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# ORDERING CACTI WITH $n$ VERTICES AND $k$ CYCLES BY THEIR LAPLACIAN SPECTRAL RADII

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Abstract. A graph is a cactus if any two of its cycles have at most one common vertex. In this paper, we determine the first sixteen largest Laplacian spectral radii together with the corresponding graphs among all connected cacti with $n$ vertices and $k$ cycles, where $n \geqslant 2k + 8$.

## 1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E$. Denote by $d(v_i)$ the degree of the vertex $v_i$ of $G$. Let $A(G)$ be the adjacency matrix and $L(G) = D(G) - A(G)$ the Laplacian matrix of the graph $G$, where $D(G) = \mathrm{diag}(d(v_1), d(v_2), \dots, d(v_n))$ denotes the diagonal matrix of vertex degrees of $G$. It is easy to see that $L(G)$ is a singular, semi-positive, symmetric matrix and its rows sum to 0. Denote its eigenvalues by

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

which are always enumerated in non-increasing order. We denote the largest eigenvalue $\mu_1(G)$ of $L(G)$ by $\mu(G)$ and call it the Laplacian spectral radius of $G$. Also, let $\phi(G, \lambda)$ be the characteristic polynomial of $G$, i.e., $\phi(G, \lambda) = \det(\lambda I - L(G))$.

There are a lot of relations between the Laplacian spectral radius and numerous graph invariants, and the Laplacian spectral radius of a graph has numerous applications in theoretical chemistry, combinatorial optimization, communication networks, etc. For related reference, one may see [14]. Besides, from the known fact that $\mu_1(G) + \mu_{n-1}(\overline{G}) = n$, we can see that the Laplacian spectral radius is of relevance to the algebraic connectivity of a graph which is a good parameter to measure how well a graph is connected. There is a good deal work on algebraic

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connectivity for graphs (see [1] and references therein), and particular attention has been paid to algebraic connectivity for unicyclic graphs (see [3] for example).

Recently, the problem concerning graphs with maximal or minimal Laplacian spectral radius of a given class of graphs has been studied extensively. For related results, one may refer to [4–13] and the references therein.

A graph is a cactus, or a treelike graph, if any two of its cycles have at most one common vertex. Cacti have been studied by several authors, for example, one may see [2,15]. Let $\mathcal{T}_{n,k}$ denote the set of all connected cacti with $n$ vertices and $k$ cycles.

When $k = 1$, $\mathcal{T}_{n,1}$ is the set of all unicyclic graphs of order $n$. Let $T_i$ ($i = 1–4, 6, 8–11, 13–19$) be the unicyclic graphs with $n$ vertices shown in Fig. 2. Let $T \in \mathcal{T}_{n,1}$ and $T \notin \{T_1, T_2, T_3, T_4\}$. Guo [7] proved that

$$n = \mu(T_1) > \mu(T_2) > \mu(T_3) > \mu(T_4) > \mu(T).$$

Let $T \in \mathcal{T}_{n,1}$ and $T \notin \{T_1, T_2, T_3, T_4, T_6, T_8, T_9, T_{10}, T_{11}, T_{17}\}$. Liu, Shao and Yuan [11] proved that

$$\mu(T_6) > \mu(T_8) > \mu(T_9) > \mu(T_{10}) > \mu(T_{11}) = \mu(T_{17}) > \mu(T).$$

Let $T \in \mathcal{T}_{n,1}$ and $T \notin \{T_1, T_2, T_3, T_4, T_6, T_8, T_9, T_{10}, T_{11}, T_{17}, T_{13}–T_{16}, T_{18}, T_{19}\}$. Liu and Liu [12] proved that

$$\mu(T_{13}) > \mu(T_{14}) = \mu(T_{18}) = \mu(T_{19}) > \mu(T_{15}) > \mu(T_{16}) > \mu(T).$$

In this paper, we consider the cases when $k \geqslant 2$, and determine the first sixteen largest Laplacian spectral radii together with the corresponding graphs in $\mathcal{T}_{n,k}$.

## 2. Preliminaries

For $v \in V(G)$, $N(v)$ denotes the set of all neighbors of vertex $v$ in $G$. Then $d(v) = |N(v)|$. $\Delta$ denotes the maximum degree of $G$ and $\delta$ denotes the minimum degree of $G$. Let $G_1, G_2, \dots, G_{20}$ be the cacti with $n$ vertices and $k$ cycles shown in Fig. 1.

Lemma 2.1 (13). *We have $\mu(G) \leqslant \max\{d(v) + m(v) : v \in V\}$, where $m(v) = \sum_{u \in N(v)} d(u)/d(v)$.*

Lemma 2.2. *Suppose $k \geqslant 1$ and $G \in \mathcal{T}_{n,k}$ with $\Delta \leqslant n - 4$. If $n \geqslant 2k + 8$, then $\mu(G) \leqslant n - 2$.*

Proof. By Lemma 2.1, we only need to prove that $\max\{d(v) + m(v) : v \in V(G)\} \leqslant n - 2$. Suppose $\max\{d(v) + m(v) : v \in V(G)\}$ occurs at the vertex $u$. We consider the following three cases.

**Case 1.** $d(u) = 1$. Suppose $v \in N(u)$, then

$$d(u) + m(u) = d(u) + d(v) \leqslant 1 + n - 4 < n - 2.$$

**Case 2.** $d(u) = 2$. Suppose $v, w \in N(u)$. Note that $G$ is a cactus, then $|N(v) \cap N(w)| \leqslant 2$ and $|N(v) \cup N(w)| \leqslant n$. Therefore

$$d(u) + m(u) = 2 + \frac{d(v) + d(w)}{2} \leqslant 2 + \frac{n + 2}{2} \leqslant n - 2.$$

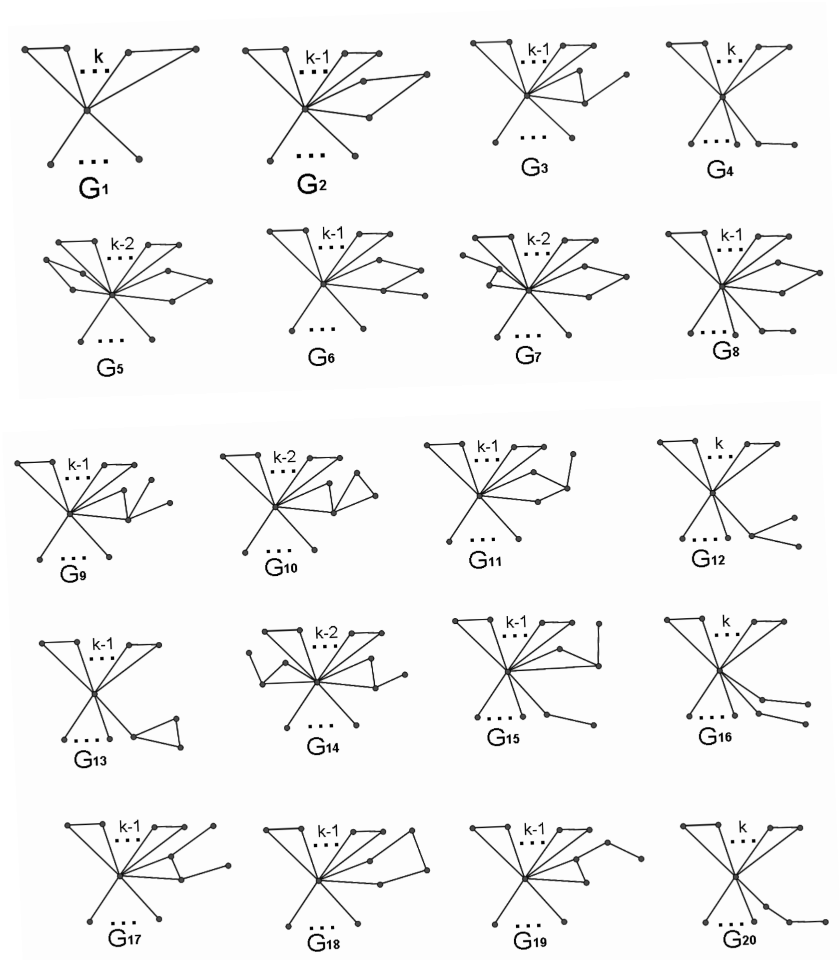


Fig. 1

**Case 3.** $3 \leqslant d(u) \leqslant n-4$. Note that $3 \leqslant d(u) \leqslant n-4$, then

$$d(u) + m(u) \leqslant d(u) + \frac{2m - d(u) - 3}{d(u)} = d(u) - 1 + \frac{2m-3}{d(u)}.$$

Next we shall prove that $d(u) - 1 + \frac{2m-3}{d(u)} \leqslant n-2$. Equivalently, $d(u)(n-1-d(u)) \geqslant 2m-3$. Once this is proved, we are done. Let $f(x) = (n-1-x)x$.

When $x \in [3, \frac{n-1}{2}]$, since $f'(x) = n-1-2x \geqslant 0$, then

$$f(x) \geqslant f(3) = 3(n-4) \geqslant 2(n+k-1) - 3 = 2m-3.$$

When $x \in [\frac{n-1}{2}, n-4]$, since $f'(x) = n-1-2x \leqslant 0$, then

$$f(x) \geqslant f(n-4) = 3(n-4) \geqslant 2(n+k-1) - 3 = 2m-3. \qquad \square$$

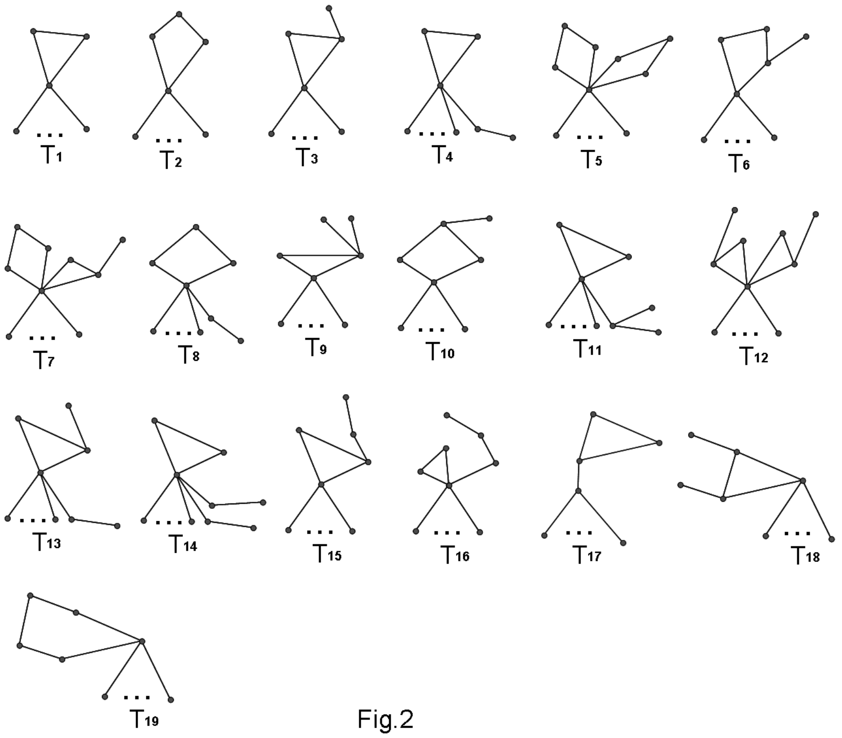


Fig.2

By combing the above discussion, the conclusion follows.

LEMMA 2.3 (4). *Let $v$ be a vertex in a connected graph $G$ and suppose that $s$ ($s \geqslant 2$) new paths (with equal length $k$) $P_i : vv_{ik}v_{i(k-1)} \dots v_{i1}$, ($i = 1, 2, \dots, s$; $k \geqslant 1$) are attached to $G$ at $v$, respectively, to form a new graph $G_s^k$, where $v_{ik}, v_{i(k-1)}, \dots, v_{i1}$, ($i = 1, 2, \dots, s$) are distinct new vertices. Let $G^k_{s;t_1,t_2,\dots,t_k}$ be the graph obtained from $G_s^k$ by adding $t_i$ ($0 \leqslant t_i \leqslant \frac{s(s-1)}{2}$) edges among vertices $v_{1i}, v_{2i}, \dots, v_{si}(1 \leqslant i \leqslant k)$, respectively. If $\Delta(G_s^k) \geqslant s+3$, then we have $\mu(G^k_{s;t_1,t_2,\dots,t_k}) = \mu(G_s^k)$.*

Let $T_1, T_2, \dots, T_{16}$ be the graphs with $n$ vertices shown in Fig. 2. Then from Lemma 2.3, we can obtain easily the following lemma.

LEMMA 2.4. *$\mu(G_1) = \mu(T_1)$, $\mu(G_2) = \mu(T_2)$, $\mu(G_3) = \mu(T_3)$, $\mu(G_4) = \mu(T_4)$, $\mu(G_5) = \mu(T_5)$, $\mu(G_6) = \mu(T_6)$, $\mu(G_7) = \mu(T_7)$, $\mu(G_8) = \mu(T_8)$, $\mu(G_9) = \mu(G_{10}) = \mu(T_9)$, $\mu(G_{11}) = \mu(T_{10})$, $\mu(G_{12}) = \mu(G_{13}) = \mu(T_{11})$, $\mu(G_{14}) = \mu(T_{12})$, $\mu(G_{15}) = \mu(T_{13})$, $\mu(G_{16}) = \mu(G_{17}) = \mu(G_{18}) = \mu(T_{14})$, $\mu(G_{19}) = \mu(T_{15})$, $\mu(G_{20}) = \mu(T_{16})$.*

LEMMA 2.5 (6). *Let $G$ be a connected graph on $n$ vertices with at least one edge; then $\mu(G) \geqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph $G$, with equality if and only if $\Delta(G) = n - 1$.*

LEMMA 2.6. *Let $n \geqslant 10$. Then $n = \mu(T_1) > \mu(T_2) > \mu(T_3) > \mu(T_4) > \mu(T_5) > \mu(T_6) > \mu(T_7) > \mu(T_8) > \mu(T_9) > \mu(T_{10}) > \mu(T_{11}) > \mu(T_{12}) > \mu(T_{13}) > \mu(T_{14}) > \mu(T_{15}) > \mu(T_{16})$.*

PROOF. From [7] we have $n = \mu(T_1) > \mu(T_2) > \mu(T_3) > \mu(T_4)$, and $\mu(T_i)$ is the largest root of the equation $h_i(\lambda) = 0$ $(i = 2, 3, 4)$, respectively, where

$$\begin{aligned}
h_2(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+4)\lambda^2 - (10n-4)\lambda + 4n,\\
h_3(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+3)\lambda^2 - (9n-5)\lambda + 3n,\\
h_4(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+4)\lambda^2 - (10n-6)\lambda + 3n.
\end{aligned}$$

It is not difficult to calculate recursively that

$$\begin{aligned}
\phi(T_5;\lambda) &= \lambda(\lambda-1)^{n-8}(\lambda^3 - 6\lambda^2 + 10\lambda - 4)h_5(\lambda),\\
\phi(T_6;\lambda) &= \lambda(\lambda-1)^{n-5}h_6(\lambda),\\
\phi(T_7;\lambda) &= \lambda(\lambda-1)^{n-8}h_7(\lambda),\\
\phi(T_8;\lambda) &= \lambda(\lambda-2)(\lambda-1)^{n-7}h_8(\lambda),\\
\phi(T_9;\lambda) &= \lambda(\lambda-1)^{n-5}h_9(\lambda),\\
\phi(T_{10};\lambda) &= \lambda(\lambda-2)(\lambda-1)^{n-6}h_{10}(\lambda),\\
\phi(T_{11};\lambda) &= \lambda(\lambda-1)^{n-5}h_{11}(\lambda),\\
\phi(T_{12};\lambda) &= \lambda(\lambda-1)^{n-8}h_{12}(\lambda),\\
\phi(T_{13};\lambda) &= \lambda(\lambda-1)^{n-7}h_{13}(\lambda),\\
\phi(T_{14};\lambda) &= \lambda(\lambda-3)(\lambda-1)^{n-7}(\lambda^2 - 3\lambda + 1)h_{14}(\lambda),\\
\phi(T_{15};\lambda) &= \lambda(\lambda-1)^{n-6}h_{15}(\lambda),\\
\phi(T_{16};\lambda) &= \lambda(\lambda-3)(\lambda-1)^{n-6}h_{16}(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
h_5(\lambda) &= \lambda^4 - (n+4)\lambda^3 + (6n-2)\lambda^2 - (10n-12)\lambda + 4n,\\
h_6(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (7n-1)\lambda^2 - (13n-19)\lambda + 4n,\\
h_7(\lambda) &= \lambda^7 - (n+10)\lambda^6 + (12n+31)\lambda^5 - (55n+14)\lambda^4 + (121n-85)\lambda^3\\
&\qquad - (132n-128)\lambda^2 + (66n-44)\lambda - 12n,\\
h_8(\lambda) &= \lambda^5 - (n+5)\lambda^4 + (7n+1)\lambda^3 - (15n-17)\lambda^2 + (10n-8)\lambda - 2n,\\
h_9(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (7n-3)\lambda^2 - (11n-17)\lambda + 3n,\\
h_{10}(\lambda) &= \lambda^4 - (n+4)\lambda^3 + (6n-4)\lambda^2 - (8n-12)\lambda + 2n,\\
h_{11}(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (7n-1)\lambda^2 - (13n-21)\lambda + 3n,\\
h_{12}(\lambda) &= \lambda^7 - (n+10)\lambda^6 + (12n+30)\lambda^5 - (54n+8)\lambda^4 + (114n-93)\lambda^3\\
&\qquad - (117n-126)\lambda^2 + (54n-39)\lambda - 9n,
\end{aligned}$$

$$\begin{aligned}h_{13}(\lambda) &= \lambda^6 - (n+7)\lambda^5 + (9n+10)\lambda^4 - (28n-18)\lambda^3 + (36n-42)\lambda^2\\ &\quad - (18n-14)\lambda + 3n,\\ h_{14}(\lambda) &= \lambda^3 - (n+1)\lambda^2 + (3n-5)\lambda - n,\\ h_{15}(\lambda) &= \lambda^5 - (n+6)\lambda^4 + (8n+4)\lambda^3 - (20n-22)\lambda^2 + (17n-26)\lambda - 3n,\\ h_{16}(\lambda) &= \lambda^4 - (n+3)\lambda^3 + (5n-4)\lambda^2 - (6n-10)\lambda + n.\end{aligned}$$

Since $n \geqslant \mu(T_i) > \Delta(T_i) + 1 \geqslant n-2 \geqslant 8$, it follows that $\mu(T_i)$ is the largest root of the equation $h_i(\lambda) = 0 (i = 5, \dots, 16)$ respectively. Next we shall divide the proof into the next 11 steps.

(1) $\mu(T_4) > \mu(T_5)$. It is easy to see that $h_4(\lambda) = h_5(\lambda) - (\lambda^2(\lambda-6) + 6\lambda + n)$ and $h_4(\mu(T_5)) < 0$. So we have $\mu(T_4) > \mu(T_5)$.

(2) $\mu(T_5) > \mu(T_6)$. Note that $h_5(\lambda) = h_6(\lambda) + \gamma_1(\lambda)$, where

$$\gamma_1(\lambda) = \lambda(\lambda^2 - (n+1)\lambda + 3n - 7).$$

Let $\alpha_1$ denote the maximum root of $\gamma_1(\lambda) = 0$. Since $\gamma_1(-1) < 0$, $\gamma_1(2) > 0$, $\gamma_1(n-2) < 0$, $\gamma_1(n-1) > 0$, it is easy to see that $n-2 < \alpha_1 < n-1$.

Note that $h_6(\lambda) = (\lambda-4)\gamma_1(\lambda) + (2\lambda^2 - (n+9)\lambda + 4n)$, then $h_6(\lambda) > 0$ for $\lambda \geqslant \alpha_1$. Thus $\mu(T_6) < \alpha_1$. Note that $\lim_{\lambda\to\infty} \gamma_1(\lambda) = +\infty$, thus $h_5(\mu(T_6)) = \gamma_1(\mu(T_6)) < 0$. This implies that $\mu(T_5) > \mu(T_6)$.

(3) $\mu(T_6) > \mu(T_7)$. It is easy to see that $h_7(\lambda) = \gamma_2(\lambda) h_6(\lambda) + \gamma_3(\lambda)$, where

$$\gamma_2(\lambda) = (\lambda-1)^2(\lambda-3), \quad \gamma_3(\lambda) = \lambda(2\lambda^2 - 8\lambda - n + 13).$$

Since $\mu(T_7) > n-2 \geqslant 8$, it follows that $\gamma_2(\mu(T_7)) > 0$ and $\gamma_3(\mu(T_7)) > 0$. This implies that $h_6(\mu(T_7)) < 0$. Thus $\mu(T_6) > \mu(T_7)$.

(4) $\mu(T_7) > \mu(T_8)$. It is easy to see that $h_7(\lambda) = (\lambda^2 - 5\lambda + 5)h_8(\lambda) - \gamma_4(\lambda)$, where

$$\begin{aligned}\gamma_4(\lambda) &= \lambda^4 - (n+3)\lambda^3 + (5n-3)\lambda^2 - (6n-4)\lambda + 2n\\ &= \lambda^2(\lambda^2 - (n+3)\lambda + 5n - 10) + (7\lambda^2 - (6n-4)\lambda + 2n).\end{aligned}$$

It is easy to see that $\gamma_4(\mu(T_8)) > 0$. So we have $h_7(\mu(T_8)) < 0$, thus $\mu(T_7) > \mu(T_8)$.

(5) From [11] we have $\mu(T_8) > \mu(T_9) > \mu(T_{10})$.

(6) $\mu(T_{10}) > \mu(T_{11})$. Note that $h_{10}(\lambda) = h_{11}(\lambda) + \gamma_5(\lambda)$, where

$$\gamma_5(\lambda) = \lambda^3 - (n+3)\lambda^2 + (5n-9)\lambda - n.$$

Let $\alpha_2$ denote the maximum root of $\gamma_5(\lambda) = 0$. Since $\gamma_5(-\infty) < 0$, $\gamma_5(\frac{1}{2}) = \frac{5n}{4} - \frac{41}{8} > 0$, $\gamma_5(n-2) < 0$, $\gamma_5(n-1) > 0$, thus $n-2 < \alpha_2 < n-1$.

Note that $h_{11}(\lambda) = (\lambda-2)\gamma_5(\lambda) + 2\lambda^2 - (2n-3)\lambda + n$, then $h_{11}(\alpha_2) = 2\alpha_2^2 - (2n-3)\alpha_2 + n > 0$ and $h_{11}(\lambda) > 0$ for $\lambda \geqslant \alpha_2$. Thus $\mu(T_{11}(\lambda)) < \alpha_2$. Note that $\lim_{\lambda\to\infty} \gamma_5(\lambda) = +\infty$, thus $h_{10}(\mu(T_{11})) = \gamma_5(\mu(T_{11})) < 0$. This implies that $\mu(T_{10}) > \mu(T_{11})$, because $\lim_{\lambda\to\infty} h_{10}(\lambda) = +\infty$.

(7) $\mu(T_{11}) > \mu(T_{12})$. Note that $h_{12}(\lambda) = \gamma_6(\lambda)h_{11}(\lambda) - \gamma_7(\lambda)$, where

$$\gamma_6(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda - 4, \gamma_7(\lambda) = 2\lambda^3 - (4n-4)\lambda^2 + (16n-45)\lambda - 3n.$$

It is easy to see that $\gamma_6(\mu(T_{12})) > 0$. Note that $\gamma_7(-\infty) < 0$, $\gamma_7(1) > 0$, $\gamma_7(n-1) < 0$, $\gamma_7(n-2) < 0$, $\gamma_7(+\infty) > 0$ and $n-2 < \mu(T_{12}) < n-1$. It follows that $\gamma_7(\mu(T_{12})) < 0$. So we have $h_{11}(\mu(T_{12})) < 0$, thus $\mu(T_{11}) > \mu(T_{12})$.

(8) $\mu(T_{12}) > \mu(T_{13})$. Note that $h_{12}(\lambda) = (\lambda-3)h_{13}(\lambda) - \gamma_8(\lambda)$, where

$$\gamma_8(\lambda) = \lambda^5 - (n+4)\lambda^4 + (6n-3)\lambda^3 - (9n-14)\lambda^2 + (3n-3)\lambda.$$

We also have $h_{12}(\lambda) = (\lambda-3)^2\gamma_8(\lambda) - \gamma_9(\lambda)$, where

$$\begin{aligned}\gamma_9(\lambda) &= 4\lambda^4 - (3n+21)\lambda^3 + (18n+18)\lambda^2 - (27n-12)\lambda + 9n\\ &= 3\lambda^4 - (3n-6)\lambda^3 + \lambda^2(\lambda^2 - 27\lambda + 18n - 15) + 33\lambda^2 - (27n-12)\lambda + 9n.\end{aligned}$$

It is easy to see that $\gamma_9(\lambda) > 0$ for $\lambda \geqslant \mu(T_{12})$. So we have $\gamma_8(\lambda) > 0$ for $\lambda \geqslant \mu(T_{12})$. Thus $h_{13}(\lambda) > 0$ for $\lambda \geqslant \mu(T_{12})$. It follows that $\mu(T_{12}) > \mu(T_{13})$.

(9) $\mu(T_{13}) > \mu(T_{14})$. Note that $h_{13}(\lambda) = (\lambda^3 - 6\lambda^2 + 9\lambda - 3)h_{14}(\lambda) - \lambda$. It is easy to see that $h_{13}(\mu(T_{14})) < 0$. Thus $\mu(T_{13}) > \mu(T_{14})$.

(10) $\mu(T_{14}) > \mu(T_{15})$. Note that $h_{15}(\lambda) = (\lambda^2 - 5\lambda + 4)h_{14}(\lambda) + (\lambda^2 - 6\lambda + n)$. It is easy to see that $h_{14}(\mu(T_{15})) < 0$. Thus $\mu(T_{14}) > \mu(T_{15})$.

(11) $\mu(T_{15}) > \mu(T_{16})$. Note that $h_{15}(\lambda) = (\lambda-3)h_{16}(\lambda) - \lambda(\lambda^2 - n\lambda + 2n - 4)$. It is easy to see that $h_{15}(\mu(T_{16})) < 0$. Thus $\mu(T_{15}) > \mu(T_{16})$.

By combining the above discussion, we have $\mu(T_4) > \mu(T_5) > \mu(T_6) > \mu(T_7) > \mu(T_8) > \mu(T_9) > \mu(T_{10}) > \mu(T_{11}) > \mu(T_{12}) > \mu(T_{13}) > \mu(T_{14}) > \mu(T_{15}) > \mu(T_{16})$. □

## 3. Main results

THEOREM 3.1. *Let $G$ be a connected cactus with $n$ vertices and $k$ cycles, and $G_1, G_2, \dots, G_{20}$ be the cacti with $n$ vertices and $k$ cycles shown in Fig. 1, where $k \geqslant 2$ and $n \geqslant 2k+8$.*

(1) *If $G \notin \{G_1, \dots, G_{20}\}$, then $\mu(G) \leqslant n-2 < \mu(G_{20})$.*

(2) *$n = \mu(G_1) > \mu(G_2) > \mu(G_3) > \mu(G_4) > \mu(G_5) > \mu(G_6) > \mu(G_7) > \mu(G_8) > \mu(G_9) = \mu(G_{10}) > \mu(G_{11}) > \mu(G_{12}) = \mu(G_{13}) > \mu(G_{14}) > \mu(G_{15}) > \mu(G_{16}) = \mu(G_{17}) = \mu(G_{18}) > \mu(G_{19}) > \mu(G_{20})$, and $\mu(G_i)$ is the largest root of the equation $f_i(\lambda) = 0 (i = 2, \dots, 20)$ respectively, where*

$$\begin{aligned}
f_2(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+4)\lambda^2 - (10n-4)\lambda + 4n,\\
f_3(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+3)\lambda^2 - (9n-5)\lambda + 3n,\\
f_4(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (6n+4)\lambda^2 - (10n-6)\lambda + 3n,\\
f_5(\lambda) &= \lambda^4 - (n+4)\lambda^3 + (6n-2)\lambda^2 - (10n-12)\lambda + 4n,\\
f_6(\lambda) &= \lambda^4 - (n+5)\lambda^3 + (7n-1)\lambda^2 - (13n-19)\lambda + 4n,\\
f_7(\lambda) &= \lambda^7 - (n+10)\lambda^6 + (12n+31)\lambda^5 - (55n+14)\lambda^4 + (121n-85)\lambda^3\\
&\qquad - (132n-128)\lambda^2 + (66n-44)\lambda - 12n,\\
f_8(\lambda) &= \lambda^5 - (n+5)\lambda^4 + (7n+1)\lambda^3 - (15n-17)\lambda^2 + (10n-8)\lambda - 2n,\\
f_9(\lambda) &= f_{10}(\lambda) = \lambda^4 - (n+5)\lambda^3 + (7n-3)\lambda^2 - (11n-17)\lambda + 3n,
\end{aligned}$$

$$
\begin{aligned}
f_{11}(\lambda) &= \lambda^4 - (n+4)\lambda^3 + (6n-4)\lambda^2 - (8n-12)\lambda + 2n,\\
f_{12}(\lambda) &= f_{13}(\lambda) = \lambda^4 - (n+5)\lambda^3 + (7n-1)\lambda^2 - (13n-21)\lambda + 3n,\\
f_{14}(\lambda) &= \lambda^7 - (n+10)\lambda^6 + (12n+30)\lambda^5 - (54n+8)\lambda^4 + (114n-93)\lambda^3\\
&\qquad - (117n-126)\lambda^2 + (54n-39)\lambda - 9n,\\
f_{15}(\lambda) &= \lambda^6 - (n+7)\lambda^5 + (9n+10)\lambda^4 - (28n-18)\lambda^3 + (36n-42)\lambda^2\\
&\qquad - (18n-14)\lambda + 3n,\\
f_{16}(\lambda) &= f_{17}(\lambda) = f_{18}(\lambda) = \lambda^3 - (n+1)\lambda^2 + (3n-5)\lambda - n,\\
f_{19}(\lambda) &= \lambda^5 - (n+6)\lambda^4 + (8n+4)\lambda^3 - (20n-22)\lambda^2 + (17n-26)\lambda - 3n,\\
f_{20}(\lambda) &= \lambda^4 - (n+3)\lambda^3 + (5n-4)\lambda^2 - (6n-10)\lambda + n.
\end{aligned}
$$

PROOF. It is easy to check that $G_1, G_2, \dots, G_{20}$ are all the cacti with $n$ vertices, $k$ cycles and maximum degree greater than $n-4$. Then $\Delta(G) \leqslant n-4$ for $G \notin \{G_1, \dots, G_{20}\}$. By Lemma 2.2, we have $\mu(G) \leqslant n-2$. By Lemmas 2.4 and 2.6, $\mu(G_{20}) = \mu(T_{16})$ is the largest root of the equation

$$f_{20}(\lambda) = h_{16}(\lambda) = \lambda^4 - (n+3)\lambda^3 + (5n-4)\lambda^2 - (6n-10)\lambda + n = 0.$$

Since $h_{16}(n-2) = -n+4 < 0$, it follows that $\mu(G_{20}) > n-2$. Thus

$$\mu(G) \leqslant n-2 < \mu(G_{20}).$$

This completes the proof of (1).

The proof of (2) follows from Lemma 2.4, Lemma 2.6 and the proof of Lemma 2.6 immediately. $\square$

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