) arises: find functions $u_1(x,t),\ u_2(x,t)$ that satisfy the parabolic equations

$$\frac{\partial u_1}{\partial t}-\frac{\partial}{\partial x}\Big(p_1(x)\frac{\partial u_1}{\partial x}\Big)+q_1(x)u_1=f_1(x,t),\quad x\in\Omega_1\equiv(a,b),\quad t>0, \tag{1.1}$$

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$$\frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x}\Big(p_2(x)\frac{\partial u_2}{\partial x}\Big) + q_2(x)u_2 = f_2(x,t), \quad x \in \Omega_2 \equiv (c,d), \;\; t > 0, \tag{1.2}$$

where $-\infty < a < b < c < d < +\infty$, and the internal boundary (or nonlocal interface jump) conditions

$$p_1(b)\frac{\partial u_1(b,t)}{\partial x} + \alpha_1 u_1(b,t) = \beta_1 u_2(c,t) + \gamma_1(t), \tag{1.3}$$

$$-p_2(c)\frac{\partial u_2(c,t)}{\partial x} + \alpha_2 u_2(c,t) = \beta_2 u_1(b,t) + \gamma_2(t). \tag{1.4}$$

These two conditions have the form of Robin–Dirichlet mixed boundary conditions (see the review of Givoli [**7**], where Robin–Dirichlet conditions have been incorporated in a finite element formulation in order to eliminate an infinite domain, a singular domain, or a substructure from computational domain). Finally, in order to complete the IBVP we pose the simplest external boundary conditions

$$u_1(a,t) = 0, \quad u_2(d,t) = 0, \tag{1.5}$$

and initial conditions

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x). \tag{1.6}$$

Throughout the paper we assume that the data satisfy the usual regularity and ellipticity conditions

$$p_i(x),\; q_i(x) \in L_\infty(\Omega_i), \quad i = 1,2, \tag{1.7}$$

$$p_i(x) \geqslant p_{0i} > 0, \quad \text{a.e. in } \Omega_i, \quad i = 1,2, \tag{1.8}$$

$$\alpha_i > 0\,, \quad \beta_i > 0, \quad i = 1,2. \tag{1.9}$$

In some cases we shall require that

$$q_i(x) \geqslant 0, \quad \text{a.e. in } \Omega_i, \quad i = 1,2, \tag{1.10}$$

and/or

$$\beta_1\beta_2 \leqslant \alpha_1\alpha_2. \tag{1.11}$$

The aim of the present paper is to study the well-posedness of the IBVP (1.1)–(1.6) and some of its properties as nonnegativity and energy stability of the solutions. Finite difference schemes for approximation of (1.1)–(1.6) are considered in [**10, 11**].

An outline of the paper is as follows. In Section 2 we discuss a heat-mass transfer problem and show how in a natural way can arise problem (1.1)–(1.6). In Section 3 the well-posedness (existence, uniqueness and regularity of the solution) for problem (1.1)–(1.6) is studied. A specific spectral problem is used. Energy stability for solutions to problem (1.1)–(1.6) is also discussed. The nonnegativity of solutions to (1.1)–(1.11) is studied in Section 4.

2. A Motivated Heat-Mass Transfer Problem

We know that conductive heat transfer is nothing else but the movement of thermal energy through the corresponding medium (material) from the more energetic particles to the others [**3, 4, 17**]. Of course, the local temperature is given by the energy of the molecules situated at that place. So, thermal energy is transferred from points with higher temperature to other points. If the temperature of some area of the medium increases, then the random molecular motion becomes more intense in that area. Thus a transfer of thermal energy is produced, which is called *heat diffusion*. The corresponding conductive heat flux $Q(x,t)$, is given by the Fourier's rate law $Q = -pu_x = -p\frac{\partial u}{\partial x}$, where p is the thermal conductivity of the medium, and $u = u(x,t)$ is the temperature at point x at time instant t. We shall assume that p depends on x only. The first law of thermodynamics (conservation of energy) gives in a standard manner the transformation of the energy from one form (e.g. mechanical, electrical, etc.) into heat. The right hand side of the above equation represents the stored heat. In other words, the heat equation looks like

$$\varepsilon\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(p(x)\frac{\partial u}{\partial x}\Big) = f(x,t),$$

where $\varepsilon = \rho c$, ρ is the density, and c is the specific heat capacity. Note that the same partial differential equation is a model of mass transfer, but in this case $u(x,t)$ represents the mass density of the material at a point x, at time t. Instead of Fourier law, a similar law is available in this case, which is called Fick's law of mass diffusion. An additional term $q(x)u$ depending on the density may appear in the mass equation due to the reaction

$$\varepsilon\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(p(x)\frac{\partial u}{\partial x}\Big) + q(x)u = f(x,t). \tag{2.1}$$

For the case of an interface joining two media with thermal conductivities p_i ($i = 1, 2$) the heat flows Q_i (even temperatures u_i) have a discontinuity jump at the interface (a point in $1D$ case, a curve in $2D$ case and a surface in the $3D$ case). We shall now discuss a $1D$ model of two media of non-stationary transfer separated by a layer where the temperature field is almost stationary, i.e., $\partial u/\partial t \equiv 0$ or the quantity $\varepsilon = \rho c$ is very small. Then equation (2.1) in the layer takes the form:

$$-\frac{\partial}{\partial x}\Big(p_L(x)\frac{\partial u_L}{\partial x}\Big) + q_L(x)u_L = f_L(x,t). \tag{2.2}$$

Let us suppose that the heat (or mass) equations of media 1, 2 are given by (1.1), (1.2) respectively, and in the intermediate layer (b,c), by (2.2). On the two ends of the layer we have the continuity conditions

$$u_1(b,t) = u_L(b,t), \quad u_L(c,t) = u_2(c,t), \tag{2.3}$$

$$p_1(b)\frac{\partial u_1}{\partial x}(b,t) = p_L(b)\frac{\partial u_L}{\partial x}(b,t), \quad p_L(c)\frac{\partial u_L}{\partial x}(c,t) = p_2(c)\frac{\partial u_2}{\partial x}(c,t). \tag{2.4}$$

Conditions (2.3) enforce the continuity of the primary variable u (e.g., temperature), whereas conditions (2.4) require the continuity of the 'flux' Q. In the layer

(b, c) we solve equation (2.2) analytically. Its general solution is of the form

$$u_L(x,t) = C_1(t)v_1(x) + C_2(t)v_2(x) + w(x,t), \tag{2.5}$$

where v_1, v_2, w are known functions, and $C_1(t), C_2(t)$ are unknown functions. From (2.5) we can write

$$\begin{bmatrix} v_1(b) & v_2(b) \\ v_1(c) & v_2(c) \end{bmatrix} \cdot \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} u_L(b,t) - w(b,t) \\ u_L(c,t) - w(c,t) \end{bmatrix}.$$

By solving this set of equations we express $C_1(t)$ and $C_2(t)$ in terms of $u_L(b,t)$, $u_L(c,t)$ and $w(b,t)$, $w(c,t)$. Then, from (2.3)–(2.5) we obtain conditions (1.3) and (1.4), where

$$\alpha_1 = \frac{p_L(b)\Delta_1(c,b)}{\Delta(b,c)}, \quad \beta_1 = \frac{p_L(b)\Delta_1(b,b)}{\Delta(b,c)},$$

$$\alpha_2 = \frac{p_L(c)\Delta_1(b,c)}{\Delta(b,c)}, \quad \beta_2 = \frac{p_L(c)\Delta_1(c,c)}{\Delta(b,c)},$$

$$\gamma_1(t) = \frac{p_L(b)}{\Delta(b,c)}\Big(\Delta_1(c,b)\, w(b,t) - \Delta_1(b,b)\, w(c,t) + \Delta(b,c)\, \frac{\partial w}{\partial x}(b,t)\Big),$$

$$\gamma_2(t) = \frac{p_L(c)}{\Delta(b,c)}\Big(\Delta_1(c,c)\, w(b,t) - \Delta_1(b,c)\, w(c,t) + \Delta(b,c)\, \frac{\partial w}{\partial x}(c,t)\Big),$$

$$\Delta(b,c) = \begin{vmatrix} v_1(b) & v_2(b) \\ v_1(c) & v_2(c) \end{vmatrix} = v_1(b)v_2(c) - v_1(c)v_2(b),$$

$$\Delta_1(r,s) = \begin{vmatrix} v_1(r) & v_2(r) \\ \frac{dv_1}{dx}(s) & \frac{dv_2}{dx}(s) \end{vmatrix} = v_1(r)\frac{dv_2}{dx}(s) - v_2(r)\frac{dv_1}{dx}(s).$$

LEMMA 2.1. *For the constants α_i and β_i inequalities* (1.9) *hold.*

PROOF. The functions $v_1(x)$ and $v_2(x)$ are two linearly independent solutions of the corresponding homogeneous ordinary differential equation

$$Lv \equiv -\frac{d}{dx}\Big(p_L(x)\frac{dv}{dx}\Big) + q_L(x)v = 0, \quad x \in (b,c).$$

For example, we can choose $v_1(x)$ and $v_2(x)$ as the solutions of the Cauchy problems

$$Lv_1 = 0, \;\; b < x < c, \quad v_1(b) = 0, \;\; p_L(b)\frac{dv_1}{dx}(b) = 1,$$

$$Lv_2 = 0, \;\; b < x < c, \quad v_2(c) = 0, \;\; p_L(c)\frac{dv_2}{dx}(c) = -1.$$

Then maximum principle arguments [**6**] imply $v_1(c) > 0$, $\frac{dv_1}{dx}(b) > 0$, $\frac{dv_1}{dx}(c) > 0$, $v_2(b) > 0$, $\frac{dv_2}{dx}(b) < 0$ and $\frac{dv_2}{dx}(c) < 0$, wherefrom follows (1.9). □

REMARK 2.1. If one formally sets $\beta_1 = 0$ (or $\beta_2 = 0$) problem (1.1)–(1.6) can be decoupled into two independent initial-boundary-value problems. Indeed, in that case u_1 satisfies equation (1.1), boundary conditions of the Dirichlet and Robin type

$$u_1(a,t) = 0, \quad p_1(b)\frac{\partial u_1(b,t)}{\partial x} + \alpha_1 u_1(b,t) = \gamma_1(t),$$

and the initial condition $u_1(x,0) = u_{10}(x)$. Such problems are already well studied (see e.g., [**16, 18, 20**]), where, for the well-posedness, $\alpha_1 > 0$ is assumed. When u_1 is determined, one obtains an analogous mixed Dirichlet–Robin initial-boundary-value problem for u_2.

Remark 2.2. Introducing new unknown functions

$$U_1(x,t) = u_1(x,t) - A_1(t)(x-a)(x-b), \quad U_2(x,t) = u_2(x,t) - A_2(t)(x-c)(x-d),$$

where $A_1(t) = \gamma_1(t)/[(b-a)p_1(b)]$ and $A_2(t) = \gamma_2(t)/[(d-c)p_2(c)]$, one obtains an analogous initial-boundary-value problem with homogeneous internal boundary conditions. In such a way, without loss of generality, we can set $\gamma_1(t) = \gamma_2(t) = 0$ in (1.3)–(1.4).

3. Well-Posedness of Problem (1.1)–(1.6)

After the formulation and explanation of the physical meaning of problem (1.1)–(1.6), we come up to the following theoretical questions: (a) existence, uniqueness and qualitative properties (as smoothness, positivity, energy stability, etc.) of the solution; (b) construction of analytical and numerical methods for approximate solution of the problem. This paper is concerned with the first group of questions. We begin with the well-posedness, i.e., existence and uniqueness of solution in appropriate Sobolev spaces.

3.1. An Auxiliary Spectral Problem. The problem is to find the triple $[\lambda; v = (v_1, v_2)] \in R \times H^1(\Omega_1) \times H^1(\Omega_2)$ which satisfies the differential equations

$$L_1 v_1 \equiv -\frac{d}{dx}\Big(p_1(x)\frac{dv_1}{dx}\Big) + q_1(x)v_1(x) = \lambda v_1(x), \quad x \in \Omega_1, \tag{3.1}$$

$$L_2 v_2 \equiv -\frac{d}{dx}\Big(p_2(x)\frac{dv_2}{dx}\Big) + q_2(x)v_2(x) = \lambda v_2(x), \quad x \in \Omega_2, \tag{3.2}$$

with Dirichlet's classical boundary conditions

$$v_1(a) = 0, \quad v_2(d) = 0, \tag{3.3}$$

as well as the nonlocal Robin–Dirichlet boundary conditions

$$\begin{aligned} l_1(v_1, v_2) &\equiv p_1(b)\frac{dv_1}{dx}(b) + \alpha_1 v_1(b) - \beta_1 v_2(c) = 0, \\ l_2(v_1, v_2) &\equiv -p_1(c)\frac{dv_2}{dx}(c) + \alpha_2 v_2(c) - \beta_2 v_1(b) = 0. \end{aligned} \tag{3.4}$$

Assuming that conditions (1.9) hold, consider the product space $L = L_2(\Omega_1) \times L_2(\Omega_2)$, endowed with the inner product and associated norm

$$(u,v)_L = \sum_{i=1}^{2} \beta_{3-i}(u_i, v_i)_{L_2(\Omega_i)}, \quad \|v\|_L = (v,v)_L^{1/2},$$

where

$$(u_i, v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i \, dx, \quad i = 1, 2.$$

We can identify $v \in L$ with a scalar function in $\Omega = \Omega_1 \cup \Omega_2$, by $v : \Omega \to R$, $v|_{\Omega_i} = v_i$, $i = 1, 2$. We introduce the product space

$$H^1 = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_i) \text{ and } v_1(a) = 0,\ v_2(d) = 0\}$$

endowed with the inner product and associated norm

$$(u, v)_{H^1} = \sum_{i=1}^{2} \beta_{3-i} \Big[(u_i, v_i)_{L_2(\Omega_i)} + \Big(\frac{du_i}{dx}, \frac{dv_i}{dx} \Big)_{L_2(\Omega_i)} \Big], \quad \|u\|_{H^1} = (u, u)_{H^1}^{1/2}.$$

We have the following assertion.

Lemma 3.1. *Under the regularity conditions* (1.7)–(1.9)*, the spectral problem* (3.1)–(3.4) *is formally equivalent to the following variational problem: find* $[\lambda, v] \in R \times H^1$ *such that*

$$A(v, w) \equiv [v, w] + Z(v, w) = \lambda (v, w)_L, \quad \forall w \in H^1, \tag{3.5}$$

where

$$[u, v] = \sum_{i=1}^{2} \beta_{3-i} [u_i, v_i]_i, \quad [u_i, v_i]_i = \int_{\Omega_i} \Big(p_i \frac{du_i}{dx} \frac{dv_i}{dx} + q_i u_i v_i \Big) dx, \ \ i = 1, 2,$$

$$Z(v, w) = \beta_2 \alpha_1 v_1(b)\, w_1(b) + \beta_1 \alpha_2 v_2(c)\, w_2(c) - \beta_1 \beta_2 \big[v_1(b)\, w_2(c) + v_2(c)\, w_1(b) \big].$$

Proof. Consider the eigentriple $[\lambda; v_1, v_2]$ of (3.1)–(3.4). Multiplying (3.1) by $w_1 \in H^1(\Omega_1)$ such that $w_1(a) = 0$ and integrating by parts using the first condition in (3.4) we obtain the identity:

$$[v_1, w_1]_1 + \alpha_1 v_1(b)\, w_1(b) - \beta_1 v_2(c)\, w_1(b) = \lambda\, (v_1, w_1)_{L_2(\Omega_1)}.$$

Analogously, multiplying (3.2) by $w_2 \in H^1(\Omega_2)$ such that $w_2(d) = 0$ and integrating by parts we obtain:

$$[v_2, w_2]_2 + \alpha_2 v_2(c)\, w_2(c) - \beta_2 v_1(b)\, w_2(b) = \lambda\, (v_2, w_2)_{L_2(\Omega_2)}.$$

Now multiplying the first of these identities by β_2, the second by β_1 and summing up we get (3.5). Conversely, consider the eigenpair $[\lambda, v]$ of the variational spectral problem (3.5). Choosing successively test functions of the form $w = (\varphi_1, 0)$ and $w = (0, \varphi_2)$, $\varphi_i \in C_0^\infty(\Omega_i)$, and using the standard rules of differentiation for Schwartz distributions [**15, 22**] we recover differential equations (3.1) and (3.2). To recover boundary conditions (3.4) we choose the test functions $w = (w_1, 0) \in H^1$ and $w = (0, w_2) \in H^1$. $\square$

We state the following important properties of the spaces H^1 and L:

(i) H^1 and L are Hilbert spaces, (ii) H^1 is compactly embedded in L.

In the following lemma we deal with some properties of the bilinear form $A(u, v)$.

Lemma 3.2. [**11**] *Under condition* (1.7)*, the bilinear form* A*, defined by* (3.5)*, is symmetric and bounded on* $H^1 \times H^1$*. If conditions* (1.8)–(1.9) *also hold, the bilinear form* A *satisfies the Gårding inequality*

$$A(v, v) \geqslant c_1 \|v\|_{H^1}^2 - c_2 \|v\|_L^2, \quad \forall\, v \in H^1, \quad c_1, c_2 > 0.$$

Remark 3.1. If besides (1.7)–(1.9), conditions (1.10)–(1.11) are satisfied, the bilinear form A is coercive, i.e., there exists a constant $c_0 > 0$ such that

$$A(v, v) \geqslant c_0 \|v\|^2_{H^1}, \quad \forall\, v \in H^1.$$

Lemmas 2.1, 3.1, 3.2 and properties (i), (ii), allow us to recast problem (3.1)–(3.4) into the general theory of abstract eigenvalue problems for bilinear forms in Hilbert spaces, see e.g. [**15, 18, 20**]. This ensures the existence of exact eigenpairs as stated in the following theorem.

Theorem 3.1. *Under conditions* (1.7) *and* (1.8) *problem* (3.1)–(3.4) *has a countable sequence of real eigenvalues* $-c_2 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$. *The corresponding eigenfunctions* $v^k \equiv (v_1^k, v_2^k)$, $k = 1, 2\ldots$, *can be chosen to be orthonormal in* L. *They constitute a Hilbert basis for* H^1 *as well as for* L.

3.2. Existence and Uniqueness of Weak Solutions. Let us introduce the cylinders $Q_{iT} = \{(x, t) \mid x \in \Omega_i,\ 0 < t < T\}$, $i = 1, 2$.

Theorem 3.2. *Assume that* $f = (f_1, f_2) \in L_2(0, T; L)$, $u_0 = (u_{10}, u_{20}) \in L$ *and* $\gamma_1(t) = \gamma_2(t) = 0$. *Then problem* (1.1)–(1.9) *has a unique solution* $u = (u_1, u_2) \in H^{1,0} \equiv L_2(0, T; H^1)$ *which satisfies the following weak formulation:*

$$\begin{aligned} -\beta_2 \int_{Q_{1T}} u_1 \frac{\partial v_1}{\partial t} dx\, dt - \beta_1 \int_{Q_{2T}} u_2 \frac{\partial v_2}{\partial t} dx\, dt + \int_0^T A(u(\cdot, t), v(\cdot, t))\, dt \\ = \beta_2 \int_{\Omega_1} u_{10}(x) v_1(x, 0)\, dx + \beta_1 \int_{\Omega_2} u_{20}(x) v_2(x, 0)\, dx \\ + \beta_2 \int_{Q_{1T}} f_1(x, t) v_1 dx\, dt + \beta_1 \int_{Q_{2T}} f_2(x, t) v_2 dx\, dt, \end{aligned}$$

$$\forall v = (v_1, v_2) \in H^{1,1} \equiv L_2(0, T; H^1) \cap H^1(0, T; L), \ \ v_i(x, T) = 0 \ \ \textit{a.e. in}\ \Omega_i.$$

Proof. The existence proof, for example, can be accomplished by the Fourier method [**15, 18, 20**]. We first construct a family of approximate solutions, using the spectral problem (3.1)–(3.4). Since $u_{i0} \in L_2(\Omega_i)$, and $f_i(x, t) \in L_2(\Omega_i)$, $i = 1, 2$, for almost all $t \in (0, T)$ we have

$$u_{i0}(x) = \sum_{k=1}^{\infty} u_{i0}^k v_i^k(x), \quad f_i(x, t) = \sum_{k=1}^{\infty} f_i^k(t) v_i^k(x), \quad i = 1, 2,$$

where $u_{i0}^k = (u_{i0}, v_i^k)_{L_2(\Omega_i)}$, $f_i^k(t) = (f_i(\cdot, t), v_i^k)_{L_2(\Omega_i)}$ and $f_i^k(t) \in L_2(0, T)$. Let us consider for each $k = 1, 2, \ldots$ the functions

$$U_i^k(t) = u_{i0}^k e^{-\lambda_k t} + \int_0^t f_i^k(\tau) e^{\lambda_k (t-\tau)} d\tau, \quad i = 1, 2,$$

which satisfy almost everywhere on $(0, T)$ the equations

$$\frac{dU_i^k}{dt} + \lambda_k U^k = f_i^k(t), \quad U_i^k(0) = u_{i0}^k, \quad i = 1, 2.$$

The pair of sums $S^K \equiv (S_1^K, S_2^K)$, $S_i^K(x, t) = \sum_{k=1}^K U_i^k(t) v_i^k(x)$, $i = 1, 2$, is a weak solution of the problem (1.1)–(1.9) with initial functions $\sum_{k=1}^K u_{i0}^k v_i^k(x)$ and right hand sides $\sum_{k=1}^{K} f_i^k v_i^k(x)$, $i = 1, 2$. Further, following the known techniques [**15, 18, 20**], we show that the series S^K is convergent in $H^{1,0}$.

Let $u^1(x,t) \equiv (u_1^1(x,t), u_2^1(x,t))$ and $u^2(x,t) \equiv (u_1^2(x,t), u_2^2(x,t))$ be two generalized solutions of problem (1.1)–(1.9). Then $u = u^1 - u^2$ is a generalized solution of the corresponding homogeneous problem. Let introduce the functions

$$v_i(x,t) = \begin{cases} \int_t^l u_i(x,\tau)\, d\tau, & x \in \Omega_i,\ t \in (0,l), \\ 0, & x \in \Omega_i,\ t \in (l,T), \end{cases}$$

where $i = 1, 2$ and $0 < \varepsilon \leqslant l \leqslant T$. It is easy to check that the functions v_i have generalized derivatives

$$\frac{\partial v_i}{\partial t} = \begin{cases} -u_i, & x \in \Omega_i,\ t \in (0,l), \\ 0, & x \in \Omega_i,\ t \in (l,T), \end{cases}$$

$$\frac{\partial v_i}{\partial x} = \begin{cases} \int_t^l \frac{\partial u_i}{\partial x}(x,\tau)\, d\tau, & x \in \Omega_i,\ t \in (0,l), \\ 0, & x \in \Omega_i,\ t \in (l,T), \end{cases}$$

$v = (v_1, v_2) \in H^{1,1}$, $v_i(x,T) = 0$, $i = 1, 2$, and $v_1(a,t) = 0$, $v_2(d,t) = 0$, $t \in [0,T]$. Plugging these formulas in the identity of Theorem 3.2, that defines the solution u, and assuming that $u_i(x,t) = 0$ for $x \in \Omega_i$, $t \in (0, l-\varepsilon)$we get the equality

$$\int_{l-\varepsilon}^{l} \|u(\cdot,t)\|_L^2 dt + \int_{l-\varepsilon}^{l} A(u(\cdot,t), v(\cdot,t))\, dt = 0. \tag{3.6}$$

Following the technique from [**18**, p. 395], and using Lemma 3.2 we show that

$$\begin{aligned} \int_{l-\varepsilon}^{l} A(u(\cdot,t), v(\cdot,t))\, dt &= -\frac{1}{2} \int_{l-\varepsilon}^{l} \frac{d}{dt}\Big(A(v(\cdot,t), v(\cdot,t))\Big) dt \\ &= \frac{1}{2} A(v(\cdot,l-\varepsilon), v(\cdot,l-\varepsilon)) \geqslant \frac{c_1}{2} \|v(\cdot,l-\varepsilon)\|_{H^1}^2 - \frac{c_2}{2} \|v(\cdot,l-\varepsilon)\|_L^2. \end{aligned} \tag{3.7}$$

Further, we have

$$\begin{aligned} \|v(\cdot,l-\varepsilon)\|_L^2 &= \sum_{i=1}^{2} \beta_{3-i} \int_{\Omega_i} \left(\int_{l-\varepsilon}^{l} u_i(x,t)\, dt \right)^2 dx \\ &\leqslant \varepsilon \sum_{i=1}^{2} \beta_{3-i} \int_{\Omega_i} \int_{l-\varepsilon}^{l} u_i^2(x,t)\, dt\, dx = \varepsilon \int_{l-\varepsilon}^{l} \|u(\cdot,t)\|_L^2\, dt. \end{aligned} \tag{3.8}$$

Choosing $\varepsilon = T/N < 2/c_2$ from (3.6)–(3.8) follows $\int_{l-\varepsilon}^{l} \|u(\cdot,t)\|_L^2\, dt \leqslant 0$, which imply that $u_i = 0$ a.e. in $\Omega_i \times (l-\varepsilon, l)$, $i = 1, 2$.

Repeating this procedure for $l = \varepsilon,\ 2\varepsilon, \dots,\ N\varepsilon = T$ we obtain that $u_i = 0$ a.e. in Q_{iT}, $i = 1, 2$, wherefrom follows $u = 0$ and $u^1 = u^2$. □

3.3. A Priori Estimates and Energy Stability of the Solutions. Let $H^{-1} = (H^1)^*$ be the dual space for H^1. The spaces H^1, L and H^{-1} form a Gelfand triple $H^1 \subset L \subset H^{-1}$ [**22**], with continuous and dense embeddings. We also introduce the space $W = \{u \mid u \in L_2(0,T;H^1),\ \frac{\partial u}{\partial t} = \big(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\big) \in L_2(0,T;H^{-1})\}$ with the inner product and associated norm

$$(u,v)_W = \int_0^T \Big[(u(\cdot,t),\, v(\cdot,t))_{H^1} + \Big(\frac{\partial u}{\partial t}(\cdot,t),\, \frac{\partial v}{\partial t}(\cdot,t)\Big)_{H^{-1}}\Big]dt, \quad \|u\|_W = (u,u)_W^{1/2}.$$

Let us consider the following weak form of (1.1)–(1.5):

$$\Big\langle \frac{\partial u}{\partial t}(\cdot,t),\, v\Big\rangle + A(u(\cdot,t),\, v) = \langle f(\cdot,t),\, v\rangle, \quad \forall v \in H^1, \tag{3.9}$$

where $\langle\cdot,\cdot\rangle$ denotes duality pairing in $H^{-1}\times H^1$.

Problem (3.9) fit into the general theory of abstract parabolic problems [**22**]. Applying Theorem 26.1 from [**22**] to (3.9) we immediately obtain the following assertion.

THEOREM 3.3. *Let the assumptions* (1.7)–(1.9) *hold and suppose that* $u_0 = (u_{10}, u_{20}) \in L$, $f = (f_1, f_2) \in L_2(0,T;H^{-1})$, $\gamma_1(t) = \gamma_2(t) = 0$. *Then the problem* (1.1)–(1.6) *has a unique weak solution* $u \in W$, *and Hadamard's a priori estimate holds:*

$$\|u\|_W^2 \leqslant C(T)\Big(\|u_0\|_L^2 + \int_0^T \|f(\cdot,t)\|_{H^{-1}}^2 dt\Big), \quad C(T) = Ce^{2c_2T}.$$

THEOREM 3.4. *Let the assumptions* (1.7)–(1.11) *hold and suppose that* $u_0 = (u_{10}, u_{20}) \in L$, $f = (f_1, f_2) \in L_2(0,T;L)$, $\gamma_1(t) = \gamma_2(t) = 0$. *Then the solution of problem* (1.1)–(1.6) *satisfies the a priori estimate:*

$$\|u(\cdot,t)\|_L^2 \leqslant e^{-2\delta t}\Big(\|u_0\|_L^2 + C\int_0^t e^{2\delta\tau}\|f(\cdot,\tau)\|_L^2\, d\tau\Big), \quad \delta > 0.$$

PROOF. Setting in (3.9) $v = u$ and using Theorem 3.1, and the Cauchy–Schwarz inequality with $\varepsilon > 0$ one obtains

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\|u(\cdot,t)\|_L^2\Big) + \lambda_1\,\|u(\cdot,t)\|_L^2 \leqslant \varepsilon\,\|u(\cdot,t)\|_L^2 + \frac{1}{4\varepsilon}\,\|f(\cdot,t)\|_L^2,$$

where $\lambda_1 > 0$ is the minimal eigenvalue of the problem (3.1)–(3.4). For $\varepsilon < \lambda_1$, result with $\delta = \lambda_1 - \varepsilon$ and $C = 1/(2\varepsilon)$ follows by integration. □

REMARK 3.2. Functional $\|u(\cdot,t)\|_L^2$ express the kinetic energy of the system. Therefore, results of such type are usually referred to as energy stability (cf. [**9**]).

REMARK 3.3. The minimal eigenvalue of problem (3.1)–(3.4) may be positive even if (1.10)–(1.11) is not satisfied. Using Lemma 3.2 and the Poincaré inequality one obtains

$$A(v,v) \geqslant c_3\,\|v\|_L^2, \quad c_3 = \frac{2c_1}{\max\{(b-a)^2,\,(d-c)^2\}} + c_1 - c_2.$$

In such a way, instead of (1.10)–(1.11) we can require that $c_3 > 0$.

4. Nonnegativity Preservation

Considering the heat conduction problem, the temperature of a body can not be negative if the temperature is nonnegative in the initial state and on the boundary of the body. This property is called nonnegativity preservation. In this section we analyze the nonnegativity preservation for the problem (1.1)–(1.11).

For a function $v_i \in H^1(\Omega_i)$, $i = 1, 2$, we denote its positive and negative parts by v_i^+ and v_i^-. That is $v_i = v_i^+ + v_i^-$, where $v_i^+ = \max\{v_i, 0\} \geqslant 0$ and $v_i^- = \min\{v_i, 0\} \leqslant 0$. By [**6**, Lemma 7.6, p. 150], we have that

$$\frac{dv_i^+}{dx} = \begin{cases} \frac{dv_i}{dx}, & \text{if } v \geqslant 0, \\ 0, & \text{if } v \leqslant 0, \end{cases} \quad \frac{dv_i^-}{dx} = \begin{cases} \frac{dv_i}{dx}, & \text{if } v \leqslant 0, \\ 0, & \text{if } v \geqslant 0. \end{cases}$$

It follows that

$$v_i^+ v_i^- = \frac{dv_i^+}{dx}\frac{dv_i^-}{dx} = v_i^- \frac{dv_i^+}{dx} + v_i^+ \frac{dv_i^-}{dx} = 0 \quad \text{a.e. in } \Omega_i, \quad i = 1,\, 2.$$

For some $T > 0$ let us consider the following inequalities

$$\frac{\partial u_i}{\partial t} + L_i u_i \geqslant 0 \quad \text{in } Q_{iT}, \quad i = 1, 2, \tag{4.1}$$

$$l_1(u_1, u_2) \geqslant 0, \quad l_2(u_1, u_2) \geqslant 0, \quad 0 < t < T. \tag{4.2}$$

Let V (see [**16**, p. 6]) be the Banach space consisting of all functions in $H^{1,0}$ having the finite norm

$$\|u\|_V = \max_i \bigg(\sup_{0 \leqslant t \leqslant T} \|u_i(\cdot, t)\|_{L_2(\Omega_i)} + \Big\|\frac{\partial u_i}{\partial x}\Big\|_{L_2(Q_{iT})} \bigg).$$

A function $u = (u_1, u_2) \in V$ is said to satisfy weakly (4.1) and (4.2) if for any $v = (v_1, v_2) \in H^{1,1}$, $v_i \geqslant 0$, $i = 1, 2$:

$$\begin{aligned} &\beta_2 \int_a^b u_1(x,t) v_1(x,t)\,dx + \beta_1 \int_c^d u_2(x,t) v_2(x,t)\,dx \\ &-\beta_2 \int_a^b u_1(x,0) v_1(x,0)\,dx - \beta_1 \int_c^d u_2(x,0) v_2(x,0)\,dx \\ &-\beta_2 \int_0^t \int_a^b u_1(x,\tau) \frac{\partial v_1}{\partial t}(x,\tau)\,dx\,d\tau - \beta_1 \int_0^t \int_c^d u_2(x,\tau) \frac{\partial v_2}{\partial t}(x,\tau)\,dx\,d\tau \\ &\qquad + \int_0^t A(u(\cdot,\tau), v(\cdot,\tau))\,d\tau \geqslant 0 \end{aligned} \tag{4.3}$$

for almost every $t \in (0, T)$. It is easy to see that if $u_i \in C^{2,1}(\overline{Q}_{iT})$, $i = 1, 2$, satisfy (4.1), (4.2) pointwise then (4.3) holds. Conversely, if $u \in V$ satisfy (4.3) and u_i are sufficiently smooth, then they also satisfy (4.1), (4.2) pointwise. We shall now prove the following theorem.

Theorem 4.1. *Let $u = (u_1, u_2) \in V$ satisfy* (4.1) *and* (4.2) *weakly, and let* (1.7)–(1.11) *hold. If $u_i(x, 0) \geqslant 0$, then $u_i(x, t) \geqslant 0$ a.e. in Q_{iT}, $i = 1, 2$.*

PROOF. Using the Steklov average and passing to the limit (see [**6**, Ch. 3]) we can formally take $v = -u^- \geqslant 0$ in (4.3) to obtain

$$
\begin{aligned}
&\frac{\beta_2}{2}\int_a^b (u_1^-(x,t))^2dx + \frac{\beta_1}{2}\int_c^d (u_2^-(x,t))^2dx - \frac{\beta_2}{2}\int_a^b (u_1^-(x,0))^2dx \\
&-\frac{\beta_1}{2}\int_c^d (u_2^-(x,0))^2dx + \beta_2\int_0^t\int_a^b \left(p_1(x)\frac{\partial u_1}{\partial x}\frac{\partial u_1^-}{\partial x} + q_1(x)u_1u_1^-\right)dx\,dt \\
&\qquad + \beta_1\int_0^t\int_c^d \left(p_2(x)\frac{\partial u_2}{\partial x}\frac{\partial u_2^-}{\partial x} + q_2(x)u_2u_2^-\right)dx\,dt \\
&\leqslant \int_0^t \Big\{ -\beta_2\alpha_1u_1(b,\tau)u_1^-(b,\tau) - \beta_1\alpha_2u_2(c,\tau)u_2^-(c,\tau) \\
&\qquad + \beta_1\beta_2\big[u_1(b,\tau)u_2^-(c,\tau) + u_2(c,\tau)u_1^-(b,\tau)\big]\Big\}\,d\tau
\end{aligned}
\tag{4.4}
$$

Further

$$
\frac{\partial u_i}{\partial x}\frac{\partial u_i^-}{\partial x} = \left(\frac{\partial u_i^-}{\partial x}\right)^2, \quad (u_iu_i^-) = (u_i^-)^2,
$$

$$
\begin{aligned}
u_1(b,t)u_2^-(c,t) + u_2(c,t)u_1^-(b,t) &= 2u_1^-(b,t)u_2^-(c,t) + u_1^+(b,t)u_2^-(c,t) \\
&\quad + u_1^-(b,t)u_2^+(c,t) \leqslant 2u_1^-(b,t)u_2^-(c,t).
\end{aligned}
$$

From here and (4.4), since by our assumption $u_i^-(x,0) = 0$, $i = 1,2$, we obtain:

$$
\begin{aligned}
&\frac{\beta_2}{2}\int_a^b (u_1^-(x,t))^2dx + \beta_2\int_0^t\int_a^b \left(p_1(x)\Big(\frac{\partial u_1^-}{\partial x}\Big)^2 + q_1(x)(u_1^-)^2\right)dx\,d\tau \\
&+\frac{\beta_1}{2}\int_c^d (u_2^-(x,t))^2dx + \beta_1\int_0^t\int_c^d \left(p_2(x)\Big(\frac{\partial u_2^-}{\partial x}\Big)^2 + q_2(x)(u_2^-)^2\right)dx\,d\tau \\
&\leqslant -\int_0^t \Big[\beta_2\alpha_1(u_1^-(b,\tau))^2 + \beta_1\alpha_2(u_2^-(c,\tau))^2 - 2\beta_1\beta_2u_1^-(b,\tau)u_2^-(c,\tau)\Big]d\tau \leqslant 0.
\end{aligned}
$$

For the last inequality we used (1.9) and (1.11). From here it follows $u_i^-(x,t) = 0$. We then conclude that $u_i(x,t) = u_i^+(x,t) \geqslant 0$, a.e. in Q_{iT}, $i = 1,2$. □

5. Concluding Remarks

We have proposed a simple parabolic problem on disjoint intervals to model layers with material properties which significantly differ from those of the surrounding medium. We assumed that the differential equation that describe the process in the layer is monotone, i.e., it satisfies the maximum principle which implies the positivity conditions (1.9). Under some additional natural assumptions on the input data we proved well-posedness of the IBVP that describes the process outside the layer as well as nonnegativity and energy stability of the solutions of the problem. The results obtained in this paper should be a good base for analysis of numerical approximations of the problems. The preservation of characteristic properties of different phenomena is a very important requirement in the construction of reliable numerical methods [**5**]. The nonnegativity preservation of finite element solutions to 1D elliptic problem on disjoint domains is discussed in [**13**]. A further development of the present modelling of layer problems should include cases when the differential operator corresponding to the process in the layer is nonmonotonous and conditions (1.9) fail. The process in the intermediate layer is considered to be stationary (Section 2) and as a result of this the derived nonlocal boundary conditions are of the Robin–Dirichlet type [**7, 8**]. But when the process in the intermediate layer is nonstationary, the arising boundary conditions will be nonlocal in time and it will be interesting to study this new problem for well-posedness and qualitative behavior of the solutions. Nonlinear 1D elliptic problems on disjoint domains are studied in [**21**]. Using the exact solutions of heat mass transfer equations [**19**] the results from the present paper could be extended to nonlinear parabolic problems.

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