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COMPLEXES OF DIRECTED TREES OF COMPLETE MULTIPARTITE GRAPHS

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Abstract. For every directed graph D we consider the complex of all directed subforests $\Delta(D)$. The investigation of these complexes was started by D. Kozlov. We generalize a result of Kozlov and prove that complexes of directed trees of complete multipartite graphs are shellable. We determine the h-vector of $\Delta(\overrightarrow{K}_{m,n})$ and the homotopy type of $\Delta(\overrightarrow{K}_{n_1,n_2,\dots,n_k})$.

1. Introduction

A *directed tree* is a tree in which one vertex is selected as the *root* and all edges are oriented away from the root. If $T = (V(T), E(T))$ is a directed tree with root r, then for every $x \in V(T)$ there exists a unique directed path from r to x. We say that a vertex y is *below* vertex x in a directed tree T if there exists a unique directed path from x to y. A *directed forest* is a family of disjoint directed trees. In this paper we write $\overrightarrow{xy}$ for a directed edge from x to y.

An *abstract simplicial complex* Δ is a collection of finite nonempty sets such that $A \subseteq B \in \Delta \Rightarrow A \in \Delta$. The element A of Δ is called a *face* (*simplex*) of Δ and its dimension is $|A| - 1$. The *vertex set* of Δ is the union of all faces of Δ. The *dimension* of the complex Δ is defined as the largest dimension of any of its faces. A *facet* of Δ is any simplex that is not a face of any larger simplex of Δ. A simplicial complex is *pure* if every of its facets has the same dimension. We denote the number of i-dimensional faces of Δ by f_i, and $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1}, f_d)$ is called the *f-vector*. A new invariant, the *h-vector* of a d-dimensional complex Δ is $h(\Delta) = (h_0, h_1, \dots, h_d, h_{d+1})$ defined by the formula

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.$$

We refer the reader to [**8**] for definitions of topological concepts used in this paper.

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Definition 1.1. Let D be a directed graph. The vertices of the *complex of directed trees* $\Delta(D)$ are oriented edges of D. The faces of $\Delta(D)$ are all directed forests that are subgraphs of D.

The question of shellability of complexes of directed trees was posed by R. Stanley. Kozlov in [**6**] showed that the existence of a complete source in a directed graph provides a shelling of its complex of directed trees. The complex of directed trees of a graph G is recognized in [**3**] as a discrete Morse complex of a 1-dimensional complex. These complexes are also studied in [**4**] and [**7**].

Geometrically, a shelling of a cell complex is a way of gluing it together from its maximal cells in a well-behaved way. In this paper we use the following definition of shellability for pure simplicial complexes.

Definition 1.2. A pure simplicial complex Δ is *shellable* if there exists a linear ordering (*shelling order*) $F_1, F_2, \dots, F_k$ of facets of Δ such that for every $i < j \leqslant k$ there exist some $l < j$ and a vertex v of F_j such that

$$F_i \cap F_j \subseteq F_l \cap F_j = F_j \setminus \{v\}.$$

For a fixed shelling order $F_1, F_2, \dots, F_k$ of Δ, the *restriction* $\mathcal{R}(F_j)$ of the facet F_j is defined by $\mathcal{R}(F_j) = \{v \text{ is a vertex of } F_j : F_j \setminus \{v\} \subset F_i \text{ for some } 1 \leqslant i < j\}$. The *type* of the facet F in the given shelling order is the cardinality of $\mathcal{R}(F)$, i.e., $\mathrm{type}(F) = |\mathcal{R}(F)|$. If we build up Δ according to a shelling order, then $\mathcal{R}(F)$ is the unique minimal new face that appears when we add the facet F. For a shellable simplicial complex we have the following combinatorial interpretation of its h-vector: $h_k(\Delta) = |\{F \text{ is a facet of } \Delta : \mathrm{type}(F) = k\}|$. Further, we know that a shellable d-dimensional simpilicial complex Δ is homotopy equivalent to a wedge of h_d spheres of dimension d. A set of maximal simplices $\mathcal{G}$ of Δ is a *generating set* of simplices if the removal of interiors of all simplices from $\mathcal{G}$ makes Δ contractible. For a shellable simplicial complex Δ the set of simplices $\mathcal{G} = \{F \text{ is a facet of } \Delta : \mathcal{R}(F) = F\}$ is a generating set of Δ, i.e., the simplicial complex $\Delta \setminus \left(\bigcup_{F \in \mathcal{G}} F\right)$ is contractible.

For more information on shellability see [**1**], [**2**] and chapter 8 of [**9**].

2. Graphs with a dominant pair

Kozlov in [**6**] used the following variant of a shelling. Let $\mathcal{F}(\Gamma)$ denote the set of all facets of a pure simplicial complex Γ. Assume that we can partition $\mathcal{F}(\Gamma)$ into the blocks $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ such that the following holds:

$$\begin{gathered} |\mathcal{F}_0| = 1; \text{ for all } i \leqslant j \text{ and two different facets } F \in \mathcal{F}_i, F' \in \mathcal{F}_j, \\ \text{there exists } k < j, \text{ a facet } F'' \in \mathcal{F}_k, \text{ and a vertex } v \in F' \\ \text{such that } F \cap F' \subseteq F'' \cap F' = F' \setminus \{v\}. \end{gathered} \tag{2.1}$$

It is easy to check that any linear order that refines partition $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ (for $i < j$ we list facets from $\mathcal{F}_i$ before facets from $\mathcal{F}_j$) is a shelling order of Γ in the sense of Definition 1.2.

A vertex c is a *complete source* in a digraph D if $\overrightarrow{cx} \in E(D)$ for all $x \in V(D) \setminus \{c\}$. The partition of $\mathcal{F}(\Delta(D))$ defined by Kozlov in [**6**] substantially uses the out-degree $d_T(c) = |\{x : \overrightarrow{cx} \in E(T)\}|$ of c:

$$\mathcal{F}_i = \{T \in \mathcal{F}(\Delta(D)) : d_T(c) - 1 - i\}.$$

It is not complicated to prove for a digraph with a complete source, that the above partition of the facets of $\Delta(D)$ satisfies the condition described in (2.1) (see the proof of Theorem 3.1 in [**6**]).

We now describe a broader family of graphs whose complexes of directed trees are shellable. For a directed graph D and a vertex $u \in V(D)$ we set $N(u) = \{x \in V(D) : \overrightarrow{ux} \in E(D)\}$. We say that a directed graph D has a *dominant pair of vertices* if there exist vertices $u, v \in V(D)$ such that

(i) $V(D) = N(u) \cup N(v)$. Therefore, we have that $\overrightarrow{uv}, \overrightarrow{vu} \in E(D)$.
(ii) For all $x \in N(u) \smallsetminus N(v)$ and $y \in N(v) \smallsetminus N(u)$ we have that $\overrightarrow{xy}, \overrightarrow{yx} \in E(D)$.

Theorem 2.1. *If a directed graph D has a dominant pair of vertices the complex $\Delta(D)$ is shellable.*

Proof. We will define a partition of the facets of $\Delta(D)$ and show that this partition satisfies (2.1). Recall that facets of the complex $\Delta(D)$ correspond to subtrees of D.

Let D be a graph with a dominant pair of vertices u, v. For a directed tree T with the root r let $h_T(x)$ denotes the length of the unique directed path from r to x. We classify directed trees of D by using $d_T(u)$, $d_T(v)$, $h_T(u)$ and $h_T(v)$. The trees of D in which the above defined parameters are the same form a block

$$\mathcal{F}_{p,q,r,s} = \{T : d_T(u) = p, d_T(v) = q, h_T(u) = r, h_T(v) = s\}$$

of our partition of the facets of $\Delta(D)$. We say that $\mathcal{F}_{p,q,r,s}$ is before $\mathcal{F}_{p',q',r',s'}$, and write $\mathcal{F}_{p,q,r,s} < \mathcal{F}_{p',q',r',s'}$, if and only if $p > p'$, or $p = p'$ and $q > q'$, or $p' = p$, $q = q'$ and $r < r'$, or $p' = p$, $q = q'$, $r = r'$ and $s < s'$. Note that the first block in this partition $\mathcal{F}_{|N(u)|,|N(v)\smallsetminus N(u)|-1,0,1}$ contains only the tree with the edge set $\{\overrightarrow{ux} : x \in N(u)\} \cup \{\overrightarrow{vy} : y \in N(v) \smallsetminus (N(u) \cup \{u\})\}$.

Now, we consider two different directed trees $T \in \mathcal{F}_{p,q,r,s}, T' \in \mathcal{F}_{p',q',r',s'}$ such that $\mathcal{F}_{p,q,r,s} \leqslant \mathcal{F}_{p',q',r',s'}$. Assume that the edges $E(T) \cap E(T')$ span a directed forest $F = T_1 \cup T_2 \cup \cdots \cup T_m$, and let r_i denote the root of T_i. Assume that r_{i_0} is the root of T'. Note that $E(T') \smallsetminus E(T)$ contains $m-1$ edges of the form $\overrightarrow{xr_i}$, where $i \neq i_0$.

The following analysis will show that there is a tree T'' such that T, T' and T'' satisfy the conditions described in (2.1). First, we consider the case when u, v are in the same tree of F (w.l.o.g. we assume $v, u \in T_1$).

(1) If the root of T' is a vertex $r_i \neq r_1$, then there exists $\overrightarrow{xr_1} \in E(T') \smallsetminus E(T)$ such that $x \neq u$ and $x \neq v$. We set $T'' = T' \smallsetminus \{\overrightarrow{xr_1}\} \cup \{\overrightarrow{ur_i}\}$ (if $r_i \in N(u)$) or $T'' = T' \smallsetminus \{\overrightarrow{xr_1}\} \cup \{\overrightarrow{vr_i}\}$ (if $r_i \notin N(u)$).
(2) Assume that r_1 is the root of T'. If there is a vertex $r_j \in N(u)$ for some $j > 1$, the assumption $d_T(u) \geqslant d_{T'}(u)$ guarantees that there exists an edge $\overrightarrow{xr_i} \in E(T') \smallsetminus E(T)$ such that $x \neq u$, $i > 1$ and $r_i \in N(u)$. Otherwise, if all of the edges $\overrightarrow{ur_i}$ (for all $r_i \in N(u)$, $i > 1$) are contained in $E(T')$, then we obtain that $d_T(u) < d_{T'}(u)$. In the above described situation we set

$T'' = T' \setminus \{\overrightarrow{xr_i}\} \cup \{\overrightarrow{ur_i}\}$.

If $r_i \in N(v) \setminus N(u)$ for all $i = 2, 3, \ldots, m$, then there exists $\overrightarrow{yr_i} \in E(T') \setminus E(T)$, such that $y \neq v$ (otherwise we obtain that $d_T(u) = d_{T'}(u)$ and $d_T(v) < d_{T'}(v)$). Then we set $T'' = T' \setminus \{\overrightarrow{yr_i}\} \cup \{\overrightarrow{vr_i}\}$.

Now, we consider the situation when the vertices u and v belong to different trees of F (w.l.o.g. we assume that $u \in T_1$ and $v \in T_2$).

(3) If the root of T' is r_i, $r_i \neq r_1$ and $r_i \in N(u)$, then there exists $\overrightarrow{xr_1} \in E(T') \setminus E(T)$ such that $x \neq u$ and we set $T'' = T' \setminus \{\overrightarrow{xr_1}\} \cup \{\overrightarrow{ur_i}\}$.

(4) If the root r_i of T' (again $r_i \neq r_1$) is not contained in $N(u)$ and $r_i \neq r_2$, then there exists $\overrightarrow{xr_2} \in E(T') \setminus E(T)$. If $x \neq u$ we set $T'' = T' \setminus \{\overrightarrow{xr_2}\} \cup \{\overrightarrow{vr_i}\}$.

If $x = u$, then v is below u and there exists $\overrightarrow{yr_1} \in E(T')$ such that $y \neq u$ and $y \neq v$. In that case we set $T'' = T' \setminus \{\overrightarrow{yr_1}\} \cup \{\overrightarrow{vr_i}\}$.

(5) If r_2 is the root of T' (recall that $r_2 \in N(v) \setminus N(u)$ and therefore $r_2 \neq v$), then there exists an edge $\overrightarrow{xr_1} \in E(T') \setminus E(T)$. If $r_1 \in N(v)$ and $x \neq v$, we set $T'' = T' \setminus \{\overrightarrow{xr_1}\} \cup \{\overrightarrow{vr_1}\}$.

If $x = v$ (and therefore $r_1 \in N(v)$), then we find an edge $\overrightarrow{yr_i} \in E(T') \setminus E(T)$ such that $y \neq u$, $i > 2$ and $r_i \in N(u)$ (or $\overrightarrow{zr_j} \in E(T') \setminus E(T)$ such that $z \neq v$, $r_j \in N(v) \setminus N(u)$, $j > 2$) by using the same arguments as in the proof of (2). Then we set $T'' = T' \setminus \{\overrightarrow{yr_i}\} \cup \{\overrightarrow{ur_i}\}$ or $T'' = T' \setminus \{\overrightarrow{zr_j}\} \cup \{\overrightarrow{vr_j}\}$.

If $r_1 \notin N(v)$, then there exists $\overrightarrow{xr_1} \in E(T') \setminus E(T)$, $x \neq u$, $x \neq v$ and we set $T'' = T' \setminus \{\overrightarrow{xr_1}\} \cup \{\overrightarrow{r_1r_2}\}$. In that case we obtain that $d_{T''}(u) = d_{T'}(u), d_{T''}(v) = d_{T'}(v)$ and $h_{T''}(u) < h_{T'}(u)$.

If the root of T' is r_1, then we have the following possibilities.

(6) There exists $i > 1$ such that $r_i \in N(u)$. Because we have that $d_T(u) \geqslant d_{T'}(u)$ it follows that there exists $\overrightarrow{xr_j} \in E(T') \setminus E(T)$ such that $x \neq u$, $r_j \in N(u)$ and $j > 1$. In that case we set $T'' = T' \setminus \{\overrightarrow{xr_j}\} \cup \{\overrightarrow{ur_j}\}$.

(7) If all vertices r_i for $i = 2, 3, \ldots, m$ are contained in $N(v) \setminus N(u)$ and $r_1 \in N(v)$ (recall that r_1 is the root of T'), then there exists $\overrightarrow{xr_2} \in E(T') \setminus E(T)$, $x \neq u$ and we set $T'' = T' \setminus \{\overrightarrow{xr_2}\} \cup \{\overrightarrow{vr_1}\}$.

(8) Finally, we assume that $r_1 \in N(u) \setminus N(v)$, $r_i \in N(v) \setminus N(u)$ for all $i > 1$, and r_1 is the root of T'. In that case we have that $d_T(u) = d_{T'}(u)$.

If $m > 2$, from $d_T(v) \geqslant d_{T'}(v)$ we conclude that there exists $r_i \in N(v)$ for $i > 2$ and an edge $\overrightarrow{yr_i} \in E(T') \setminus E(T)$ such that $y \neq v$. Then we set $T'' = T' \setminus \{\overrightarrow{yr_i}\} \cup \{\overrightarrow{vr_i}\}$.

For $m = 2$, we again consider the edge $\overrightarrow{xr_2} \in E(T') \setminus E(T)$. If $x = r_1$, we have that $d_T(u) = d_{T'}(u), d_T(v) = d_{T'}(v)$ and

$$T = T' \setminus \{\overrightarrow{r_1r_2}\} \cup \{\overrightarrow{zr_1}\} \text{ or } T = T' \setminus \{\overrightarrow{r_1r_2}\} \cup \{\overrightarrow{yr_2}\}.$$

But, then we have that $h_T(u) > h_{T'}(u)$ or $h_T(u) = h_{T'}(u), h_T(v) > h_{T'}(v)$ which is a contradiction with the assumption. Therefore, in this case ($m = 2$) we have that $x \neq r_1$. If we set $T'' = T' \setminus \{\overrightarrow{xr_2}\} \cup \{\overrightarrow{r_1r_2}\}$, then we obtain that $d_{T'}(u) = d_{T''}(u), d_{T'}(v) = d_{T''}(v)$, $h_{T''}(u) = h_{T'}(u)$ and $h_{T''}(v) < h_{T'}(v)$.

$\square$

3. A complete multipartite graph

Let $K_{n_1,n_2,\ldots,n_k}$ denote a complete multipartite graph. Assume that its vertex set is $V = V_1 \cup V_2 \cup \cdots \cup V_k$, where $|V_i| = n_i$. Furthermore, we assume that all sets V_i are linearly ordered. We may choose one vertex in V_1 and one in V_2 and denote them by 1 and -1.

Let $\overrightarrow{K}_{n_1,n_2,\ldots,n_k}$ denote a directed graph obtained from a complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ when one replaces all edges by pairs of directed edges going in opposite directions. Note that $1, -1$ is a dominant pair of vertices in $\overrightarrow{K}_{n_1,n_2,\ldots,n_k}$ and from Theorem 2.1 we know that $\Delta\big(\overrightarrow{K}_{n_1,n_2,\ldots,n_k}\big)$ is shellable. We use a slight modification of the algorithm described in [**5**] to encode directed trees in $\overrightarrow{K}_{n_1,n_2,\ldots,n_k}$.

Remark 3.1. For each directed tree T of $\overrightarrow{K}_{n_1,n_2,\ldots,n_k}$ we associate the set of sequences $\{C_0, C_1, \ldots, C_k\}$ of the vertex set such that

(i) The length of the sequence C_0 is $k-1$ and any $x \in V$ can occur in C_0.
(ii) For any $i > 0$ the length of C_i is $n_i - 1$ and C_i contains vertices from $V \smallsetminus V_i$.

Let r denote the root of T. For a vertex $v \in V$, $v \neq r$, let $U_T(v)$ denote the unique vertex u such that $\overrightarrow{uv} \in E(T)$. We say that the *depth* of a vertex v in T (denoted by $\mathrm{depth}(v)$) is the length of the longest directed path from v to a leaf of T. For all $i = 1, 2, \ldots, k$ let v_i' denote the vertex from V_i with the maximal depth in T (if there are more than one vertex in V_i with maximal depth for v_i', we choose the greatest one among them in the linear order of V_i).
If the root of T is a vertex that belongs to V_{i_0}, then we have that $v_{i_0}' = r$.

The sequence C_0 contains vertices $\{U_T(v_i') : i \neq i_0\}$, and the vertex $U_T(v_j')$ is before $U_T(v_s')$ in C_0 if and only if $\mathrm{depth}(v_j') < \mathrm{depth}(v_s')$ or $\mathrm{depth}(v_j') = \mathrm{depth}(v_s')$ and $j < s$. For any $i > 0$ the entries of the sequence C_i are $n_i - 1$ vertices $\{U_T(v) : v \in V_i, v \neq v_i'\}$ and we order the set of these vertices in the same way as in C_0. Vertices from V_j that appear in C_i and have the same depth, we order in C_i by using the linear order defined on V_j. We say that $\{C_0, C_1, \ldots, C_k\}$ is the code for the tree T. The proof that the map $T \mapsto \{C_0, C_1, \ldots, C_k\}$ is a bijection, as well as more details about this construction can be found in [**5**].

It is easily seen from the above remark that there are

$$n^{k-1}(n-n_1)^{n_1-1}(n-n_2)^{n_2-1}\cdots(n-n_k)^{n_k-1}$$

directed trees in $\overrightarrow{K}_{n_1,n_2,\ldots,n_k}$. These are the facets of $\Delta\big(\overrightarrow{K}_{n_1,n_2,\ldots,n_k}\big)$.

Theorem 3.1. *The h-vector of $\Delta\big(\overrightarrow{K}_{m,n}\big)$ is given by*

$$\begin{aligned} h_k(\Delta(K_{m,n})) = &\sum_{p+q=k} \binom{m-1}{p}(n-1)^p \binom{n-1}{q}(m-1)^q \\ &+ (m+n-1) \sum_{p+q=k-1} \binom{m-1}{p}(n-1)^p \binom{n-1}{q}(m-1)^q. \end{aligned}$$

Proof. Note that $\Delta(\overrightarrow{K}_{m,n})$ is $(m+n-2)$-dimensional complex. We consider the shelling order of $\Delta(\overrightarrow{K}_{m,n})$ described in Theorem 2.1. Recall that

$$\mathcal{R}(T) = \{\overrightarrow{xy} \in E(T) : E(T) \setminus \{\overrightarrow{xy}\} \subset E(S) \text{ for some tree } S \text{ that precedes } T\}.$$

In other words, an edge $\overrightarrow{xy} \in E(T)$ is in $\mathcal{R}(T)$ if it can be replaced with another edge $\overrightarrow{zw} \notin E(T)$ such that $(T \setminus \{\overrightarrow{xy}\}) \cup \{\overrightarrow{zw}\}$ is a new directed tree which precedes T in considered shelling order.

It is easy to check that the following statements hold:

(i) The restriction $\mathcal{R}(T)$ does not contain the edge $\overrightarrow{1x}$. A replacement of $\overrightarrow{1x}$ will decrease the out-degree of 1.

(ii) A replacement of the edge $\overrightarrow{-1x}$ in T will decrease the out-degree of -1. A new tree $T' = (T \setminus \{\overrightarrow{-1x}\}) \cup \{\overrightarrow{yz}\}$ precedes T in the considered shelling order only if we increase the out-degree of 1. We can do this if and only if the vertex $y = 1$ is below -1 in T and $z \in V_2$ is the root of T. Other edges $\overrightarrow{-1x'}$ can not be replaced.

(iii) Let r be the root of T. For a vertex $x \in V_1, x \neq 1$, and an edge $\overrightarrow{xy}$ we have:
   (a) if 1 is not below y the tree $(T \setminus \{\overrightarrow{xy}\}) \cup \{\overrightarrow{1y}\}$ precedes T.
   (b) If 1 is below y and if r belongs V_2, we have that $(T \setminus \{\overrightarrow{xy}\}) \cup \{\overrightarrow{1y}\}$ is before T.
   (c) If $r \in V_1$ (recall that 1 is below y) we set $S = (T \setminus \{\overrightarrow{xy}\}) \cup \{\overrightarrow{yr}\}$. Then we have $d_T(1) = d_S(1)$, $d_T(-1) = d_S(-1)$, $h_T(1) > h_S(1)$ and therefore the tree S precedes T.

   So, any of the considered edges $\overrightarrow{xy}$ is contained in $\mathcal{R}(T)$.

A similar analysis shows that an edge $\overrightarrow{xy}$, where $x \in V_2$, $x \neq -1$, is contained in $\mathcal{R}(T)$ except when x is the root of T, -1 is below y and 1 is not below y.

From the above remarks we have that for a directed tree T

$$\text{type}(T) = m + n - 1 - d_T(1) - d_T(-1) \tag{3.1}$$

except for the following trees:

(A1) Trees in which the root r belongs to V_2 and the vertex 1 is below of -1. The type of a such tree T is $\text{type}(T) = m + n - d_T(1) - d_T(-1)$.

(A2) Trees in which the root $r \in V_2$, $r \neq -1$, there exists an edge $\overrightarrow{rx} \in E(T)$ such that -1 is below x and 1 is not below x. The type of this tree is $\text{type}(T) = m + n - d_T(1) - d_T(-1) - 2$.

Now, we count the number of trees in $\Delta(\overrightarrow{K}_{m,n})$ with given $d_T(1) + d_T(-1)$. Let $\{\{r\}, C_1, C_2\}$ be the set of sequences of vertices associated to a tree T (r is the root of T) in Remark 3.1. We set

$$p = |\{x \in C_1 : x \neq -1\}|, \quad q = |\{y \in C_2 : y \neq 1\}|.$$

From Remark 3.1 we obtain that there are

$$(m+n-2)\sum_{p+q=k-1}\binom{m-1}{p}(n-1)^p\binom{n-1}{q}(m-1)^q \\ +2\sum_{p+q=k}\binom{m-1}{p}(n-1)^p\binom{n-1}{q}(m-1)^q$$

directed trees in $\overrightarrow{K}_{m,n}$ such that $d_T(1)+d_T(-1)=m+n-1-k$. Note that the summands in the second row correspond with the trees in which the root is 1 or -1. From the relation (3.1) we have that all of these trees are of the type k, except the trees described in (A1) and (A2).

The remaining trees of $\overrightarrow{K_{m,n}}$ of the type k are all

(B1) trees described in (A2) in which $d_T(1)+d_T(-1)=m+n-2-k$; or
(B2) trees described in (A1) in which $d_T(1)+d_T(-1)=m+n-k$.

Let T be a directed tree as considered in $(B1)$. If $\overrightarrow{ry}$ is the first edge on the path from r to 1, then $T'=(T\smallsetminus\{\overrightarrow{ry}\})\cup\{\overrightarrow{-1y}\}$ is a tree as described in $(A1)$. Note that the map $T\mapsto T'$ is an injection, and all trees described in $(A1)$ are contained in the image of this map except the trees whose root is -1. From Remark 3.1 it follows that there are

$$\sum_{p+q=k}\binom{m-1}{p}(n-1)^p\binom{n-1}{q}(m-1)^q$$

trees with -1 as the root and $d_T(1)+d_T(-1)=m+n-1-k$, which should be subtracted while calculating $h_k(\Delta(\overrightarrow{K}_{m,n}))$.

Further, if T is a tree described in $(A2)$, and $\overrightarrow{rx}$ is the first edge of the path from r to 1, then $T'=(T\smallsetminus\{\overrightarrow{rx}\})\cup\{\overrightarrow{-1x}\}$ is a tree as in $(B2)$. This map is an injection, and a tree from $(B2)$ is not in the image of this map if and only if its root is -1.

There are

$$\sum_{p+q=k-1}\binom{m-1}{p}(n-1)^p\binom{n-1}{q}(m-1)^q$$

trees with -1 as the root and $d(1)+d(-1)=m+n-k$ that should be added when determining $h_k\big(\Delta(\overrightarrow{K}_{m,n})\big)$. □

From the above theorem we obtain that the generating facets for $\Delta\big(\overrightarrow{K}_{m,n}\big)$ are:

(i) All directed trees of $\overrightarrow{K}_{m,n}$ in which the vertices 1 and -1 are leaves and the root of such a tree is a vertex contained in V_1.
(ii) All directed trees of $\overrightarrow{K}_{m,n}$ in which the root is from V_2, the vertex 1 is a leaf below -1, and the out-degree of the vertex -1 in such a tree is one.

Corollary 3.1. *The complex $\Delta\big(\overrightarrow{K}_{m,n}\big)$ is homotopy equivalent to a wedge of $(m+n-1)(m-1)^{n-1}(n-1)^{m-1}$ spheres of dimension $m+n-2$.*

Theorem 3.2. *The complex* $\Delta\big(\overrightarrow{K}_{n_1,n_2,\dots,n_k}\big)$ *is homotopy equivalent to a wedge of* $(n-1)^{k-1}(n-n_1-1)^{n_1-1}(n-n_2-1)^{n_2-1}\cdots(n-n_k-1)^{n_k-1}$ *spheres of dimension* $n-2$.

Proof. We use a shelling of $\Delta\big(\overrightarrow{K}_{n_1,n_2,\dots,n_k}\big)$ described in Theorem 2.1 to recognize generating faces. These are

(**A**) directed trees in which the vertex 1 is a leaf, and there does not exist an edge $\overrightarrow{-1v}$, for a vertex $v \in V_1$

except the tress of the above form in which

(A_1) the root is a vertex $v_2 \in V_2$, there is an edge $\overrightarrow{v_2v_1}$ for a vertex $v_1 \in V_1$, the vertex -1 is below v_1, and 1 is not below v_1; and

(A_2) the root is a vertex $v_1 \in V_1$ and the leaf 1 is below -1.

Generating facets of $\Delta\big(\overrightarrow{K}_{n_1,n_2,\dots,n_k}\big)$ are also:

(**B**) directed trees in which the root is a vertex $r \in V \setminus V_1$, there is only one edge of the form $-1v_1$ for a vertex $v_1 \in V_1$ and 1 is a leaf below v_1.

Now, we define a map between a subset of the trees of the type (B) and directed trees of type (A_1) or (A_2). If T is a tree of the type (B) with the root $r \in V \setminus V_1$ and $-1 \to v_1 \to x \to y \to \cdots \to 1$ is the unique path from -1 to 1, then

$$T' = T \setminus \big\{\overrightarrow{-1v_1}, \overrightarrow{v_1x}\big\} \cup \{\overrightarrow{xv_1}, \overrightarrow{v_1r}\} \text{ is a tree of type } A_1 \text{ if } x \in V_2,$$
$$T'' = T \setminus \big\{\overrightarrow{-1v_1}, \overrightarrow{v_1x}\big\} \cup \big\{\overrightarrow{-1x}, \overrightarrow{v_1r}\big\} \text{ is a tree of type } A_2 \text{ if } x \in V \setminus (V_1 \cup V_2).$$

The above map is a bijection that exhausts all trees of type (B) except the trees in which $\overrightarrow{-11}$ is an edge. Therefore, in order to estimate the number of the generating simplices of $\Delta(\overrightarrow{K}_{n_1,n_2,\dots,n_k})$ we have to count directed trees in $\overrightarrow{K}_{n_1,n_2,\dots,n_k}$ in which

(*) 1 is a leaf, there are no other edges of the form $\overrightarrow{-1v}$, for a vertex $v \in V_1$; or

(**) 1 is a leaf, $\overrightarrow{-11}$ is an edge, there are no other edges of the form $\overrightarrow{-1v}$, for a vertex $v \in V_1$, and the root is a vertex $r \in V \setminus V_1$.

From Remark 3.1 we obtain that the code of a tree described in (*) or (**) does not contain label -1 in the sequences C_0 at the place reserved for the deepest vertex of V_1. Also, a tree described in (*) does not contain -1 in the sequence C_1. For a tree described in (**) the vertex -1 appears in C_1 only in the first place, and the last entry of C_0 (the root of such a tree) is not from V_1. Therefore, in the code of such a tree there exists $v \in V \setminus V_1$ that appears in C_0 as $U(v_1')$. We replace this vertex v with -1 and obtain the bijection between generating simplices of $\Delta\big(\overrightarrow{K}_{n_1,n_2,\dots,n_k}\big)$ and directed trees of $\overrightarrow{K}_{n_1,n_2,\dots,n_k}$ in which -1 does not occur in C_1 and 1 does not occur at all. For a tree described in (*) the code remains unchanged. The number of these trees is

$$(n-1)^{k-1}(n-n_1-1)^{n_1-1}(n-n_2-1)^{n_2-1}\cdots(n-n_k-1)^{n_k-1}. \qquad \square$$

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