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GENERALIZED STUMMEL CLASS AND MORREY SPACES

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Abstract. We revisit the Stummel class and its relation with Morrey spaces. We reformulate a result of Ragusa and Zamboni [**11**] and then discuss its generalization, as proposed by Eridani and Gunawan [**4**]. An improvement of the results previously obtained by Eridani and Gunawan is obtained and some extensions are presented.

1. Introduction

In 1971, Adams [**1**] studied traces of potential arising from translation invariant operators and proved the following inequality

$$\|u \cdot V^{1/p} \,:\, L^p\| \leqslant C\, \|V \,:\, L^{1,\lambda}\|^{1/p}\, \|\nabla u \,:\, L^\alpha\| \tag{1.1}$$

for $u \in C_0^\infty(\mathbb{R}^d)$, $p = \frac{\alpha\lambda}{d-\alpha}$, $\lambda > d-\alpha$, $1 < \alpha < d$. Here V is a nonnegative function in the Morrey space $L^{1,\lambda} = L^{1,\lambda}(\mathbb{R}^d)$, which we shall define below. Adams' inequality looks like (but not the same as) Olsen's inequality [**8**]

$$\|u \cdot W \,:\, L^{p,\lambda}\| \leqslant C\, \|W \,:\, L^{(d-\lambda)/\alpha,\lambda}\|\, \|(-\Delta)^{\alpha/2} u \,:\, L^{p,\lambda}\|$$

for $1 < p < \frac{d}{\alpha}$, $0 \leqslant \lambda < d - \alpha p$. It is well known that if $f := (-\Delta)^{\alpha/2} u \in L^{p,\lambda}$, then $u = (-\Delta)^{-\alpha/2} f \in L^{q,\lambda}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d-\lambda}$ [**2**]. Related works may also be found in [**3, 12, 15**].

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For $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant d$, the Morrey space $L^{p,\lambda}$ consists of locally integrable functions f for which

$$\|f \,:\, L^{p,\lambda}\| := \sup_{x\in\mathbb{R}^d,\, r>0} \left(\frac{1}{r^\lambda} \int_{|x-y|<r} |f(y)|^p \, dy \right)^{1/p} < \infty.$$

Note that $L^{p,0} = L^p$, the usual Lebesgue space. For historical background of Morrey spaces, see [**6**].

As we learn from the definition of $L^{p,\lambda}$, the parameter p describes the local integrability, while λ seems to measure the global integrability. Hence $L^{1,\lambda}$ may be used to describe the global integrability without taking into account the local integrability. The aspect can be seen from the fact that $L^{1,\lambda}$ supplements $L^{p,\lambda}$ in the sharp maximal inequality [**13**, Theorem 1.1]. The space $L^{1,\lambda}$ is a function space which is difficult to grasp. For example, unlike $L^{p,\lambda}$ with $p > 1$, it is not the case that we can characterize $L^{1,\lambda}$ in terms of the Littlewood–Paley decomposition. This is because the singular integral operators like the Riesz transforms are not bounded on $L^{1,\lambda}$. Nevertheless, this space can be compared with other function spaces. This is what we do in the present paper.

Our entry point is the work of Ragusa and Zamboni [**11**], which offers a similar imbedding to (1.1) by assuming more hypotheses on the function V. For $0 < \alpha < d$, they define *the Stummel modulus* of $f \in L^1_{\mathrm{loc}} = L^1_{\mathrm{loc}}(\mathbb{R}^d)$, denoted by $\eta_\alpha f$, by

$$\eta_\alpha f(r) := \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{d-\alpha}} \, dy, \quad r > 0.$$

They then define *the Stummel class* S_α by

$$S_\alpha := \{f \in L^1_{\mathrm{loc}} : \lim_{r\to 0} \eta_\alpha f(r) = 0\}.$$

Note that the definitions also make sense for $\alpha = d$. For $\alpha = 2$, S_α is known as the Stummel–Kato class. Ragusa and Zamboni obtain the following relation between the Stummel class S_α and the Morrey space $L^{1,\lambda}$.

THEOREM 1.1. *If f belongs to $L^{1,\lambda}$, $d - \alpha < \lambda < d$, then f belongs to S_α with*

$$\eta_\alpha f(r) \leqslant C\, r^{\lambda-d+\alpha} \|f \,:\, L^{1,\lambda}\|, \quad r > 0.$$

Conversely, if f belongs to S_α and $\eta_\alpha f(r) \sim r^\beta$, then f belongs to $L^{1,d-\alpha+\beta}$.

The second part of the theorem tells us that if $\eta_\alpha f(r)$ behaves like a power of r, then the improvement of the integrability of u in Adams' inequality can be seen either in terms of V in $L^{1,\lambda}$ or its local Riesz potential given by $\eta_\alpha f(r)$. From another point of view, the above theorem provides a characterization of the Morrey space $L^{1,\lambda}$ in terms of the Stummel modulus.

In this paper, we discuss a generalized version of the Stummel class and its relation with generalized Morrey spaces. Our aim is to reprove and improve the results of Eridani and Gunawan [**4**] by using weaker assumptions. (At the same time, an analogous result in the nonhomogeneous setting is obtained by Setya-Budhi et al. [**14**].) Moreover, we introduce two (generalized) Stummel classes with variable growth condition and with nonradially symmetric condition, and establish their relation with generalized Morrey spaces.

Throughout the paper, the letter C denotes a positive constant, which may vary from line to line.

2. Preliminaries

2.1. Definitions. For $\psi : (0,\infty) \to (0,\infty)$ with $\int_0^1 t^{d-1}\psi(t)\,dt < \infty$, we define the Stummel modulus of $f \in L^1_{\rm loc}$ by

$$\eta_\psi f(r) := \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} |f(y)|\,\psi(|x-y|)\,dy, \quad r > 0.$$

Accordingly, we define the Stummel class S_ψ by

$$S_\psi := \{f \in L^1_{\rm loc} : \lim_{r\to 0} \eta_\psi f(r) = 0\}.$$

Just like $\eta_\alpha f$, we see that $\eta_\psi f(r)$ is a nondecreasing function of r. Moreover, we note that if $\psi(t) = t^{\alpha-d}$, $0 < \alpha \leqslant d$, then $\eta_\psi f = \eta_\alpha f$ and $S_\psi = S_\alpha$. One may also observe that for $0 < \alpha \leqslant d$, we have $S_\alpha \subseteq S_\psi$ provided that $\psi(t) \leqslant Ct^{\alpha-d}$ for some positive constant C. In general, the condition $\int_0^1 t^{d-1}\psi(t)\,dt < \infty$ guarantees that S_ψ contains all locally bounded functions on $\mathbb{R}^d$.

Along with the generalized Stummel class, we also have the generalized Morrey spaces (as in [**5**, **7**]). For $1 \leqslant p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, the generalized Morrey space $\mathcal{M}^{p,\phi} := \mathcal{M}^{p,\phi}(\mathbb{R}^d)$ is defined to be the space of all functions $f \in L^p_{\rm loc}$ for which

$$\|f : \mathcal{M}^{p,\phi}\| := \sup_{B=B(x,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|}\int_B |f(y)|^p dy\right)^{1/p} < \infty.$$

Here $|B|$ denotes the usual Lebesgue measure of $B = B(x,r)$, which is a constant times r^d. We notice that if $\phi(t) = t^{(\lambda-d)/p}$, $0 \leqslant \lambda \leqslant d$, then $\mathcal{M}^{p,\phi} = L^{p,\lambda}$.

2.2. Assumptions. We say that a function $\varphi : (0,\infty) \to (0,\infty)$ satisfies *the doubling condition* if there exists a constant $C > 0$ such that

$$1 \leqslant \frac{r}{s} \leqslant 2 \Rightarrow \frac{1}{C} \leqslant \frac{\varphi(r)}{\varphi(s)} \leqslant C.$$

For example, for any $a \in \mathbb{R}$, the function $\varphi(t) = t^a$ satisfies the doubling condition. It is not hard to see that if φ_1 and φ_2 satisfy the doubling condition, so do their product and quotient.

Observe that if φ satisfies the doubling condition, then we have

$$\int_R^{2R} \frac{\varphi(t)}{t}\,dt \sim \varphi(R),$$

that is, there exists a constant $C > 0$ such that

$$\frac{1}{C}\,\varphi(R) \leqslant \int_R^{2R} \frac{\varphi(t)}{t}\,dt \leqslant C\,\varphi(R), \quad \text{for every } R > 0.$$

Now, the doubling condition can be decomposed into two conditions, namely the *left-doubling condition* $1 \leqslant \frac{r}{s} \leqslant 2 \Rightarrow \frac{1}{C} \leqslant \frac{\varphi(r)}{\varphi(s)}$ and the *right-doubling condition* $1 \leqslant \frac{r}{s} \leqslant 2 \Rightarrow \frac{\varphi(r)}{\varphi(s)} \leqslant C$.

If φ satisfies the right-doubling condition, then we have

$$\frac{1}{C}\,\varphi(2R) \leqslant \int_R^{2R} \frac{\varphi(t)}{t}\,dt \leqslant C\,\varphi(R) \quad \text{for every } R > 0.$$

Meanwhile, if φ satisfies the left-doubling condition, then we have

$$\frac{1}{C}\,\varphi(R) \leqslant \int_R^{2R} \frac{\varphi(t)}{t}\,dt \leqslant C\,\varphi(2R) \quad \text{for every } R > 0.$$

Note also that φ satisfies the right-doubling condition if and only if $1/\varphi$ satisfies the left-doubling condition.

Example 2.1. Consider the function $\varphi_1(t) = e^{-t}$, $t \in (0,\infty)$. This function satisfies the right-doubling condition but fails to satisfy the left-doubling condition (and hence it does not satisfy the doubling condition). Meanwhile, the function $\varphi_2(t) = e^t$, $t \in (0,\infty)$, satisfies the left-doubling condition but not the right-doubling condition. Consequently, for any $a \in \mathbb{R}$, the function $\varphi_3(t) = t^a e^{-t}$ satisfies the right-doubling condition only, while the function $\varphi_4(t) = t^a e^t$ satisfies the left-doubling condition only.

Example 2.2. Let $\gamma < \beta < 0$. Define $r_1 := \frac{1}{2}$ and $r_{j+1} := r_j^{\gamma/\beta}$ for $j = 1, 2, \ldots$. Then $r_j \to 0$ as $j \to \infty$. Now define the function ψ on $(0,\infty)$ by

$$\psi(r) := \begin{cases} r_j^\gamma, & \text{if } r_{j+1} < r \leqslant r_j; \\ r^\gamma, & \text{if } r_1 < r. \end{cases}$$

Then $r^\beta < \psi(r) \leqslant r^\gamma$ for $0 < r < 1$. Since ψ is nonincreasing, ψ satisfies the right-doubling condition. But

$$\frac{\psi(r_j+)}{\psi(r_j)} = \frac{r_{j-1}^\gamma}{r_j^\gamma} = \frac{r_j^\beta}{r_j^\gamma} = r_j^{\beta-\gamma} \to 0, \quad \text{as } j \to \infty.$$

Hence ψ does not satisfy the left-doubling condition. If in the definition above we let $0 < \beta < \gamma$, then the resulting function ψ satisfies the left-doubling condition but not the right-doubling condition. Moreover, the function $r^{-d}\psi(r)$ has the same property as ψ.

In general, nonincreasing functions satisfy the right-doubling condition, while nondecreasing functions satisfy the left-doubling condition. The function ϕ in the definition of the Morrey space $\mathcal{M}^{p,\phi}$ is assumed to be nonincreasing, and hence it satisfies the right-doubling condition. In addition, $t^{d/p}\phi(t)$ is assumed to be nondecreasing, so that — after division by $t^{d/p}$ — the function ϕ must satisfy the left-doubling condition. Thus ϕ here satisfies the doubling condition.

In [**4**], the function ψ in the definition of the Stummel class S_ψ is also assumed to satisfy the doubling condition. We find, however, that this is not necessary: we can obtain the same result as in [**4**] by assuming that ψ satisfies the right-doubling condition only or the left-doubling condition only. This, of course, enlarges the coverage of the functions ψ in the definition of the Stummel class S_ψ.

3. Main Results

Our first result below shows the inclusion of the generalized Morrey space $\mathcal{M}^{1,\phi}$ in the Stummel class S_ψ, under some conditions on ψ.

THEOREM 3.1. *Suppose that* $\int_0^1 t^{d-1}\psi(t)\phi(t)\,dt < \infty$. *Then* $\mathcal{M}^{1,\phi} \subseteq S_\psi$ *provided that* ψ *satisfies the right-doubling condition* or *the left-doubling condition.*

PROOF. The method used is the same as that employed in the proof of Theorem 1 in [**10**]. Let $f \in \mathcal{M}^{1,\phi}$. Suppose first that ψ satisfies the right-doubling condition. For $x \in \mathbb{R}^d$ and $r > 0$, we have

$$\begin{aligned}\int_{|x-y|<r} |f(y)|\psi(|x-y|)\,dy &= \sum_{j=-\infty}^{-1} \int_{2^j r \leqslant |x-y| < 2^{j+1}r} |f(y)|\,\psi(|x-y|)\,dy \\ &\leqslant C \sum_{j=-\infty}^{-1} \psi(2^j r) \int_{|x-y|<2^{j+1}r} |f(y)|\,dy \\ &\leqslant C \sum_{j=-\infty}^{-1} (2^{j+1}r)^d \psi(2^j r)\,\phi(2^{j+1}r) \|f : \mathcal{M}^{1,\phi}\|.\end{aligned}$$

Recall that ϕ is assumed to satisfy the doubling condition. Thus, we have

$$\begin{aligned}\int_{|x-y|<r} |f(y)|\,\psi(|x-y|)\,dy &\leqslant C\,\|f : \mathcal{M}^{1,\phi}\| \sum_{j=-\infty}^{-1} (2^j r)^d \psi(2^j r)\,\phi(2^j r) \\ &\leqslant C\,\|f : \mathcal{M}^{1,\phi}\| \sum_{j=-\infty}^{-1} \int_{2^{j-1}r}^{2^j r} t^{d-1}\psi(t)\,\phi(t)\,dt \\ &= \|f : \mathcal{M}^{1,\phi}\| \int_0^{r/2} t^{d-1}\psi(t)\,\phi(t)\,dt.\end{aligned}$$

The last inequality implies that

$$\eta_\psi f(r) \leqslant C\,\|f : \mathcal{M}^{1,\phi}\| \int_0^{r/2} t^{d-1}\psi(t)\,\phi(t)\,dt.$$

Since $\int_0^1 t^{d-1}\psi(t)\,\phi(t)\,dt < \infty$, we have $\lim_{r\to 0}\int_0^{r/2} t^{d-1}\psi(t)\phi(t)\,dt = 0$. Hence we find that $\lim_{r\to 0}\eta_\psi f(r) = 0$, that is, $f \in S_\psi$.

Suppose now that ψ satisfies the left-doubling condition. Then, for $x \in \mathbb{R}^d$ and $r > 0$, we have

$$\int_{|x-y|<r} |f(y)|\psi(|x-y|)\,dy = \sum_{j=-\infty}^{-1} \int_{2^j r \leqslant |x-y| < 2^{j+1}r} |f(y)|\psi(|x-y|)\,dy$$

$$
\begin{aligned}
&\leqslant C \sum_{j=-\infty}^{-1} \psi(2^{j+1}r) \int_{|x-y|<2^{j+1}r} |f(y)|\, dy \\
&\leqslant C \sum_{j=-\infty}^{-1} (2^{j+1}r)^d \psi(2^{j+1}r)\phi(2^{j+1}r)\|f : \mathcal{M}^{1,\phi}\| \\
&\leqslant C\, \|f : \mathcal{M}^{1,\phi}\| \sum_{j=-\infty}^{-1} \int_{2^{j+1}r}^{2^{j+2}r} t^{d-1}\psi(t)\phi(t)\, dt \\
&= C\, \|f : \mathcal{M}^{1,\phi}\| \int_0^{2r} t^{d-1}\psi(t)\phi(t)\, dt.
\end{aligned}
$$

The last inequality implies that

$$\eta_\psi f(r) \leqslant C\, \|f : \mathcal{M}^{1,\phi}\| \int_0^{2r} t^{d-1}\psi(t)\phi(t)\, dt.$$

But $\lim\limits_{r\to 0} \int_0^{2r} t^{d-1}\psi(t)\phi(t)\, dt = 0$ gives $\lim\limits_{r\to 0} \eta_\psi f(r) = 0$, so that $f \in S_\psi$. $\square$

Corollary 3.1. *Suppose that for some $\epsilon > 0$, the function $t^{d-\epsilon}\psi(t)\phi(t)$ is almost increasing, that is, there exists a constant $C > 0$ such that*

$$r \leqslant s \Rightarrow r^{d-\epsilon}\psi(r)\phi(r) \leqslant C s^{d-\epsilon}\psi(s)\phi(s).$$

If $f \in \mathcal{M}^{1,\phi}$, then $f \in S_\psi$ with

(i) *$\eta_\psi f(r) \leqslant C\, r^d \psi\big(\frac{r}{2}\big)\phi\big(\frac{r}{2}\big)$ for every $r > 0$, provided that ψ satisfies the right-doubling condition, or*

(ii) *$\eta_\psi f(r) \leqslant C\, r^d \psi(2r)\phi(2r)$ for every $r > 0$, provided that ψ satisfies the left-doubling condition.*

Remark. The constant C above depends on d, ϵ, f, ψ and ϕ, but not on r.

Proof. If ψ satisfies the right-doubling condition, then we have already shown that there exists a constant $C > 0$ such that for every $r > 0$,

$$\eta_\psi f(r) \leqslant C \int_0^{r/2} t^{d-1}\psi(t)\phi(t)\, dt.$$

Since $t^{d-\epsilon}\psi(t)\phi(t)$ is almost increasing, we have

$$
\begin{aligned}
\int_0^{r/2} t^{d-1}\psi(t)\phi(t)\, dt &= \int_0^{r/2} t^{d-\epsilon}\psi(t)\phi(t) t^{\epsilon-1} dt \\
&\leqslant C r^{d-\epsilon}\psi\big(\tfrac{r}{2}\big)\phi\big(\tfrac{r}{2}\big) \int_0^{r/2} t^{\epsilon-1} dt = C\, r^d \psi\big(\tfrac{r}{2}\big)\phi\big(\tfrac{r}{2}\big).
\end{aligned}
$$

Hence $\eta_\psi f(r) \leqslant C\, r^d \psi\big(\frac{r}{2}\big)\phi\big(\frac{r}{2}\big)$ for every $r > 0$. In particular, $\eta_\psi f(r) \leqslant C\, r^\epsilon$ for $0 < r < 1$, so that $f \in S_\psi$. The second estimate for $\eta_\psi f$ is obtained in a similar way when ψ satisfies the left-doubling condition. $\square$

In [**11**], the converse of the above theorem for $\psi(t) = t^{\alpha-d}$, $0 < \alpha \leqslant d$, is obtained by first showing that the Stummel modulus $\eta_\alpha f$ satisfies the right-doubling condition. As we show in our next theorem, it is not necessary to do so, since we know that the Stummel modulus is nondecreasing.

THEOREM 3.2. *If $f \in S_\psi$, then $f \in \mathcal{M}^{1,\phi}$ provided that*

(i) *ψ satisfies the right-doubling condition and $\int_0^r \frac{\eta_\psi f(t)}{t\psi(t)}\,dt \leqslant Cr^d\phi(r)$ for every $r > 0$, or*

(ii) *ψ satisfies the left-doubling condition and $\int_0^r \frac{\eta_\psi f(4t)}{t\psi(t)}\,dt \leqslant Cr^d\phi(r)$ for every $r > 0$.*

PROOF. Suppose that ψ satisfies the right-doubling condition. For a given ball $B := B(x,r)$, where $x \in \mathbb{R}^d$ and $r > 0$, we have

$$\begin{aligned}\int_{|x-y|<r} |f(y)|\,dy &= \sum_{j=-\infty}^{-1} \int_{2^j r \leqslant |x-y| < 2^{j+1}r} |f(y)|\,dy \\ &\leqslant C \sum_{j=-\infty}^{-1} \frac{1}{\psi(2^{j+1}r)} \int_{|x-y|<2^{j+1}r} |f(y)|\,\psi(|x-y|)\,dy \\ &\leqslant C \sum_{j=-\infty}^{-1} \frac{\eta_\psi f(2^{j+1}r)}{\psi(2^{j+1}r)}.\end{aligned}$$

Now $\frac{1}{\psi(2^{j+1}r)} \leqslant \frac{C}{\psi(t)}$ for $2^{j+1}r \leqslant t \leqslant 2^{j+2}r$. Meanwhile, $\eta_\psi f$ is nondecreasing, so that $\eta_\psi f(2^{j+1}r) \leqslant \eta_\psi f(t)$ for $2^{j+1}r \leqslant t \leqslant 2^{j+2}r$. Hence, from the previous inequality, we obtain

$$\begin{aligned}\int_{|x-y|<r} |f(y)|\,dy &\leqslant C \sum_{j=-\infty}^{-1} \int_{2^{j+1}r}^{2^{j+2}r} \frac{\eta_\psi f(t)}{t\psi(t)}\,dt \\ &= C \int_0^{2r} \frac{\eta_\psi f(t)}{t\psi(t)}\,dt \leqslant C\,(2r)^d\phi(2r) \leqslant C\,r^d\phi(r).\end{aligned}$$

The last inequality may now be rewritten as $\frac{1}{\phi(r)\,|B|}\int_B |f(y)|\,dy \leqslant C$. Taking the supremum over all balls B in $\mathbb{R}^d$, we obtain $f \in \mathcal{M}^{1,\phi}$.

We leave the proof of the second part — when ψ satisfies the left-doubling condition — to the readers. $\square$

COROLLARY 3.2. *Suppose that for some $\epsilon > 0$, the function $t^\epsilon\psi(t)$ is almost decreasing, that is, there exists a constant $C > 0$ such that*

$$r \leqslant s \Rightarrow r^\epsilon\psi(r) \geqslant C\,s^\epsilon\psi(s).$$

If $f \in S_\psi$, then $f \in \mathcal{M}^{1,\phi}$ provided that

(i) *ψ satisfies the right-doubling condition and $\eta_\psi f(r) \leqslant Cr^d\psi(r)\phi(r)$ for every $r > 0$, or*

(ii) *ψ satisfies the left-doubling condition and $\eta_\psi f(4r) \leqslant Cr^d\psi(r)\phi(r)$ for every $r > 0$.*

PROOF. If ψ satisfies the right-doubling condition $\eta_\psi f(r) \leqslant Cr^d\psi(r)\phi(r)$ for every $r > 0$, then we have

$$\int_0^r \frac{\eta_\psi f(t)}{t\psi(t)}\,dt = \int_0^r \frac{\eta_\psi f(t)}{t^\epsilon\psi(t)}\,t^{\epsilon-1}\,dt$$
$$\leqslant C\,\frac{\eta_\psi f(r)}{r^\epsilon\psi(r)}\int_0^r t^{\epsilon-1}dt = C\,\frac{\eta_\psi f(r)}{\psi(r)} \leqslant Cr^d\phi(r),$$

for every $r > 0$. This is precisely the condition in the first part of Theorem 3.2. Similarly, if ψ satisfies the left-doubling condition and $\eta_\psi f(4r) \leqslant C\,r^d\psi(r)\phi(r)$ for every $r > 0$, then we obtain the condition in the second part of Theorem 3.2. $\square$

REMARK. For $\delta > 0$, let $\psi(t) \sim t^{-d}(\log t^{-1})^{-1-\delta}$ for small $t > 0$. Then one may observe that for some $\epsilon > 0$, the function $t^\epsilon\psi(t)$ is almost decreasing.

4. Further Results

We shall now present some variants of the previous results.

4.1. First Variant. For $1 \leqslant p < \infty$ and a suitable function $\phi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, we define the generalized Morrey space $\mathcal{M}_*^{p,\phi} := \mathcal{M}_*^{p,\phi}(\mathbb{R}^d)$ to be the space of all functions $f \in L^p_{\rm loc}$ for which

$$\|f\,:\,\mathcal{M}_*^{p,\phi}\| := \sup_{B=B(x,r)} \frac{1}{\phi(x,r)}\left(\frac{1}{|B|}\int_B |f(y)|^p dy\right)^{1/p} < \infty.$$

Next, for $\psi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$, we define the generalized Stummel modulus by

$$\eta^*_\psi f(r) := \sup_{x\in\mathbb{R}^d}\int_{|x-y|<r} |f(y)|\psi(x,|x-y|)\,dy, \quad r > 0,$$

and the generalized Stummel class S^*_ψ by $S^*_\psi := \{f \in L^1_{\rm loc} : \lim_{r\to 0}\eta^*_\psi f(r) = 0\}$. Then we have the following results, which can be proved in the same way as Theorems 3.1 and 3.2.

THEOREM 4.1. *Suppose that* $\lim_{r\to0}\sup_{x\in\mathbb{R}^d}\int_0^r t^{d-1}\psi(x,t)\phi(x,t)\,dt < \infty$. *Then we have* $\mathcal{M}_*^{1,\phi} \subseteq S^*_\psi$ *provided that* ψ *satisfies the right-doubling condition, that is, there exists a constant* $C > 0$ *such that*

$$x \in \mathbb{R}^d \text{ and } 1 \leqslant \frac{r}{s} \leqslant 2 \Rightarrow \frac{\psi(x,r)}{\psi(x,s)} \leqslant C$$

or the left-doubling condition, that is, there exists a constant $C > 0$ *such that*

$$x \in \mathbb{R}^d \text{ and } 1 \leqslant \frac{r}{s} \leqslant 2 \Rightarrow \frac{1}{C} \leqslant \frac{\psi(x,r)}{\psi(x,s)}.$$

EXAMPLE 4.1. Let $\alpha : \mathbb{R}^d \to (0,\infty)$ with $0 < \inf_{x\in\mathbb{R}^d}\alpha(x) \leqslant \sup_{x\in\mathbb{R}^d}\alpha(x) \leqslant d$, and put $\psi(x,t) = t^{\alpha(x)-d}$, $x \in \mathbb{R}^d$, $t > 0$. Then one may check that ψ satisfies the right- and left-doubling conditions.

THEOREM 4.2. *If* $f \in S^*_\psi$, *then* $f \in \mathcal{M}_*^{1,\phi}$ *provided that*

(i) *ψ satisfies the right-doubling condition and* $\int_0^r \frac{\eta_\psi^* f(t)}{t\psi(x,t)} dt \leqslant Cr^d\phi(x,r)$ *for every* $x \in \mathbb{R}^d$ *and* $r > 0$*, or*

(ii) *ψ satisfies the left-doubling condition and* $\int_0^r \frac{\eta_\psi^* f(4t)}{t\psi(x,t)} dt \leqslant Cr^d\phi(x,r)$ *for every* $x \in \mathbb{R}^d$ *and* $r > 0$.

Remark. We leave it to the readers to formulate the consequences of the above theorems that are analogous to Corollaries 3.1 and 3.2.

4.2. Second Variant. We go back to the generalized Morrey spaces $\mathcal{M}^{p,\phi}$ we considered earlier, but for a nonnegative locally integrable function Ψ we now define the generalized Stummel modulus by

$$\eta_\Psi f(r) := \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} |f(y)|\Psi(x-y)\,dy, \quad r > 0,$$

and the generalized Stummel class S_Ψ by $S_\Psi := \{f \in L^1_{\mathrm{loc}} : \lim_{r\to 0} \eta_\Psi f(r) = 0\}$. Notice that Ψ may not be a radial function. As in [**9**], the function Ψ is assumed to satisfy the following property: there are constants $C, \delta > 0$ and $0 \leqslant \epsilon < 1$ such that

$$\sup_{R\leqslant|x|<2R} \Psi(x) \leqslant \frac{C}{R^d} \int_{\delta(1-\epsilon)R\leqslant|y|<2\delta(1+\epsilon)R} \Psi(y)\,dy \tag{4.1}$$

for every $R > 0$. Then, we have the following inclusion of the Morrey space $\mathcal{M}^{1,\phi}$ in the Stummel class S_Ψ.

Theorem 4.3. *Suppose that Ψ satisfies* (4.1). *If* $\lim\limits_{r\to 0} \int_{|x|<r} \Psi(x)\phi(|x|)\,dx = 0$, *then* $\mathcal{M}^{1,\phi} \subseteq S_\Psi$.

Proof. Let $f \in \mathcal{M}^{1,\phi}$, $x \in \mathbb{R}^d$ and $r > 0$. For convenience, put $c_1 := \delta(1-\epsilon)$ and $c_2 := \delta(1+\epsilon)$. Then, we have

$$\begin{aligned}
&\int_{|x-y|<r} |f(y)|\Psi(x-y)\,dy \\
&\qquad = \sum_{j=-\infty}^{-1} \int_{2^j r\leqslant|x-y|<2^{j+1}r} |f(y)|\Psi(x-y)\,dy \\
&\qquad \leqslant \sum_{j=-\infty}^{-1} \sup_{2^j r\leqslant|x-y|<2^{j+1}r} \Psi(x-y) \int_{2^j r\leqslant|x-y|<2^{j+1}r} |f(y)|\,dy \\
&\qquad \leqslant C \sum_{j=-\infty}^{-1} \int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r} \Psi(z)\,dz \cdot \frac{1}{(2^{j+1}r)^d} \int_{|x-y|<2^{j+1}r} |f(y)|\,dy \\
&\qquad \leqslant C\,\|f : \mathcal{M}^{1,\phi}\| \sum_{j=-\infty}^{-1} \phi(2^{j+1}r) \int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r} \Psi(z)\,dz.
\end{aligned}$$

Using the assumption that ϕ satisfies the doubling condition (when $c_2 > 1$) and that ϕ is nonincreasing, we have $\phi(2^{j+1}r) \leqslant C\phi(c_2 2^{j+1}r) \leqslant C\phi(|z|)$ for $|z| < c_2 2^{j+1}r$.

Hence

$$\phi(2^{j+1}r)\int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r}\Psi(z)\,dz\leqslant C\int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r}\Psi(z)\phi(|z|)\,dz.$$

Next, we have $\sum\limits_{j=-\infty}^{\infty}\chi_{[c_1,2c_2)}(2^j r)\sim 1+\log_2\frac{1+\epsilon}{1-\epsilon}$ (the overlapping property), so that

$$\sum_{j=-\infty}^{-1}\int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r}\Psi(z)\phi(|z|)\,dz\leqslant C\int_{|z|<c_2 r}\Psi(z)\phi(|z|)\,dz.$$

Thus we obtain

$$\int_{|x-y|<r}|f(y)|\Psi(x-y)\,dy\leqslant C\,\|f\,:\,\mathcal{M}^{1,\phi}\|\int_{|z|<c_2 r}\Psi(z)\phi(|z|)\,dz.$$

Since the integral on the right hand side tends to 0 as $r\to 0$, we conclude that $f\in S_\Psi$. □

To prove the converse, we assume that Ψ satisfies the following property: there are constants $C,\delta>0$ and $0\leqslant\epsilon<1$ such that

$$\sup_{R\leqslant|x|<2R}\frac{1}{\Psi(x)}\leqslant\frac{C}{R^d}\int_{\delta(1-\epsilon)R\leqslant|y|<2\delta(1+\epsilon)R}\frac{1}{\Psi(y)}\,dy \tag{4.2}$$

for every $R>0$.

THEOREM 4.4. *Suppose that Ψ satisfies (4.2). If $f\in S_\Psi$, then $f\in\mathcal{M}^{1,\phi}$ provided that*

$$\int_{|x|<r}\frac{\eta_\Psi f(\kappa|x|)}{|x|^d\Psi(x)}\,dx\leqslant C\,r^d\phi(r)$$

for every $r>0$, where $\kappa:=\frac{2}{\delta(1-\epsilon)}$.

PROOF. Let $f\in S_\Psi$, $x\in\mathbb{R}^d$ and $r>0$. The same as before, we let $c_1:=\delta(1-\epsilon)$, $c_2:=\delta(1+\epsilon)$, and put $\kappa:=\frac{2}{c_1}$. It follows that

$$\begin{aligned}\int_{|x-y|<r}|f(y)|\,dy&=\sum_{j=-\infty}^{-1}\int_{2^j r\leqslant|x-y|<2^{j+1}r}|f(y)|\,dy\\&\leqslant\sum_{j=-\infty}^{-1}\sup_{2^j r\leqslant|x-y|<2^{j+1}r}\frac{1}{\Psi(x-y)}\int_{|x-y|<2^{j+1}r}|f(y)|\Psi(x-y)\,dy\\&\leqslant C\sum_{j=-\infty}^{-1}\int_{c_1 2^j r\leqslant|z|<c_2 2^{j+1}r}\frac{1}{|z|^d\Psi(z)}\,dz\cdot\eta_\Psi f(2^{j+1}r)\\&\leqslant C\int_{|z|<c_2 r}\frac{\eta_\Psi f(\kappa|z|)}{|z|^d\Psi(z)}\,dz\leqslant C\,r^d\phi(r),\end{aligned}$$

where we have used the fact that $\eta_\Psi f$ is nondecreasing, the overlapping property, and the doubling property of ϕ. This proves the theorem. □

5. Concluding Remarks

We have improved the results of Eridani and Gunawan [**4**] by using weaker assumptions. We have also found that the doubling property of the Stummel modulus is unnecessary, and used its increasing property instead. Moreover, we have added some variants and proved similar results.

We end the paper with the following proposition, which tells us that given a function in the Stummel class S_ψ, we can actually have extra information about its integrability.

PROPOSITION 5.1. *If* $f \in S_\psi$ *and* $\int_0^1 [\eta_\psi f(t)]^{1-\theta} t^{-1} dt < \infty$, *for* $0 < \theta < 1$, *then for every* $x \in \mathbb{R}^d$ *and* $r > 0$

$$\int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{[\eta_\psi f(2|x-y|)]^\theta} \, dy \leqslant \int_0^{2r} [\eta_\psi f(t)]^{1-\theta} t^{-1} dt,$$

and hence

$$\lim_{r \to 0} \int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{[\eta_\psi f(2|x-y|)]^\theta} \, dy = 0.$$

PROOF. Indeed, since $\eta_\psi f$ is an increasing function of r, so are $[\eta_\psi f]^\theta$ and $[\eta_\psi f]^{1-\theta}$ for $0 < \theta < 1$. It thus follows that for every $x \in \mathbb{R}^d$ and $r > 0$

$$\begin{aligned}
&\int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{[\eta_\psi f(2|x-y|)]^\theta \, dy} \\
&\qquad = \sum_{j=-\infty}^{-1} \int_{2^j r \leqslant |x-y| < 2^{j+1} r} |f(y)| \frac{\psi(|x-y|)}{[\eta_\psi f(2|x-y|)]^\theta} \, dy \\
&\qquad \leqslant \sum_{j=-\infty}^{-1} \frac{1}{[\eta_\psi f(2^{j+1} r)]^\theta} \int_{2^j r \leqslant |x-y| < 2^{j+1} r} |f(y)| \psi(|x-y|) \, dy \\
&\qquad \leqslant \sum_{j=-\infty}^{-1} [\eta_\psi f(2^{j+1} r)]^{1-\theta} \leqslant \sum_{j=-\infty}^{-1} \int_{2^{j+1} r}^{2^{j+2} r} [\eta_\psi f(t)]^{1-\theta} t^{-1} dt \\
&\qquad = \int_0^{2r} [\eta_\psi f(t)]^{1-\theta} t^{-1} dt.
\end{aligned}$$

But $\int_0^1 [\eta_\psi f(t)]^{1-\theta} t^{-1} dt < \infty$, and so we have $\lim_{r \to 0} \int_0^{2r} [\eta_\psi f(t)]^{1-\theta} t^{-1} dt = 0$. This proves the proposition. □

REMARK. Proposition 5.1 is also valid for the generalized Stummel classes S_ψ^* and S_Ψ defined in Section 4.

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