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ON THE EXTREME POINTS AND STRONGLY EXTREME POINTS IN KÖTHE–BOCHNER SPACES

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Abstract. We give the necessary conditions of extreme points and strongly extreme points in the unit ball of Köthe–Bochner spaces. The conditions have been shown to be sufficient earlier.

1. Introduction

Let (Ω, Σ, μ) denote a measure space with σ-finite and complete measure μ and $L^0 = L^0(\Omega)$ denote the space of all (equivalence classes of) Σ-measurable real-valued functions, equipped with the topology of convergence in measure on μ-finite sets. In what follows, if $x, y \in L^0$, then $x \leqslant y$ means $x(t) \leqslant y(t)$ μ-almost everywhere in Ω. A Banach subspace E of L^0 is said be a Köthe function space (over (Ω, Σ, μ)) if

(i) $|x| \leqslant |y|$, $x \in L^0$, $y \in E$ imply $x \in E$ and $\|x\| \leqslant \|y\|$.
(ii) $\operatorname{supp} E := \bigcup\{\operatorname{supp} x : x \in E\} = \Omega$, where $\operatorname{supp} x = \{t \in \Omega : x(t) \neq 0\}$.

Köthe spaces are Banach lattices and detailed studies of Banach lattices and Köthe spaces may be found in [**4**]. A Köthe function space E is said to be order continuous provided $x_n \searrow 0$ implies $\|x_n\| \to 0$. E is said to be strictly monotone if $x \leqslant y$ and $x \neq y$ implies $\|x\| < \|y\|$.

In this paper we always denote by E a Köthe function space on (Ω, Σ, μ), X a Banach space. By $E(X)$ we denote the Banach space of all equivalence classes of strongly measurable function $x : \Omega \to X$ such that $\tilde{x} = \|x(\cdot)\|_X \in E$ equipped with the norm $|||x||| = \|\tilde{x}\|_E$.

The space $(E(X), |||\cdot|||)$ is called the Köthe–Bochner space. The most important class of Köthe–Bochner function spaces $E(X)$ are the Lebesgue–Bochner spaces $L^p(X)$, $(1 \leqslant p < \infty)$ and their generalization the Orlicz–Bochner spaces $L^\phi(X)$.

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They have been studied by many authors. The geometric properties of the Köthe–Bochner spaces have been also studied by many authors.

For any Banach space X, let $S(X)$ and $B(X)$ stand for the unit sphere and unit ball of X, respectively. An $x \in S(X)$ is called an extreme point of $B(X)$ if $2x = y+z$, $y, z \in B(X)$ implies $y = z$. Denote the set of all extreme points of $B(X)$ by $\operatorname{Ext} B(X)$. An X is called rotund (R) provided that $S(X) = \operatorname{Ext} B(X)$. An $x \in S(X)$ is said to be strongly extreme point of $B(X)$ if for every sequence $\{x_n\}$ in X, $\lim_n \|x_n \pm x\| = 1$ implies $\lim_n x_n = 0$, or equivalently, for $\{x_n\}, \{y_n\} \subset X$, $\lim_n \|x_n\| = \lim_n \|y_n\| = 1$ and $2x = x_n+y_n$ $(n = 1, 2, \dots)$ implies $\lim_n \|x_n-y_n\| = 0$. An X is called mid-point locally uniformly rotund (MLUR) if the set of all strongly extreme points of $B(X)$ is equal to $S(X)$. A point $x \in S(X)$ is said to be a point of local uniform rotundity (or a LUR-point) if for arbitrary sequence $\{x_n\} \subset B(X)$, $\lim_n \|x_n + x\| = 2$ and $\lim_n \|x_n\| = 1$ imply $\lim_n x_n = x$. A space X is locally uniformly uniformly rotund (LUR) if every point of $S(X)$ is a LUR-point. Clearly, (LUR) $\Rightarrow$ (MLUR) $\Rightarrow$ (R).

In 2006, Ren, Feng and Wu [**5**] obtained the sufficient conditions for extreme points of $B(E(X))$. The sufficient or necessary conditions for strongly extreme points of $B(E(X))$ were discussed by Hudzik and Mastylo [**2**] and Cerdá, Hudzik, and Mastylo [**1**]. We list the results here:

Proposition 1.1. [**5**] *Let E be a strictly monotone Köthe function space and X a real Banach space. Assume that $x \in S(E(X))$ have the following properties:*
(a) $\tilde{x} = \|x(\cdot)\|_X$ *is an extreme point of $B(X)$.*
(b) $\frac{x(t)}{\|x(t)\|_X}$ *is an extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*
Then x is an extreme point of $B(E(X))$.

Proposition 1.2. [**2**] *Let E be a locally uniformly rotund Köthe function space and X a real Banach space. If $x \in S(E(X))$ is such that $\frac{x(t)}{\|x(t)\|_X}$ is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$, then x is a strongly extreme point of $B(E(X))$.*

Proposition 1.3. [**2**] *Let E be as in Proposition* 1.2 *and X a separable Banach space. If $x \in S(E(X))$ is a strongly extreme point, then $\frac{x(t)}{\|x(t)\|_X}$ is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*

Proposition 1.4. [**1**] *Let E be an order continuous Banach function lattice and X a real Banach space. Assume that $x \in S(E(X))$ have the following properties:*
(a) $\tilde{x} = \|x(\cdot)\|_X$ *is a strongly extreme point of $B(X)$.*
(b) $\frac{x(t)}{\|x(t)\|_X}$ *is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*
Then x is a strongly extreme point of $B(E(X))$.

In this paper, we will show that the two properties in Proposition 1 and Proposition 4 are exactly the necessary conditions for $x \in S(E(X))$ to be extreme point and strongly extreme point of $B(E(X))$, respectively.

2. Main Results

THEOREM 2.1. *Let E be a Köthe function space and X a real Banach space. Assume that $x \in S(E(X))$ is an extreme point of $B(E(X))$. Then we have:*
(a) $\tilde{x} = \|x(\cdot)\|_X$ *is an extreme point of* $B(X)$.
(b) $\frac{x(t)}{\|x(t)\|_X}$ *is an extreme point of* $B(X)$ *for a.e.* $t \in \operatorname{supp} x$.

PROOF. (a) Suppose that $\|x(\cdot)\|_X$ is not an extreme point of $B(E)$. Then there exist two real measurable functions $y(t)$, $z(t)$, such that $y(\cdot)$, $z(\cdot) \in S(E)$ and $2\|x(t)\|_X = y(t) + z(t)$ but $y \neq z$. Define

$$g(t) = \begin{cases} \frac{x(t)}{\|x(t)\|_X}, & t \in \operatorname{supp} x, \\ x_0, & \text{otherwise}, \end{cases}$$

where x_0 is a fixed point in $S(X)$. Obviously $g(\cdot)$ is measurable. Let $f_1(t) = y(t)g(t)$, $f_2(t) = z(t)g(t)$. Then $|||f_1||| = \big\|\|y(t)g(t)\|_X\big\|_E = \|y(t)\|_E = 1$.

Similarly, $|||f_2||| = 1$. Moreover, $f_1(t) + f_2(t) = (y(t) + z(t))g(t) = 2x(t)$ but $f_1 \neq f_2$. This contradicts the assumption that x is an extreme point.

(b) Observe that $\frac{x(t)}{\|x(t)\|_X} \in S(X)$. Let $A = \big\{t : \frac{x(t)}{\|x(t)\|_X} \subsetneqq \operatorname{Ext} S(X)\big\}$. Suppose that $\mu A > 0$. Then there are $y_1, z_1 \in S(X)$ such that $2\frac{x(t)}{\|x(t)\|_X} = y_1(t) + z_1(t)$, $(\in A)$ but $y_1 \neq z_1$. Define

$$(y(t), z(t)) = \begin{cases} (x(t), x(t)), & t \in \Omega \smallsetminus A, \\ \big(\|x(t)\|_X y_1(t), \|x(t)\|_X z_1(t)\big), & t \in A. \end{cases}$$

Therefore, $2x(t) = y(t) + z(t)$ a.e. $t \in \Omega$ but $z \neq y$. Moreover,

$$\begin{aligned} |||y||| &= \big\|\|y(t)\|_X\big\|_E \\ &= \big\|\|y(t)\|_X \chi_A + \|y(t)\|_X \chi_{\Omega \smallsetminus A}\big\|_E \\ &= \big\|\big\|\|x(t)\|_X y_1(t)\big\|_X \chi_A + \|x(t)\|_X \chi_{\Omega \smallsetminus A}\big\|_E \\ &= \big\|\|x(t)\|_X \chi_A + \|x(t)\|_X \chi_{\Omega \setminus A}\big\|_E \\ &= \big\|\|x(t)\|_X\big\|_E = |||x||| = 1. \end{aligned}$$

Similarly, $|||z||| = 1$. This means that x is not an extreme point of $B(E(X))$, a contradiction. So $\mu A = 0$, which finishes the proof. □

THEOREM 2.2. *Let E be a Köthe function space and X a real Banach space. Assume that x is a strongly extreme point of $B(E(X))$. Then*
(a) $\|x(\cdot)\|_X$ *is a strongly extreme point of* $B(E)$;
(b) $\frac{x(t)}{\|x(t)\|_X}$ *is a strongly extreme point of* $B(X)$ *a.e.* $t \in \operatorname{supp} x$.

PROOF. (a) Assume $\lim_{n\to\infty} \big\|\|x(t)\|_X \pm x_n(t)\big\|_E = 1$, where $\{x_n\}$ is a sequence of real valued functions. Define

$$g(t) = \begin{cases} \frac{x(t)}{\|x(t)\|_X}, & t \in \operatorname{supp} x, \\ x_0, & \text{otherwise}, \end{cases}$$

where $x_0 \in S(X)$ is a fixed point. Then $g(t)$ is measurable. Moreover, $\|g(t)\|_X \equiv 1$ and $\|x(t)\|_X g(t) \equiv x(t)$ for every $t \in \Omega$. Observe that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \big\| \|x(t)\|_X \pm x_n(t) \big\|_E \\ &= \lim_{n\to\infty} \big|\!\big|\!\big| \left[\, \|x(t)\|_X \pm x_n(t) \right] g(t) \big|\!\big|\!\big| \\ &= \lim_{n\to\infty} |\!|\!| x(t) \pm x_n(t) g(t) \,|\!|\!| \,. \end{aligned}$$

Since x is a strongly extreme point in $B(E(X))$, we have

$$\lim_{n\to\infty} |\!|\!| x_n(t) g(t) |\!|\!| = \lim_{n\to\infty} \|x_n(t)\|_E = 0$$

which implies that $\|x(\cdot)\|_X$ is a strongly extreme point of $B(E)$.

(b) Denote by $z(t) = \frac{x(t)}{\|x(t)\|_X}$. Define

$$A = \{t \in \operatorname{supp} x : z(t) \text{ is not a strongly extreme point of } B(X)\}.$$

We should prove $\mu A = 0$. Suppose it is not the case, e.g., $\mu A > 0$. Define

$$\begin{aligned} A_{mk} = \big\{ t \in A :\ & \text{there exists a sequence } \{z_n(t)\} \text{ such that} \\ & 1 - \tfrac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \tfrac{1}{n},\ \|z_n(t)\|_X \geqslant \tfrac{1}{m} \text{ for } n \geqslant k \big\}. \end{aligned}$$

Then $\bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} A_{mk} = A$. There are $m_0, k_0 \geqslant 1$ such that $\mu(A_{m_0 k_0}) > 0$. Let $\delta = \frac{1}{m_0}$. We deduce that in the set $B = A_{m_0 k_0}$, if $n \geqslant k_0$, then $1 - \frac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \frac{1}{n}$ and $\|z_n(t)\|_X \geqslant \delta > 0$.

However the functions $z_n(\cdot)$ need not be measurable, so we have to make some modification.

$z(\cdot)$ is measurable, it is almost a uniform limit of a sequence of countably valued functions. That is, there is a subset $C \subset B$ with $\mu C = \mu B$ and a sequence of countably valued functions $\{y_n(\cdot)\}$ such that $y_n(t) \to z(t)$ uniformly on C. There exist a natural number k_1 satisfying $\|y_n(t) - z(t)\|_X < \frac{1}{n}$ whenever $n > k_1$. Denote by $y_n(t) = \sum_{i=1}^{\infty} e_{in} \chi_{C_{in}}, (\mu(C_{in}) > 0)$. y_n is single valued on C_{in}, i.e., $y_n(t) \equiv e_{in} (t \in C_{in})$. Let $k = \max\{k_0, k_1\}$. When $n > k$, we have $1 - \frac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \frac{1}{n}$, and $\|z_n(t)\|_X \geqslant \delta$. Select a point $t_{i_0} \in C_{in}$, then define $x_n(t) = \sum_{i=1}^{\infty} z_n(t_{i_0}) \chi_{C_{in}}$. Then $\{x_n(\cdot)\}$ is a sequence of measurable functions since they are countably valued.

For every $t \in C_{in}$ $(i = 1, 2, \dots)$, if $n > k$, then

$$\begin{aligned} \|x_n(t) \pm z(t)\|_X &\geqslant \|x_n(t) \pm z(t_{i_0})\|_X - \|z(t_{i_0}) - y_n(t_{i_0})\|_X - \|y_n(t_{i_0}) - z(t)\|_X \\ &= \|z_n(t_{i_0}) \pm z(t_{i_0})\|_X - \|z(t_{i_0}) - y_n(t_{i_0})\|_X - \|y_n(t) - z(t)\|_X \\ &> 1 - \tfrac{1}{n} - \tfrac{1}{n} - \tfrac{1}{n} = 1 - \tfrac{3}{n} \to 1. \end{aligned}$$

Analogously, $\|x_n(t) \pm z(t)\|_X \leqslant 1 + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = 1 + \frac{3}{n} \to 1$. Moreover, $\|x_n(t)\|_X = \|z_n(t_{i_0})\|_X \geqslant \delta$.

Now we define a sequence of measurable functions on G as follows:

$$x'_n(t) = \begin{cases} 0, & t \in G \setminus C \\ \|x(t)\|_X x_n(t), & t \in C. \end{cases}$$

Thus, for $n > k$,

$$\begin{aligned}\big|\|x_n'(t)\|_X\big| &= \big\|\|x_n'(t)\|_X\big\|_E \\ &= \big\|\|\|x(t)\|_X x_n(t)\|_X\chi_C\big\|_E = \big\|\|x(t)\|_X\|x_n(t)\|_X\chi_C\big\|_E \\ &= \big\|\|x(t)\|_X\chi_C\|x_n(t)\|_X\big\|_E \geqslant \delta\big\|\|x(t)\|_X\chi_C\big\|_E = \delta' > 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\|x_n'(t) \pm x(t)\|\| &= \big\|\|x_n'(t) \pm x(t)\|_X\big\|_E \\ &= \big\|\|x(t)\|_X\chi_{G\smallsetminus C} + \big\|x(t) \pm \|x(t)\|_X x_n(t)\big\|_X\chi_C\big\|_E \\ &= \Big\|\|x(t)\|_X\chi_{G\smallsetminus C} + \|x(t)\|_X\Big\|\frac{x(t)}{\|x(t)\|_X} \pm x_n(t)\Big\|_X\chi_C\Big\|_E \\ &\geqslant \big\|\|x(t)\|_X\chi_{G\smallsetminus C} + (1-\tfrac{3}{n})\|x(t)\|_X\chi_C\big\|_E \\ &> (1-\tfrac{3}{n})\big\|\|x(t)\|_X\big\|_E = (1-\tfrac{3}{n})\ \|\|\ x(t)\|\| = 1-\tfrac{3}{n} \to 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\|x_n'(t) \pm x(t)\|\| &\leqslant \big\|\|x(t)\|_X\chi_{G\smallsetminus C} + (1+\tfrac{3}{n})\|x(t)\|_X\chi_C\big\|_E \\ &\leqslant (1+\tfrac{3}{n})\big\|\|x(t)\|_X\big\|_E = (1+\tfrac{3}{n})\ \|\|\ x(t)\|\| = 1+\tfrac{3}{n} \to 1.\end{aligned}$$

This contradicts the fact that x is a strongly extreme point of $B(E(X))$. The proof is completed. □

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