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NOTE ON A QUESTION OF REINHOLD BAER ON PREGROUPS II

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Abstract. Reinhold Baer asked the relationship between certain properties in a nonempty set P with a partial operation (called an "add" by Baer [1]). The first paper in our sequence [Paper I] answered his question for a special type of an add called a pregroup by Stallings [12]. This paper [Paper II] answers an analogous question for a wider class of adds.

1. Introduction

Let P be a nonempty set with a partial operation, called an "add" by Baer [1]. Formally, a partial operation on P is a mapping $m : D \to P$ where $D \subseteq P \times P$. If (a, b) belongs to D, we denote $m(a, b)$ by ab and say that ab is *defined* or *exists*. [Baer [1] denoted $m(a, b)$ by $a + b$.] We also say that a sequence $X = [a_1, a_2, \ldots, a_n]$ is *defined* if each pair $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n$ is defined. By a *triple* in X, we mean a subsequence $[a_i, a_{i+1}, a_{i+2}]$. The *universal group* $G(P)$ of an add P is the group with presentation: $G(P) = \mathrm{gp}(P; \text{operation } m)$ That is, P is the set of generators, and the defining relations are of the form $ab = c$ where $m(a, b) = c$. P is said to be *group-embeddable* or simply *embeddable* if P can be embedded in its universal group $G(P)$.

Next follows classical examples of embeddable adds.

Example 1.1. Let K and L be groups with isomorphic subgroups A or, equivalently, which intersect in a subgroup A. Then the amalgam $P = K \cup_A L$ is an add which is embeddable in $G(P) = K *_A L$, the free product of K and L with A amalgamated.

Example 1.2. Let K, H, L be groups. Suppose K and H have isomorphic subgroups A, and suppose H and L have isomorphic subgroups B. Then the amalgam $P = K \cup_A H \cup_B L$ is an add which is embeddable in $G(P) = K *_A H *_B L$, the free product of K, H, L with subgroups A and B amalgamated.

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Example 1.3. Let $T = (K_i; A_{rs})$ be a tree graph of groups with vertex groups K_i, and with edge groups A_{rs}. [Here A_{rs} is a subgroup of vertex groups K_r and K_s.] Let $P = \bigcup_i (K_i; A_{rs})$, the amalgam of the groups in T. Then P is an add which is embeddable in $G(P) = *(K_i; A_{rs})$, the tree product of the vertex groups K_i with the A_{rs} amalgamated.

Example 1.4. Let $G = (K_i; A_{rs})$ be a graph of groups with vertex groups K_i and with edge groups A_{rs}. Again A_{rs} is a subgroup of vertex groups K_r and K_s. Let $P = \bigcup_i (K_i; A_{rs})$. Then P is an add but, when the graph is not a tree, P need not be embeddable in $G(P) = *(K_i; A_{rs})$, the free product of groups K_i with the A_{rs} amalgamated. In fact, there are examples where $G(P) = \{e\}$.

Let P be an add. Then P will be called a pree if it satisfies the following three axioms of Stallings [12]:

[P1] (Identity) There exists $1 \in P$ such that for all a, we have $1a$ and $a1$ are defined and $1a = a1 = a$.

[P2] (Inverses) For each $a \in P$, there exists a^{-1} in P such that aa^{-1} and $a^{-1}a$ are defined,and $aa^{-1} = a^{-1}a = 1$.

[P4] (Weak Associative Law) If ab and bc are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case $(ab)c = a(bc)$. [We then say that the triple $abc = (ab)c = a(bc)$ is defined.]

Remark. Stallings also listed the following axiom:

[P3] If ab is defined, then $b^{-1}a^{-1}$ is defined and $(ab)^{-1} = b^{-1}a^{-1}$.

However, one can show that [P3] follows from [P1], [P2], and [P4] (see [3]). Thus [P3] is true in a pree P.

The following is also true in a pree P (See, e.g., Paper I [7]):

Proposition. *If ab is defined, then $(ab)b^{-1}$ is defined and $(ab)b^{-1} = a$. Dually, if ab is defined, then $a^{-1}(ab)$ is defined and $a^{-1}(ab) = b$.*

Each add P in the above examples are prees. Example 1.4 shows that a pree P need not be embeddable in its universal group $G(P)$.

Stallings [12] invented the name “pregroup" for a pree P which also satisfies the following axiom:

[T1]=[P5] If ab, bc, and cd are defined, then abc or bcd is defined.

Theorem A. (Stallings, 1971) *A pregroup P is embedded in $G(P)$.*

We note that a pregroup is a generalization of the add in Example 1.1, but not of the add $P = K \cup_A H \cup_B L$ in Example 1.2. For example, let $x \in K \setminus A$, $y \in L \setminus B$, $a \in A$, $b \in B$. Then xa, ab, and by are defined in P, but neither xab nor aby need be defined. However the add P in Example 1.2 does satisfies the axiom:

[T2] If ab, bc, cd, de are defined, then abc, bcd, or cde is defined.

Notation: Let A be a set of axioms for an add P. We will let A-pree denote a pree P which also satisfies the axioms A. Thus a pregroup is a T1-pree.

[We note that in Paper I [10], the term pree was used synonymously for an add, and hence a pree did not include axioms [P1], [P2], [P4]. However here a pree P denotes an add which does satisfy axioms [P1], [P2], [P4]. Also, in Paper I, we denoted an A-pree by A-pregroup.]

Consider now Baer's Postulate XI (Consists of three parts):

(a) If ab, bc, cd exist, then $a(bc)$ or $(bc)d$ exist.
(b) If bc, cd and $a(bc)$ exist, then ab or $(bc)d$ exist.
(c) If ab, bc and $(bc)d$ exist, then $a(bc)$ or cd exist.

Baer then states: "In certain instances it is possible to deduce properties (b), (c) from (a); but whether or not this is true in general, the author does not know."

The content of the following, given in four parts, appears in Paper I [10]; the first two parts answer Baer's question.

Theorem T1. *In a pree P, axiom* T1 *is equivalent to each of the following axioms:*

(1) [B1-1] *If bc, cd, $a(bc)$ are defined, then ab or $(bc)d$ is defined.*
(2) [B1-2] *If ab, bc, $(bc)d$ are defined, then $a(bc)$ or cd is defined.*
(3) [B1-3] *If ab, $(ab)c$, $((ab)c)d$ are defined, then bc or cd is defined.*
(4) [B1-4] *If cd, $b(cd)$, $a(b(cd))$ are defined, then ab or bc is defined*

Note [T1] is Baer's Part (a), [B1-1] is Baer's Part (b) and [B1-2] is Baer's Part (c).

Here we generalize Theorem T1 using the axiom [T2] instead of [T1].

Theorem T2. *In a pree P, axiom* [T2] *is equivalent to each of the following axioms:*

(1) [B2-1] *If bc, cd, $a(bc)$, $(cd)e$ are defined, then ab, $(bc)d$, or de is defined.*
(2) [B2-2] *If ab, $(ab)c$, de, $c(de)$ are defined, then bc, cd, or $(ab)c(de)$ is defined.*

2. Proof of Theorem T2

First we restate axioms [T2], [B2-1], [B2-2] using different letters. Here a_i and b_i are in the pree P.

[T2] If $X = [a_1, a_2, a_3, a_4, a_5]$ is defined, then a triple in X is defined.

[B2-1] If b_2b_3, b_3b_4, $b_1(b_2b_3)$, $(b_3b_4)b_5$ are defined, then b_1b_2, $(b_2b_3)b_4$, or b_4b_5 is defined.

[B2-2] If b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$ are defined, then b_2b_3, b_3b_4, or $(b_1b_2)b_3(b_4b_5)$ is defined.

Proof that [T2] and [B2-1] are equivalent. (1) Assume [T2] holds. Suppose the hypothesis of [B2-1] holds, that is, suppose $b_2b_3, b_3b_4, b_1(b_2b_3), (b_3b_4)b_5$ are defined. Let $a_1 = b_1$, $a_2 = b_2b_3$, $a_3 = b_3^{-1}$, $a_4 = b_3b_4$, $a_5 = b_5$. Then the hypothesis of [T2] holds, that is, $[a_1, a_2, a_3, a_4, a_5]$ is defined. By [T2], one of the following is defined: $a_1a_2a_3 = b_1b_2$, $a_2a_3a_4 = (b_2b_3)b_4$, or $a_3a_4a_5 = b_4b_5$. This is the conclusion of [B2-1]. Thus [T2] implies [B2-1].

(2) Assume [B2-1] holds. Suppose the hypothesis of [T2] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5]$ is defined. Let $b_1 = a_1$, $b_2 = a_2a_3$, $b_3 = a_3^{-1}$, $b_4 = a_3a_4$, $b_5 = a_5$.

Then the hypothesis of [B2-1] holds, that is, b_2b_3, b_3b_4, $b_1(b_2b_3)$, $(b_3b_4)b_5$ are defined. By [B2-1], one of the following is defined: $b_1b_2 = a_1a_2a_3$, $(b_2b_3)b_4 = a_2a_3a_4$, or $b_4b_5 = a_3a_4a_5$ This is the conclusion of [T2]. Thus [B2-1] implies [T2].

By (1) and (2), [T2] and [B2-1] are equivalent in a pree P. □

Proof that [T2] and [B2-2] are equivalent. (1) Assume [T2] holds. Suppose the hypothesis of [B2-2] holds, that is, suppose b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$ are defined. Let $a_1 = b_1^{-1}$, $a_2 = b_1b_2$, $a_3 = b_3$, $a_4 = b_4b_5$, $a_5 = b_5^{-1}$. Then the hypothesis of [T2] holds, that is, $[a_1, a_2, a_3, a_4, a_5]$ is defined. By [T2], one of the following is defined: $a_1a_2a_3 = b_2b_3$, $a_2a_3a_4 = (b_1b_2)b_3(b_4b_5)$, or $a_3a_4a_5 = b_3b_4$. This is the conclusion of [B2-2]. Thus [T2] implies [B2-2].

(2) Assume [B2-2] holds. Suppose the hypothesis of [T2] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5]$ is defined. Let $b_1 = a_1^{-1}$, $b_2 = a_1a_2$, $b_3 = a_3$, $b_4 = a_4a_5$, $b_5 = a_5^{-1}$. Then the hypothesis of [B2-2] holds, that is, b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$ are defined. By [B2-2], one of the following is defined: $b_2b_3 = a_1a_2a_3$, $b_3b_4 = a_3a_4a_5$, or $(b_1b_2)b_3(b_4b_5) = a_2a_3a_4$ This is the conclusion of [T2]. Thus [B2-2] implies [T2].

By (1) and (2), [T2] and [B2-1] are equivalent in a pree P. □

Accordingly, Theorem T2 is proved.

3. Previous Results

Many authors have generalized the Stallings pregroup [T1-pree] by giving a weaker set of axioms than [P5]=[T1] which also guarantees that a pree P is embeddable in $G(P)$. First we restate these axioms, which also appear in Paper I, and then we restate the relevant Theorem B which also appears in Paper I.

[Sn, $n \geqslant 4$] (Baer 1953) Suppose $a_1a_2^{-1}, a_2a_3^{-1}, \ldots, a_na_1^{-1}$ are defined. Then, for some i, $a_ia_{i+2}^{-1}$ (mod n) is defined.

[K] (Kushner 1988) If ab, bc, cd and $(ab)(cd)$ are defined, then abc or bcd is defined.

[Tn] (Kushner and Lipschutz 1993) If $X = [a_1, a_2, \ldots, a_{n+3}]$ is defined, then a triple in X is defined.

[L] (Lipschutz 1994) Suppose ab, bc, cd are defined, but $[ab, cd]$ and $[a, bc, d]$ are reduced. If $(ab)z$ and $z^{-1}(cd)$ are defined, then bz and $z^{-1}c$ are defined.

[M] (Baer (1950 and Lipschutz 1994) Equivalent fully reduced words have the same length.

We note that the axiom [Tn] holds in the tree pree P in Example 1.3 when the tree has diameter $\leqslant n$. Thus [T2] holds for a star graph, that is, a graph of diameter 2. We also note that Axiom [M] is analogous to the following axiom of Baer [1, page 684]: "Similar irreducible vectors have the same length".

Theorem B. *Each of the following prees P is embeddable in $G(P)$:*

(1) Sn-*pree* (Baer 1953, [1]);

(2) KT2-*pree* (Kushner and Lipschutz 1988, [7]);

(3) T2-*pree* (Kushner 1978, [6], and Hoare 1992, [4]);

(4) KT3-*pree* (Kushner and Lipschutz 1993, [8];

(5) KLM-*pree* (Lipschutz 1994, [9]);

(6) KL-*pree* $=\mathrm{S}_4\mathrm{S}_5$-*pree* (Gilman 1998, [2], and Hoare 1998, [5]).

We note that Gilman and Hoare proved (6) independently. In fact, Gilman [2] proved (6) using small-cancellation theory, and Hoare [5] proved (6) by showing that [M] follows from [K] and [L].

4. Generalizations

One of the purposes in this paper is to generalize Theorem T2. We have the following axioms:

[T6] Suppose $X = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined. Then a triple in X is defined.

[B6-1] Suppose all the following are defined: (1) b_2b_3, b_3b_4, $b_1(b_2b_3)$, $(b_3b_4)b_5$, (2) b_6b_7, b_7b_8, $b_5(b_6b_7)$, $(b_7b_8)b_9$. Then one of the following is defined: b_1b_2, $(b_2b_3)b_4$, b_4b_5, $(b_3b_4)b_5(b_6b_7)$, b_5b_6, $(b_6b_7)b_8$, or b_8b_9

[B6-2] Suppose all the following are defined: (1) b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$, (2) b_5b_6, $(b_5b_6)b_7$, b_8b_9, $b_7(b_8b_9)$. Then one of the following is defined: b_2b_3, $(b_1b_2)b_3(b_4b_5)$, b_3b_4, $(b_4b_5)b_6$, b_6b_7, $(b_5b_6)b_7(b_8b_9)$, or b_7b_8,

(Note that (2), in both cases, can be obtained from (1) by adding 4 to each subscript.)

Theorem T6. (1) *In a pree P, axiom* [T6] *is equivalent to* [B6-1]. (2) *In a pree P, axiom* [T6] *is equivalent to* [B6-2].

Proof of Theorem T6(1). (1) Proof that [T6] implies [B6-1]. Assume [T6] holds. Suppose the hypothesis of [B6-1] holds, that is, the following are defined: $b_2b_3, b_3b_4, b_1(b_2b_3), (b_3b_4)b_5, b_6b_7, b_7b_8, b_5(b_6b_7), (b_7b_8)b_9$ Let $a_1 = b_1$, $a_2 = b_2b_3$, $a_3 = b_3^{-1}$, $a_4 = b_3b_4$, $a_5 = b_5$, $a_6 = b_6b_7$, $a_7 = b_7^{-1}$, $a_8 = b_7b_8$, $a_9 = b_9$. Then the hypothesis of [T6] holds, that is, $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined. By [T6], one of the following is defined: $a_1a_2a_3 = b_1b_2$, $a_2a_3a_4 = (b_2b_3)b_4$, $a_3a_4a_5 = b_4b_5$, $a_4a_5a_6 = (b_3b_4)b_5(b_6b_7)$, $a_5a_6a_7 = b_5b_6$, $a_6a_7a_8 = (b_6b_7)b_8$, or $a_7a_8a_9 = b_8b_9$. This is the conclusion of [B6-1]. Thus [T6] implies [B6-1].

(2) Proof that [B6-1] implies [T6]. Assume [B6-1] holds. Suppose the hypothesis of [T6] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined. Let $b_1 = a_1$, $b_2 = a_2a_3$, $b_3 = a_3^{-1}$, $b_4 = a_3a_4$, $b_5 = a_5$, $b_6 = a_6a_7$, $b_7 = a_7^{-1}$, $b_8 = a_7a_8$, $b_9 = a_9$. Then the hypothesis of [B6-1] holds, that is, the following are defined: b_2b_3, b_3b_4, $b_1(b_2b_3)$, $(b_3b_4)b_5$, b_6b_7, b_7b_8, $b_5(b_6b_7)$, $(b_7b_8)b_9$. By [B6-1], one of the following is defined: $b_1b_2 = a_1a_2a_3$, $(b_2b_3)b_4 = a_2a_3a_4$, $b_4b_5 = a_3a_4a_5$, $(b_3b_4)b_5(b_6b_7) = a_4a_5a_6$, $b_5b_6 = a_5a_6a_7$, $(b_6b_7)b_8 = a_6a_7a_8$, or $b_8b_9 = a_7a_8a_9$. This is the conclusion of [T2]. Thus [B2-1] implies [T2].

By (1) and (2), Theorem T6(1) is proved. □

Proof of Theorem T6(2). (1) Proof that [T6] implies [B6-2]. Assume [T6] holds. Suppose the hypothesis of [B6-2] holds, that is, that the following are defined: b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$, b_5b_6, $(b_5b_6)b_7$, b_8b_9, $b_7(b_8b_9)$. Let $a_1 = b_1^{-1}$, $a_2 = b_1b_2$, $a_3 = b_3$, $a_4 = b_4b_5$, $a_5 = b_5^{-1}$, $a_6 = b_5b_6$, $a_7 = b_7$, $a_8 = b_8b_9$, $a_9 = b_9^{-1}$. Then the hypothesis of [T6] holds, that is, $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined. By [T6], one of the following is defined: $a_1a_2a_3 = b_2b_3$, $a_2a_3a_4 = (b_1b_2)b_3(b_4b_5)$,

$a_3a_4a_5 = b_3b_4$, $a_4a_5a_6 = (b_4b_5)b_6$, $a_5a_6a_7 = b_6b_7$, $a_6a_7a_8 = (b_5b_6)b_7(b_8b_9)$, or $a_7a_8a_9 = b_7b_8$. This is the conclusion of [B6-2]. Thus [T6] implies [B6-2].

(2) Proof that [B6-2] implies [T6]. Assume [B6-2] holds. Suppose the hypothesis of [T6] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined. Let $b_1 = a_1^{-1}$, $b_2 = a_1a_2$, $b_3 = a_3$, $b_4 = a_4a_5$, $b_5 = a_5^{-1}$, $b_6 = a_5a_6$, $b_7 = a_7$, $b_8 = a_8a_9$, $b_9 = a_9^{-1}$. Then the hypothesis of [B6-2] holds, that is, the following are defined: b_1b_2, $(b_1b_2)b_3$, b_4b_5, $b_3(b_4b_5)$, b_5b_6, $(b_5b_6)b_7$, b_8b_9, $b_7(b_8b_9)$. By [B6-2], one of the following is defined: $b_2b_3 = a_1a_2a_3$, $(b_1b_2)b_3(b_4b_5) = a_2a_3a_4$, $b_3b_4 = a_3a_4a_5$, $(b_4b_5)b_6 = a_4a_5a_6$, $b_6b_7 = a_5a_6a_7$, $(b_5b_6)b_7(b_8b_9) = a_6a_7a_8$, or $b_7b_8 = a_7a_8a_9$. This is the conclusion of [T6]. Thus [B6-2] implies [T]. By (1) and (2), Theorem T6(2) is proved. □

Accordingly, Theorem T6 is proved.

5. Questions

We have shown that the proof of Theorem T6 is very similar to the proof of Theorem T2. Likely, one can prove an analogous Theorem Tm where m≡2 (mod 4).

Question 1. Find a generalization of T2 for other Tm, especially [T3], [T4], and [T5].

The following transitive order relation on a pregroup P is due to Stallings (see [11]). Let $L(x) = \{a \in P; ax \text{ is defined.}\}$. Put $x \leqslant y$ if $L(y) \subseteq L(x)$, and put $x < y$ if $L(y) \subseteq L(x)$ and $L(y) \neq L(x)$. The following theorem is due to Hoare [5] and Rimlinger [11].

Theorem C. *The following conditions on a pree P are equivalent:*
(1) [T1] *If $X = [w, x, y, z]$ is defined, then wxy or xyz is defined.*
(2) *If $x^{-1}a$ and $a^{-1}y$ are defined but $x^{-1}y$ is not defined, then $a < x$ and $a < y$.*
(3) *If $x^{-1}y$ is defined, then $x \leqslant y$ or $y \leqslant x$.*

Question 2. Find an analogous Theorem C2 for the axiom [T2].

We note that the following axioms are a direct generalization of axioms B1-3 and B1-4. [B2-3] If ab, $(ab)c$, $((ab)c)d$, $(((ab)c)d)e$ are defined, then bc, cd, or de is defined. [B2-4] If de, $c(de)$, $b(c(de))$, $a(b(c(de)))$ are defined, then ab, bc, or cd is defined.

Question 3. What role, if any, do the axioms [B2-3] and [B2-4], and the analogous axioms [Bn-3] and [Bn-4], play in the embedding of a pree P in its universal group $G(P)$?

Question 4. Find alternate generalizations of [B2-3] and [B2-4], and the role they would play in the embedding of a pree P in its universal group $G(P)$.

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