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ON A RELATION BETWEEN SUMS OF ARITHMETICAL FUNCTIONS AND DIRICHLET SERIES

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Abstract. We introduce a concept called good oscillation. A function is called good oscillation, if its m-tuple integrals are bounded by functions having mild orders.

We prove that if the error terms coming from summatory functions of arithmetical functions are good oscillation, then the Dirichlet series associated with those arithmetical functions can be continued analytically over the whole plane.

We also study a sort of converse assertion that if the Dirichlet series are continued analytically over the whole plane and satisfy a certain additional assumption, then the error terms coming from the summatory functions of Dirichlet coefficients are good oscillation.

1. Introduction

Let $s = \sigma + it$ be a complex variable, where σ and t are real. Let $\zeta(s)$ be the Riemann zeta-function which is defined for $\sigma > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

$\zeta(s)$ can be continued analytically over the whole s-plane beyond the line $\sigma = 1$, and its only singularity is a simple pole at $s = 1$.

Many proofs of the analytic continuation of $\zeta(s)$ are known (see, for example, Titchmarsh [**6**]). Among them, we recall the proof based on the Euler–Maclaurin summation formula. Applying the partial summation formula to (1.1), one has

$$\zeta(s) = s\int_1^{\infty} \frac{\sum_{n\leqslant x} 1}{x^{s+1}}\,dx = s\int_1^{\infty} \frac{[x]}{x^{s+1}}\,dx, \quad \sigma > 1,$$

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where $[x]$ is the largest integer not exceeding x. The right-hand side of the above equation is rewritten as

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx, \tag{1.2}$$

and (1.2) gives the analytic continuation of $\zeta(s)$ for $\sigma > 0$ except for $s = 1$ because of the boundedness of $[x] - x + \frac{1}{2}$. Let $B_m(x)$, $m \in \mathbf{N}$, be the Bernoulli polynomials which are inductively defined from $B_1(x) = x - \frac{1}{2}$ by keeping the properties $\frac{d}{dx}\frac{B_{m+1}(x)}{(m+1)!} = \frac{B_m(x)}{m!}$ and $\int_0^1 B_m(x)dx = 0$, e.g., $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, and so on (see, for example, Apostol [**1**, p. 226]). Then, applying the integration by parts to (1.2), one has

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{m=1}^{M-1}(s)_m\frac{B_{m+1}}{(m+1)!} - \frac{(s)_M}{M!}\int_1^\infty \frac{B_M(x-[x])}{x^{s+M}}\,dx, \tag{1.3}$$

where $B_m = B_m(0)$ is the Bernoulli number and $(s)_m$ is the function defined by $(s)_m = s(s+1)\cdots(s+m-1)$. Since $B_M(x-[x])$, $M \in \mathbf{N}$, are periodic functions, $B_M(x-[x])$ are all bounded. Hence the integral on the right-hand side of (1.3) is analytic on the half-plane $\sigma > -M+1$, and, by taking M to ∞, $\zeta(s)$ is continued analytically over the whole s-plane except for $s = 1$.

Now let us consider the general Dirichlet series $F(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$, where $a(n)$, $n \in \mathbf{N}$, are complex numbers. $F(s)$ will be continued analytically over the whole s-plane except for finite poles by the same argument as above, if the following assumption is satisfied:

> Consider the summatory function of the Dirichlet coefficients, $\sum_{n\leqslant x} a(n)$. Let $\mathcal{J}(x)$ be a certain function concerning with location of poles. Define the function $E_0(x)$ by $E_0(x) = \sum_{n\leqslant x} a(n) - \mathcal{J}(x)$. Then there exists a sequence of functions $\{E_m(x)\}_{m=1}^\infty$ such that
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> $$\begin{cases} \frac{d}{dx}E_1(x) = E_0(x), & x \in (0,\infty) - \mathbf{N}, \\ \frac{d}{dx}E_{m+1}(x) = E_m(x), & x \in (0,\infty), \quad m \geqslant 1, \end{cases}$$
>
> and $E_m(x)$ are all bounded.

To prove the analytic continuation of $F(s)$ by the same argument as above, the assumption for the boundedness of $E_m(x)$ may be relaxed. We introduce the following definition.

Definition 1.1. Let $g_0(x) : (0,\infty) \to \mathbf{C}$ be a function which is continuous on $(0,\infty) - \mathbf{N}$, bounded on every finite open interval $(0,c)$, and bounded by $O(x^{\alpha_0})$ as $x \to \infty$, where α_0 is a nonnegative constant. Let C_m, $m \in \mathbf{N}$, be arbitrary constants, and $g_m(x;C_m)$, $m \in \mathbf{N}$, be the functions defined by

$$g_1(x;C_1) = \int_0^x g_0(v)\,dv + C_1, \quad x \in (0,\infty), \tag{1.4}$$

and

$$g_m(x;C_m) = \int_0^x g_{m-1}(v;C_{m-1})\,dv + C_m, \quad x \in (0,\infty), \quad m \geqslant 2. \tag{1.5}$$

Then the function $g_0(x)$ is called *good oscillation*, if there exists a nonnegative sequence $\{\alpha_m\}_{m=1}^{\infty}$ such that $g_m(x; C_m) = O(x^{\alpha_m})$ as $x \to \infty$ and $m+1-\alpha_m \to \infty$ as $m \to \infty$.

By this definition, $g_1(x; C_1)$ is differentiable on $(0, \infty) - \mathbf{N}$ and $g_m(x; C_m)$, $m \geqslant 2$, are differentiable on $(0, \infty)$, which satisfy

$$\begin{cases} \frac{d}{dx} g_1(x; C_1) = g_0(x), \quad x \in (0, \infty) - \mathbf{N}, \\ \frac{d}{dx} g_{m+1}(x; C_{m+1}) = g_m(x; C_m), \quad x \in (0, \infty), \quad m \geqslant 1. \end{cases}$$

We notice that the sequence $\{C_m\}_{m=1}^{\infty}$ in Definition 1.1 is uniquely determined. We will verify this in Section 2.

The purpose of this paper is to give a condition, for which the analytic continuation of the Dirichlet seires is valid, from a point of view of the concept of good oscillation.

Theorem 1.1. *Let $a(n)$, $n \in \mathbf{N}$, be complex numbers. Assume the following condition* (X)*:*

(X) *There exist constants $l \in \mathbf{N} \cup \{0\}$, J_h, and J such that the function $g_0(x)$ defined by*

$$g_0(x) = \sum_{n \leqslant x} a(n) - \left(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \right)$$

is good oscillation, where the empty sum $\sum_{h=0}^{l-1}$ in the case $l = 0$ is defined to be 0.

Then the following assertion (Y) *holds:*

(Y) *There exists a constant $\sigma_1 \geqslant 1$ such that the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{1.6}$$

is absolutely convergent for $\sigma > \sigma_1$. Moreover, $F(s)$ can be continued analytically over the whole s-plane beyond the line $\sigma = \sigma_1$, and its only singularity is a pole of the order l at $s = 1$.

The following result states that a sort of converse assertion holds under additional assumptions.

Theorem 1.2. *Assume the condition* (Y) *in Theorem* 1.1 *and the following conditions* (A1), (A2)*:*

(A1) *For any nonnegative integer m, there exists a nonnegative constant c_m such that $F(-m - \frac{1}{2} + it) = O((1 + |t|)^{c_m})$.*

(A2) *If $\{c_m\}_{m=0}^{\infty}$ is the sequence above, then* $\lim_{m\to\infty} c_m/m^2 = 0$

Then the assertion (X) *in Theorem* 1.1 *holds.*

The functional equation of $\zeta(s)$ and the Phragmén–Lindelöf convexity principle give the well-known estimate $\zeta(\sigma+it) = O((1+|t|)^{\frac{1}{2}-\sigma})$, where the implied constant is uniform for s in the vertical strip $-M \leqslant \sigma \leqslant \delta < 0$, and hence c_m can be chosen as $m+1$ and $\lim_{m\to\infty} \frac{c_m}{m^{1+\varepsilon}} = 0$ holds for every arbitrary small $\varepsilon > 0$. The property $\lim_{m\to\infty} \frac{c_m}{m^{1+\varepsilon}} = 0$ holds for other L-functions, e.g., Dirichlet L-functions, cusp form L-functions, and the power moments of those L-functions, because those functions have functional equations.

Since the Bernoulli polynomials $B_M(x-[x])$, where $M \in \mathbf{N}$, are all bounded, $-B_1(x-[x]) = [x] - x + \frac{1}{2}$ is good oscillation. Moreover, $B_1(x-[x])$ is expressed as

$$B_1(x-[x]) = -\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}, \quad x \notin \mathbf{Z}. \tag{1.7}$$

Using (1.2) and (1.7), one can prove the functional equation of $\zeta(s)$ [**6**, pp. 13–15].

This example shows that more properties about $g_0(x)$ than good oscillation are required to study analytic continuations with functional equations. On the other hand, the multiple L-series $L_j(s)$, discussed in the last section, can be continued analytically over the whole s-plane, but any functional equation has not been obtained. This example may suggest that the concept of good oscillation is appropriate to study analytic continuations off functional equations.

2. A note on the concept of good oscillation

As is mentioned in Introduction, the sequence $\{C_m\}_{m=1}^{\infty}$ in Definition 1.1 is uniquely determined. Here we prove this.

Let $\{C_m\}_{m=1}^{\infty}$ and $\{D_m\}_{m=1}^{\infty}$ be the sequences of constants, and $g_m(x;C_m)$ and $g_m(x;D_m)$ be the functions defined by (1.4) and (1.5) with the same $g_0(x)$. If $g_0(x)$ is good oscillation, then there exists a nonnegative sequence $\{\alpha_m\}_{m=1}^{\infty}$ (resp. $\{\beta_m\}_{m=1}^{\infty}$) such that $g_m(x;C_m) = O(x^{\alpha_m})$ (resp. $g_m(x;D_m) = O(x^{\beta_m})$) as $x \to \infty$ and $m+1-\alpha_m \to \infty$ (resp. $m+1-\beta_m \to \infty$) as $m \to \infty$. Assume $C_m \neq D_m$ for some m, and let K be the least m such that $C_m \neq D_m$. Then, by (1.4) and (1.5),

$$\begin{aligned}
g_K(x;D_K) &= g_K(x;C_K) + D_K - C_K, \quad D_K - C_K \neq 0,\\
g_{K+1}(x;D_{K+1}) &= (D_K - C_K)x + g_{K+1}(x;C_{K+1}) + D_{K+1} - C_{K+1}\\
&= (D_K - C_K)x + g_{K+1}(x;C_{K+1}) + O(1),
\end{aligned}$$

and, for $m \geqslant K+1$,

$$\begin{aligned}
g_m(x;D_m) &= (D_K - C_K)\frac{x^{m-K}}{(m-K)!} + g_m(x;C_m) + O(x^{m-K-1})\\
&= (D_K - C_K)\frac{x^{m-K}}{(m-K)!} + O(x^{\alpha_m}) + O(x^{m-K-1}).
\end{aligned}$$

From the assumption $m+1-\alpha_m \to \infty$ it follows that $\alpha_m \leqslant m-K-1$ for all large m. Hence we have

$$g_m(x;D_m) = (D_K - C_K)\frac{x^{m-K}}{(m-K)!} + O(x^{m-K-1}).$$

This shows $\beta_m \geqslant m-K$, which contradicts $m+1-\beta_m \to \infty$ as $m \to \infty$.

3. Proof of Theorem 1.1

Let N_1 and N_2 be positive integers, $f(x)$ be a C^M function defined on the closed interval $[N_1, N_2]$, and $a(n)$, $n \in \mathbf{N}$, be complex numbers. Let $g_0(x) : (0, \infty) \to \mathbf{C}$ be a function defined by

$$g_0(x) = \sum_{n \leqslant x} a(n) - \biggl(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \biggr),$$

where $l \in \mathbf{N} \cup \{0\}$, and J_h, J are constants. For this $g_0(x)$, let $g_m(x; C_m)$, $m \in \mathbf{N}$, be the functions defined by (1.4) and (1.5). Then, by the integration by parts in the sense of Stieltjes,

$$\begin{aligned}
\sum_{N_1 < n \leqslant N_2} f(n) a(n) &= \int_{N_1}^{N_2} f(x)\, d\biggl(\sum_{n \leqslant x} a(n) \biggr) \\
&= \int_{N_1}^{N_2} f(x) \frac{d}{dx} \biggl(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \biggr) dx + \int_{N_1}^{N_2} f(x)\, d(g_0(x)) \\
&= \int_{N_1}^{N_2} f(x) \frac{d}{dx} \biggl(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \biggr) dx \\
&\quad + \Bigl[f(x)\, g_0(x) \Bigr]_{N_1}^{N_2} - \int_{N_1}^{N_2} f'(x) g_0(x)\, dx.
\end{aligned}$$

Repeating the integration by parts, we have

$$\begin{aligned}
\sum_{N_1 < n \leqslant N_2} f(n)\, a(n) &= \int_{N_1}^{N_2} f(x) \frac{d}{dx} \biggl(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \biggr) dx \\
&\quad + \Bigl[f(x)\, g_0(x) \Bigr]_{N_1}^{N_2} + \sum_{m=1}^{M-1} (-1)^m \Bigl[f^{(m)}(x)\, g_m(x; C_m) \Bigr]_{N_1}^{N_2} \\
&\quad + (-1)^M \int_{N_1}^{N_2} f^{(M)}(x)\, g_{M-1}(x; C_{M-1}) dx.
\end{aligned} \tag{3.1}$$

From now on, we put $f(x) = x^{-s}$ and $N_1 = N$ in (3.1), and we abbreviate $g_m(x; C_m)$ to $g_m(x)$. Then

$$\begin{aligned}
\sum_{N < n \leqslant N_2} \frac{a(n)}{n^s} &= \int_N^{N_2} \frac{1}{x^s} \frac{d}{dx} \biggl(x \sum_{h=0}^{l-1} J_h (\log x)^h + J \biggr) dx \\
&\quad + \sum_{m=0}^{M-1} \Bigl[(s)_m \frac{g_m(x)}{x^{s+m}} \Bigr]_N^{N_2} + (s)_M \int_N^{N_2} \frac{g_{M-1}(x)}{x^{s+M}}\, dx.
\end{aligned}$$

Now let us assume the condition (X) of Theorem 1.1. Then there exists a nonnegative sequence $\{\alpha_m\}_{m=0}^{\infty}$ such that $g_m(x) = O(x^{\alpha_m})$ as $x \to \infty$ and $m+1-\alpha_m \to \infty$ as $m \to \infty$. Hence, for s with $\sigma > \max_{0 \leqslant m \leqslant M-1} \{1, \alpha_m\}$, we can take N_2 to be ∞, and have

$$F(s)-\sum_{n=1}^{N}\frac{a(n)}{n^s}=\int_N^\infty\frac{1}{x^s}\frac{d}{dx}\left(x\sum_{h=0}^{l-1}J_h(\log x)^h+J\right)dx \\ -\sum_{m=0}^{M-1}(s)_m\frac{g_m(N)}{N^{s+m}}+(s)_M\int_N^\infty\frac{g_{M-1}(x)}{x^{s+M}}\,dx. \tag{3.2}$$

Let W_r be a number defined by

$$W_h=\begin{cases}J_h+(h+1)J_{h+1}, & \text{if } 0\leqslant h\leqslant l-2,\\ J_h, & \text{if } h=l-1.\end{cases}$$

Applying

$$\frac{d}{dx}\left(x\sum_{h=0}^{l-1}J_h(\log x)^h+J\right)=\sum_{h=0}^{l-1}W_h(\log x)^h$$

and

$$\int_N^\infty\frac{(\log x)^h}{x^s}\,dx=\sum_{j=0}^{h}\binom{h}{j}j!(-1)^{j+1}\frac{N^{1-s}(\log N)^{h-j}}{(1-s)^{j+1}}$$

to (3.2), we have, for $\sigma>\max_{0\leqslant m\leqslant M-1}\{1,\alpha_m\}$, that

$$F(s)=\sum_{n=1}^{N}\frac{a(n)}{n^s}+\sum_{h=0}^{l-1}W_h h!\sum_{j=0}^{h}\frac{(-1)^{j+1}}{(h-j)!}\frac{N^{1-s}(\log N)^{h-j}}{(1-s)^{j+1}} \\ -\sum_{m=0}^{M-1}(s)_m\frac{g_m(N)}{N^{s+m}}+(s)_M\int_N^\infty\frac{g_{M-1}(x)}{x^{s+M}}\,dx. \tag{3.3}$$

The integral on the right-hand side of (3.3) is analytic on the half-plane $\sigma>1-(M-\alpha_{M-1})$, and hence (3.3) is valid for $\sigma>1-(M-\alpha_{M-1})$. Since $M-\alpha_{M-1}$ tends to ∞ as $M\to\infty$, $F(s)$ can be continued analytically over the whole s-plane except for $s=1$. This completes the proof.

Remark 3.1. In the condition (X) of Theorem 1.1, we assume existence of constants $l\in\mathbf{N}\cup\{0\}$, J_h, and J such that the function $g_0(x)$ defined by

$$g_0(x)=\sum_{n\leqslant x}a(n)-\left(x\sum_{h=0}^{l-1}J_h(\log x)^h+J\right)$$

is good oscillation. These constants $l\in\mathbf{N}\cup\{0\}$, J_h, and J are uniquely determined, i.e., if there exist $\widetilde{l}\in\mathbf{N}\cup\{0\}$, $\widetilde{J_h}$, and $\widetilde{J}$ such that the function $\widetilde{g_0}(x)$ defined by

$$\widetilde{g_0}(x)=\sum_{n\leqslant x}a(n)-\left(x\sum_{h=0}^{\widetilde{l}-1}\widetilde{J_h}(\log x)^h+\widetilde{J}\right)$$

is good oscillation, then $l = \widetilde{l}$, $J_h = \widetilde{J_h}$, and $J = \widetilde{J}$. This assertion is equivalent to that if

$$G_0(x) = x\sum_{h=0}^{L} D_h(\log x)^h + D$$

is good oscillation, then $D_h = 0$ for all $0 \leqslant h \leqslant L$ and $D = 0$. Here we prove this.

Assume that there exists an h such that $D_h \neq 0$, and let $H = \max\{h \mid D_h \neq 0\}$. Then

$$G_0(x) = D_H x(\log x)^H + O(x(\log x)^{H-1}).$$

For $G_0(x)$, let $G_m(x)$, $m \in \mathbf{N}$, be the functions defined by (1.4) and (1.5). Then

$$G_m(x) = D_H\frac{x^{m+1}}{(m+1)!}(\log x)^H + O(x^{m+1}(\log x)^{H-1}).$$

This shows $\alpha_m \geqslant m+1$, which contradicts that $G_0(x)$ is good oscillation. Thus $D_h = 0$ for all $0 \leqslant h \leqslant L$ and $G_0(x) = D$. $D = 0$ is similarly proved as above.

Remark 3.2. In Definition 1.1 we required $m+1-\alpha_m \to \infty$ as $m \to \infty$. Even if we replace this with $\limsup_{m\to\infty}(m+1-\alpha_m) = \infty$, Theorem 1.1 still holds, and Theorem 1.2 holds with the relaxed condition (A2)' $\liminf_{m\to\infty} c_m/m^2 = 0$. By this replacement, however, the properties described in Section 2 and Remark 3.1 are lost.

4. Proof of Theorem 1.2

Firstly, we only assume the condition (Y).

Let $A_m(x) : (0,\infty) \to \mathbf{C}$, $m \in \mathbf{N}\cup\{0\}$, be the function defined by the Riesz sum of the Dirichlet coefficients $a(n)$ in (1.6):

$$A_m(x) = \begin{cases} \frac{1}{m!}\sum_{n\leqslant x} a(n)(x-n)^m, & \text{if } x \geqslant 1 \text{ and } m \geqslant 1,\\ \sum_{n\leqslant x} a(n) - \widetilde{a}(x), & \text{if } x \geqslant 1 \text{ and } m = 0,\\ 0, & \text{if } 0 < x < 1, \end{cases}$$

where $\widetilde{a}(x)$ is the function defined by $\widetilde{a}(x) = \frac{a(x)}{2}$ if x is an integer, and $\widetilde{a}(x) = 0$ otherwise. Let $SR_m(x) : (0,\infty) \to \mathbf{C}$, $m \in \mathbf{N}\cup\{0\}$, be the function defined by

$$SR_m(x) = \sum_{j=-1}^{m} \operatorname{Res}_{w=-j} F(w)\frac{x^{w+m}}{w(w+1)\dots(w+m)}. \tag{4.1}$$

Let $E_m(x) : (0,\infty) \to \mathbf{C}$, $m \in \mathbf{N}\cup\{0\}$, be the function defined by

$$E_m(x) = A_m(x) - SR_m(x). \tag{4.2}$$

Then it is easily verified that

$$\begin{cases} \frac{d}{dx}E_1(x) = E_0(x), & x \in (0,\infty) - \mathbf{N},\\ \frac{d}{dx}E_{m+1}(x) = E_m(x), & x \in (0,\infty), \quad m \geqslant 1. \end{cases}$$

The function $E_m(x)$ was used in [**5**] for another purpose.

The following lemma can be proved similarly to Davenport [**2**, p.105, Lemma], and we omit the proof.

Lemma 4.1. *Let $c > 0$ and $y > 0$. It follows that, for $m \in \mathbf{N}$,*

$$I_m(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^w}{w(w+1)\cdots(w+m)}\,dw = \begin{cases} 0, & 0 < y \leqslant 1 \\ \frac{1}{m!}(1-y^{-1})^m, & y > 1, \end{cases}$$

and

$$I_0(y) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^w}{w}\,dw = \begin{cases} 0, & 0 < y < 1 \\ \frac{1}{2}, & y = 1 \\ 1, & y > 1. \end{cases}$$

For $m \in \mathbf{N}\cup\{0\}$, let

$$R_m(y,T) = I_m(y) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^w}{w(w+1)\cdots(w+m)}\,dw.$$

Then, for $T > 2m$,

$$|R_m(y,T)| < \begin{cases} y^c \min\left\{\frac{2^m}{T^m}, \frac{1}{T^{m+1}|\log y|}\right\}, & y \neq 1 \\ \frac{1}{T^m}, & y = 1. \end{cases} \tag{4.3}$$

Now we assume (Y), (A1), and the new condition (B): there exists a nonnegative λ such that $a(n) = O(n^{\lambda+\varepsilon})$, where $\varepsilon > 0$ is chosen arbitrarily small. Then we have the following.

Lemma 4.2. *Assume the conditions* (Y) *in Theorem* 1.1, (A1) *in Theorem* 1.2, *and* (B) *introduced above. Let $x > 2$ and $T > 2$. Choose c in $R_m(y,T)$ introduced above as $c = \sigma_1 + \varepsilon$, where σ_1 and ε are the same ones as in* (Y) *and* (B), *respectively. Then*

$$\left|\sum_{n=1}^{\infty} a(n)R_m(x/n,T)\right| \ll \frac{x^{\sigma_1+\varepsilon}}{T^{m+1}} + \frac{x^{1+\lambda+\varepsilon}\log x}{T^{m+1}} + \frac{x^{\lambda+\varepsilon}}{T^m}.$$

Proof. From (4.3) it follows that

$$\begin{aligned} &\left|\sum_{n=1}^{\infty} a(n)R_m(x/n,T)\right| \\ &\qquad < \sum_{\substack{n=1\\ n\neq x}}^{\infty} |a(n)|\left(\frac{x}{n}\right)^{\sigma_1+\varepsilon} \min\left\{\frac{2^m}{T^m}, \frac{1}{T^{m+1}|\log(x/n)|}\right\} + \frac{|a(x)|}{T^m}, \end{aligned} \tag{4.4}$$

where the second term on the right-hand side only appears in the case that x is an integer, and this is $O(x^{\lambda+\varepsilon}/T^m)$ by (B).

Divide the sum on the right-hand side of (4.4) into three sums $\sum_1$, $\sum_2$, and $\sum_3$ accordingly as the range of n, $n \leqslant \frac{3}{4}x$ or $n \geqslant \frac{5}{4}x$, $\frac{3}{4}x < n < x$, and $x < n < \frac{5}{4}x$, respectively. By (Y) and (B), $\sum_1$ is $O(x^{\sigma_1+\varepsilon}/T^{m+1})$.

To estimate $\sum_2$, let x_1 be the largest integer not greater than x. The contribution of the term $n = x_1$ is $O(x^{\lambda+\varepsilon}/T^m)$ by (B). For the other terms we put $n = x_1 - \nu$, $0 < \nu < \frac{1}{4}x$, to get $\log(x/n) \geqslant \nu/x_1 \gg \nu/x$. Then, by (B),

$$\sum\nolimits_2 \ll \frac{x}{T^{m+1}} \sum_{0<\nu<\frac{1}{4}x} \frac{|a(x_1-\nu)|}{\nu} + \frac{x^{\lambda+\varepsilon}}{T^m} \ll \frac{x^{1+\lambda+\varepsilon}\log x}{T^{m+1}} + \frac{x^{\lambda+\varepsilon}}{T^m}.$$

To estimate $\sum_3$, let x_2 be the least integer greater than x. The contribution of the term $n = x_2$ is $O(x^{\lambda+\varepsilon}/T^m)$ by (B). For the other terms we put $n = x_2 + \nu$, $0 < \nu < \frac{1}{4}x$, to get $-\log(x/n) \geqslant \nu/(2x_2+\nu) \gg \nu/x$. Then, by (B),

$$\sum\nolimits_3 \ll \frac{x}{T^{m+1}} \sum_{0<\nu<\frac{1}{4}x} \frac{|a(x_2+\nu)|}{\nu} + \frac{x^{\lambda+\varepsilon}}{T^m} \ll \frac{x^{1+\lambda+\varepsilon}\log x}{T^{m+1}} + \frac{x^{\lambda+\varepsilon}}{T^m}.$$

Thus we obtain the desired estimate. □

LEMMA 4.3. *Assume the conditions* (Y) *in Theorem* 1.1*,* (A1) *in Theorem* 1.2*, and* (B) *introduced above. Let $x > 2$, $T > 2$, and c_m be the same one as in* (A1)*. Then*

$$E_m(x) \ll \frac{x^{m+\sigma_1+\varepsilon}}{T^{m+1}} + \frac{x^{m+1+\lambda+\varepsilon}\log x}{T^{m+1}} + \frac{x^{m+\lambda+\varepsilon}}{T^m} + x^{-\frac{1}{2}}\max\{T^{c_m-m}, \log T\}$$
$$+ \frac{x^m}{T^{m+1}} \exp_T\Big(\frac{c_m(\sigma_1+\varepsilon)}{m+\frac{1}{2}+\sigma_1+\varepsilon}\Big) \int_{-m-\frac{1}{2}}^{\sigma_1+\varepsilon} \Big(x \exp_T\Big(-\frac{c_m}{m+\frac{1}{2}+\sigma_1+\varepsilon}\Big)\Big)^{\sigma} d\sigma.$$

PROOF. By the residue theorem the function $SR_m(x)$ defined by (4.1) can be expressed as

$$SR_m(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} F(w)\frac{x^{w+m}}{w(w+1)\cdots(w+m)}\,dw, \tag{4.5}$$

where $\mathcal{C}$ is the boundary of the rectangle with vertices $\sigma_1+\varepsilon-iT$, $\sigma_1+\varepsilon+iT$, $-m-\frac{1}{2}+iT$, and $-m-\frac{1}{2}-iT$ in positive orientation. $\mathcal{C}$ consists of four sides $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ taken in this order starting with the right side. For $\int_{\mathcal{C}_1}\ldots dw$, we have by (1.6) and Lemma 4.1 that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_1} F(w)\frac{x^{w+m}}{w(w+1)\cdots(w+m)}\,dw = A_m(x) - x^m\sum_{n=1}^{\infty} a(n)\,R_m(x/n,T). \tag{4.6}$$

By (4.5), (4.6), and Lemma 4.2, we have

$$\begin{aligned} E_m(x) = &-\frac{1}{2\pi i}\int_{\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4} F(w)\frac{x^{w+m}}{w(w+1)\cdots(w+m)}dw \\ &+ O\bigg(\frac{x^{m+\sigma_1+\varepsilon}}{T^{m+1}} + \frac{x^{m+1+\lambda+\varepsilon}\log x}{T^{m+1}} + \frac{x^{m+\lambda+\varepsilon}}{T^m}\bigg). \end{aligned} \tag{4.7}$$

For $\int_{\mathcal{C}_3} \dots dw$, we have by (A1) that

$$\int_{\mathcal{C}_3} F(w) \frac{x^{w+m}}{w(w+1)\cdots(w+m)}\, dw \ll x^{-\frac{1}{2}} \int_{-T}^{T} \frac{(1+|y|)^{c_m}}{(1+|y|)^{m+1}}\, dy \tag{4.8}$$
$$\ll x^{-\frac{1}{2}} \times \begin{cases} T^{c_m - m}/(c_m - m), & \text{if } c_m > m, \\ \log T, & \text{if } c_m \leqslant m \end{cases}$$
$$\ll x^{-\frac{1}{2}} \max\{T^{c_m - m}, \log T\}.$$

The treatments of $\int_{\mathcal{C}_2} \dots dw$ and $\int_{\mathcal{C}_4} \dots dw$ are similar, and we show the former case only. By the conditions (Y) and (A1), and by the Phragmén–Lindelöf convexity principle,

$$F(\sigma + it) = O\Big(\exp_{|t|} \Big(\frac{-c_m(\sigma - \sigma_1 - \varepsilon)}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) \Big) \tag{4.9}$$

holds uniformly for s with $-m - \frac{1}{2} \leqslant \sigma \leqslant \sigma_1 + \varepsilon$ and $|t| \geqslant 2$. Using (4.9), we have

$$\int_{\mathcal{C}_2} F(w) \frac{x^{w+m}}{w(w+1)\cdots(w+m)}\, dw \tag{4.10}$$
$$\ll \frac{x^m}{T^{m+1}} \exp_T \Big(\frac{c_m(\sigma_1 + \varepsilon)}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) \int_{-m-\frac{1}{2}}^{\sigma_1 + \varepsilon} \Big(x \exp_T \Big(-\frac{c_m}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) \Big)^{\sigma} d\sigma.$$

Substituting (4.8) and (4.10) into (4.7), we obtain the desired estimate. □

In Lemma 4.3 we have assumed the condition (B): $a(n) = O(n^{\lambda + \varepsilon})$. This λ can be chosen as $\lambda = \sigma_1$ under the condition (Y). In fact, since $\sum_{n=1}^{\infty} \frac{|a(n)|}{n^{\sigma_1 + \varepsilon}}$ converges, the terms corresponding to all large n are uniformly bounded by some constant. Thus we can put $\lambda = \sigma_1$ in Lemma 4.3, and have the following estimate under the conditions (Y) and (A1):

$$E_m(x) \ll \frac{x^{m+1+\sigma_1+\varepsilon} \log x}{T^{m+1}} + \frac{x^{m+\sigma_1+\varepsilon}}{T^m} + x^{-\frac{1}{2}} \max\{T^{c_m - m}, \log T\} \tag{4.11}$$
$$+ \frac{x^m}{T^{m+1}} \exp_T \Big(\frac{c_m(\sigma_1 + \varepsilon)}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) \int_{-m-\frac{1}{2}}^{\sigma_1 + \varepsilon} \Big(x \exp_T \Big(-\frac{c_m}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) \Big)^{\sigma} d\sigma.$$

Put $T = x^{\frac{m + \sigma_1 + \frac{1}{2}}{c_m + 1}}$ in (4.11). Then

$$x \exp_T \Big(-\frac{c_m}{m + \frac{1}{2} + \sigma_1 + \varepsilon} \Big) > x^{\frac{1}{c_m + 1}} > 1,$$

and hence the fourth term on the right-hand side of (4.11) is estimated as

$$\ll \frac{x^{m+\sigma_1+\varepsilon}}{T^{m+1}}.$$

The third term on the right-hand side of (4.11) is estimated as

$$\ll x^{-\frac{1}{2}} (\log T + T^{c_m - m}) \ll 1 + \frac{x^{-\frac{1}{2}} T^{c_m + 1}}{T^{m+1}} \ll 1 + \frac{x^{m+\sigma_1}}{T^{m+1}}.$$

Hence, with $T = x^{\frac{m+\sigma_1+\frac{1}{2}}{c_m+1}}$,

$$(4.12)\qquad \begin{aligned} E_m(x) &\ll \frac{x^{m+1+\sigma_1+\varepsilon}\log x}{T^{m+1}} + \frac{x^{m+\sigma_1+\varepsilon}}{T^m} + 1 \\ &\ll x^{m+1+\sigma_1+2\varepsilon-\frac{m+\sigma_1+\frac{1}{2}}{c_m+1}(m+1)} + x^{m+\sigma_1+\varepsilon-\frac{m+\sigma_1+\frac{1}{2}}{c_m+1}m} + 1 \\ &\ll x^{m+1+\sigma_1-\frac{m+\sigma_1+\frac{1}{2}}{c_m+1}m} + 1. \end{aligned}$$

By (4.12), we have $E_m(x) = O(x^{\alpha_m})$ with

$$m+1-\alpha_m = -\sigma_1 + \frac{m+\sigma_1+\frac{1}{2}}{c_m+1}m \quad \text{or} \quad m+1,$$

and, under the condition (A2), the right-hand side tends to ∞ as $m \to \infty$.

Now we define the function $g_0(x)$ by $g_0(x) = E_0(x) + \widetilde{a}(x)$. Then $g_0(x)$ is good oscillation under the conditions (Y), (A1), and (A2), and has the expression

$$\begin{aligned} g_0(x) &= \sum_{n\leqslant x} a(n) - \sum_{j=-1}^{0} \operatorname{Res}_{w=-j} F(w)\frac{x^w}{w} \\ &= \sum_{n\leqslant x} a(n) - \left(x\sum_{h=0}^{l-1} \frac{(-\log x)^h}{h!} \sum_{r=h}^{l-1} C_{-(r+1)}(-1)^r + F(0) \right), \end{aligned}$$

where the constants $C_{-(r+1)}$ come from the Laurent expansion of $F(s)$ at $s=1$:

$$F(s) = \frac{C_{-l}}{(s-1)^l} + \cdots + \frac{C_{-1}}{s-1} + O(1).$$

This completes the proof of Theorem 1.2.

5. An application of Theorem 1.2

Let j be a positive integer. Let s_i, $i = 1,2,\dots,j$, be complex variables, and χ_i, $i = 1,2,\dots,j$, be Dirichlet characters of the same conductor $q \geqslant 2$. The multiple L-series is defined by

$$L_j(s_1,\dots,s_j \mid \chi_1,\dots,\chi_j) = \sum_{\substack{0<n_1<\cdots<n_j \\ n_1,\dots,n_j\in\mathbf{N}}} \frac{\chi_1(n_1)}{n_1^{s_1}}\frac{\chi_2(n_2)}{n_2^{s_2}}\cdots\frac{\chi_j(n_j)}{n_j^{s_j}}.$$

If $\mathrm{Re}(s_i) \geqslant 1$, $i = 1,2,\dots,j-1$, and $\mathrm{Re}(s_j) > 1$, then the series are absolutely convergent.

When all s_i are equal to s, we abbreviate the multiple L-series to $L_j(s)$. Then

$$L_j(s) = \sum_{n=1}^{\infty} \frac{h_j(n)}{n^s}$$

with

$$(5.1)\quad h_j(n) = \sum_{\substack{m_1m_2\dots m_{j-1}\mid n \\ m_1<\cdots<m_{j-1}<\frac{n}{m_1\cdots m_{j-1}}}} \chi_1(m_1)\dots\chi_{j-1}(m_{j-1})\chi_j\left(\frac{n}{m_1m_2\dots m_{j-1}}\right).$$

The first named author studied the analytic continuation of $L_j(s)$ in [**3**]. As an application of the analytic continuation, properties of $\sum_{n\leqslant x} h_j(n)$ were studied in [**4**].

The function $L_j(s)$ can be continued meromorphically over the whole s-plane. Moreover, if $j = 2$, $\chi_1\chi_2(-1) = 1$, and χ_1, χ_2 are primitive characters, then $L_2(s)$ is continued to an entire function. Hence $L_2(s)$ satisfies the condition (Y) in Theorem 1.2 with $l = 0$. The proof of the analytic continuation of $L_2(s)$ is based on the Euler–Maclaurin summation formula and much technique. In fact, the following expression of $L_2(s)$ can be obtained (see [**3**] for details): for $\sigma > 1 - \frac{M}{2}$,

$$\begin{aligned} L_2(s) &= \frac{1}{q^{2s}} \sum_{a_1=1}^{q} \sum_{a_2=1}^{q} \chi_1(a_1)\chi_2(a_2) \\ &\quad \times \Bigg(\frac{1}{s-1}\zeta(2s-1, a_1/q) + \sum_{m=0}^{M-1} \frac{\widetilde{B}_{m+1}(\frac{a_1-a_2}{q})}{(m+1)!}(s)_m \zeta(2s+m, a_1/q) \\ &\qquad - \sum_{m_1=0}^{\infty} \frac{1}{(m_1+a_1/q)^s} \frac{(s)_M}{M!} \int_{m_1+\frac{a_1-a_2}{q}}^{\infty} \frac{\widetilde{B}_M(x)}{(x+a_2/q)^{s+M}} dx \Bigg), \end{aligned}$$

where $\zeta(s, a)$ is the Hurwitz zeta-function and $\widetilde{B}_M(x) = B_M(x - [x])$. Using this expression, we have for any nonnegative integer m that

$$L_2(-m - \frac{1}{2} + it) \ll (1 + |t|)^{4+2m}.$$

Hence $L_2(s)$, where $\chi_1\chi_2(-1) = 1$ and χ_1, χ_2 are primitive characters, satisfies the conditions (Y), (A1), and (A2) in Theorem 1.2, and consequently, there exists a constant J such that the function $\sum_{n\leqslant x} h_2(n) - J$ is good oscillation. It seems difficult to give a proof of this directly from (5.1).

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