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SOME FURTHER RESULTS ON A QUESTION OF YI

Abhijit Banerjee

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Abstract. Concerning a question of Yi [**18**], we study the problem of uniqueness of meromorphic functions sharing two sets with the notion of weighted sharing of sets and obtain four results which will not only improve the results of Lahiri [**12**], Lin-Yi [**20**] but also improve a recent result of the present author [**3**] and thus provide an answer to the question of Gross [**6**] in a more compact and convenient way. We exhibit two examples to show that a condition in one of our results is sharp. Till now our result is the best in this regard.

1. Introduction, Definitions, and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$ $(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, $r \notin E$.

We use I to denote any set of infinite linear measure of $0 < r < \infty$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [**7**].

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

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The author dedicates the paper to the memory of his respected maternal grandfather Late S. N. Chattopadhyay who germinated author's inquisition for research work in Mathematics at his early age.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity, the set $\bigcup_{a\in S}\{z : f(z) = a\}$ is denoted by $\bar{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\bar{E}_f(S) = \bar{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values. The problem of determining a meromorphic (or entire) function on $\mathbb{C}$ by its single pre-images, counting with multiplicities, of finite sets is an important one and it has been studied by many mathematicians.

In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the preimages, ignoring multiplicities, of 5 distinct values. A few years later, he showed that when multiplicities are considered, 4 points are sufficient (with one exceptional situation). In 1977 F. Gross extended the study by considering pre-images of a set and introduced the notion of unique range set. We recall that a set is called a unique range set (counting multiplicities) for a particular family of functions if the inverse image of the set counting multiplicities uniquely determines the function in the family.

Now let $\mathcal{F}$ be a nonempty subset of the set of meromorphic functions. A subset S of $\mathbb{C}\cup\{\infty\}$ is called a unique range set (a URS in short) for $\mathcal{F}$ if for any $f, g \in \mathcal{F}$ such that $E_f(S) = E_g(S)$ one has $f \equiv g$. In 1982 the first example of URS for entire functions was found by F. Gross and C. C. Yang that is

$$S = \{z \in \mathbb{C} : e^z + z = 0\}.$$

Note that S is an infinite set. Since then, the study of URS is focused mainly on two problems: finding different URS with the number of elements small as possible, and characterizing the URS. To reduce the number of elements in the range set as small as possible Gross [**6**] proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical.

In [**6**] Gross asked the following question: *Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j)$ $= E_g(S_j)$ for $j = 1, 2$ must be identical?*

During the last two decades a famous problem in value distribution theory has been to give explicitly a set S with n elements and make n as small as possible such that any two meromorphic functions f and g that share the value ∞ and the set S must be equal. Naturally several authors investigate the possible answer in the above direction and continuous efforts are being carried out to relax the hypothesis of the results; cf. [**1**]–[**5**], [**8**], [**12**], [**16**], [**18**], [**20**], [**21**].

In the direction to the question of Gross, in 1995 Yi [**18**] proved for meromorphic functions the following result.

Theorem A. [**18**] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$, where n and m are two positive integers such that $m \geqslant 2$, $n \geqslant 2m + 7$, with n and m having no common factor, a and b be two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no* *multiple root. If* f *and* g *are two nonconstant meromorphic functions satisfying* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$, *then* $f \equiv g$.

In the same paper Yi also asked the following question: *What can be said if* $m = 1$ *in Theorem* A? In connection to this question he proved the following theorem.

Theorem B. [**18**] *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$, *where* n $(\geqslant 9)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* f *and* g *are two nonconstant meromorphic functions such that* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$, *then either* $f \equiv g$ *or* $f \equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ *and* $g \equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, *where* h *is a nonconstant meromorphic function.*

To provide an answer to the question of Yi and to find under which condition $f \equiv g$ Lahiri [**8**] proved the following result.

Theorem C. [**8**] *Let* S *be defined as in Theorem* B *and* n $(\geqslant 8)$ *be an integer. If* f *and* g *are two nonconstant meromorphic functions having no simple poles such that* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$, *then* $f \equiv g$.

Fang and Lahiri [**5**] improved Theorem C by further reducing the cardinality of the same range set and proved the following theorem.

Theorem D. [**5**] *Let* S *be defined as in Theorem* B *and* $n(\geqslant 7)$ *be an integer. If* f *and* g *are two nonconstant meromorphic functions having no simple poles such that* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$, *then* $f \equiv g$.

Let $S = \{z : z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \cdots + e^{6z}}{1 + e^z + \cdots + e^{6z}}, \quad g = \frac{1 + e^z + \cdots + e^{5z}}{1 + e^z + \cdots + e^{6z}}.$$

Obviously $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ but $f \not\equiv g$. So for the validity of Theorem D, f and g must not have any simple pole.

If two meromorphic functions f and g have no simple pole, then clearly $\Theta(\infty; f) \geqslant \frac{1}{2}$ and $\Theta(\infty; g) \geqslant \frac{1}{2}$. So Fang and Lahiri did not provide the exact lower bound of $\Theta(\infty; f) + \Theta(\infty; g)$.

To proceed further we require the following definition, known as weighted sharing of sets and values, which renders a useful tool for the purpose of relaxation of the nature of sharing the sets.

Definition 1.1. [**10, 11**] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [**10**] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$, and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$. Clearly $E_f(S) = E_f(S,\infty)$ and $\bar{E}_f(S) = E_f(S,0)$.

Improving Theorem D Lahiri [**12**] proved the following theorem.

Theorem E. [**12**] *Let S be defined as in Theorem* B *and n $(\geqslant 7)$ be an integer. If for two nonconstant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, then $f \equiv g$.*

In 2006 to deal with a question of Gross, Yi and Lin [**20**] proved the following results.

Theorem F. [**20**] *Let S be defined as in Theorem* B *and n $(\geqslant 7)$ be an integer. If for two nonconstant meromorphic functions f and g, $\Theta(\infty; f) > \frac{1}{2}$, $E_f(S,\infty) = E_g(S,\infty)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, then $f \equiv g$.*

Theorem G. [**20**] *Let S be defined as in Theorem* B *and n $(\geqslant 8)$ be an integer. If for two nonconstant meromorphic functions f and g, $\Theta(\infty; f) > \frac{2}{n-1}$, $E_f(S,\infty) = E_g(S,\infty)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, then $f \equiv g$.*

Recently the present author [**3**] has not only generalized Theorem E by investigating the problem of further relaxation of the nature of sharing the set $\{\infty\}$ in Theorem E, but also given an exact lower bound of $\Theta(\infty; f) + \Theta(\infty; g)$ at the expense of allowing $n \geqslant 8$ in Theorem E in which the multiplicities of the poles cease to matter.

The present author has proved the following results.

Theorem H. [**3**] *Let S be defined as in Theorem* B *and n $(\geqslant 7)$ be an integer. If for two nonconstant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1 + \frac{29}{6nk+6n-5}$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},k) = E_g(\{\infty\},k)$, where $0 \leqslant k < \infty$, then $f \equiv g$.*

Theorem I. [**3**] *Let S be defined as in Theorem* B *and $n(\geqslant 8)$ be an integer. If for two nonconstant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, then $f \equiv g$.*

Regarding Theorems B–I following example establishes the fact that the set S can not be replaced by any arbitrary set containing six distinct elements.

Example 1.1. Let $f(z) = \sqrt{\alpha\beta\gamma}\ e^{z}$ and $g(z) = \sqrt{\alpha\beta\gamma}\ e^{-\,z}$ and $S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\alpha}, \beta\sqrt{\gamma}, \gamma\sqrt{\alpha}, \gamma\sqrt{\beta}\}$, where α, β and γ are three nonzero distinct complex numbers. Clearly $E_f(S,\infty) = E_g(S,\infty)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, but $f \not\equiv g$.

So it remains an open problem whether the degree of the equation defining S in Theorems B–I can be reduced to six. We here provide a solution. Also from the above discussion the following query is natural.

(i) *Keeping n intact in Theorem* E *and Theorem* I, *is it at all possible to further relax the conditions over ramification indexes in both theorems?*

We also provide an affirmative answer to the above question.

The following four theorems are the main results of the paper, which improve and complete all the previous results.

THEOREM 1.1. *Let S be defined as in Theorem* B, *where $n = 6$. If for two nonconstant meromorphic functions f and g, $\Theta_f + \Theta_g > 2$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, then $f \equiv g$, where $\Theta_f + \Theta_g = \Theta(0;f) + \Theta(\infty;f) + \Theta(0;g) + \Theta(\infty;g)$.*

THEOREM 1.2. *Let S be defined as in Theorem* B, *where $n = 7$. If for two nonconstant meromorphic functions f and g, $\Theta_f + \Theta_g > 1$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, then $f \equiv g$, where Θ_f, and Θ_g have the same meaning as defined in Theorem* 1.1.

THEOREM 1.3. *Let S be defined as in Theorem* B, *where $n = 7$. If for two nonconstant meromorphic functions f and g, $\Theta_f + \Theta_g > \frac{4}{3}$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, then $f \equiv g$, where Θ_f and Θ_g have the same meaning as defined in Theorem* 1.1.

THEOREM 1.4. *Let S be defined as in Theorem* B *and n ($\geqslant 8$) be an integer. If for two nonconstant meromorphic functions f and g, $\Theta_f + \Theta_g > \frac{4}{n-1}$, $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, then $f \equiv g$, where Θ_f and Θ_g have the same meaning as defined in Theorem* 1.1.

The following examples show that the condition $\Theta_f + \Theta_g > \frac{4}{n-1}$ is sharp in Theorem 1.4.

EXAMPLE 1.2. Let $f = -a\frac{1-h^{n-1}}{1-h^n}$ and $g = -ah\frac{1-h^{n-1}}{1-h^n}$, where $h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}$, $\alpha = \exp(\frac{2\pi i}{n})$ and n ($\geqslant 3$) is an integer. Then $T(r,f) = (n-1)T(r,h) + O(1)$ and $T(r,g) = (n-1)T(r,h) + O(1)$ and $T(r,h) = T(r,e^z) + O(1)$. Further we see that $h \neq \alpha, \alpha^2$ and so for any complex number $\gamma \neq \alpha, \alpha^2$, $\bar{N}(r,\gamma;h) \sim T(r,h)$. We also note that a root of $h = 1$ is not a pole and zero of f and g. Hence $\Theta(\infty;f) = \Theta(\infty;g) = \frac{2}{n-1}$. On the other hand

$$\Theta(0,f) = 1 - \limsup_{r\to\infty} \frac{\sum_{k=1}^{n-2} \bar{N}(r,\beta^k;h) + \bar{N}(r,\infty;h)}{(n-1)T(r,h) + O(1)} = 0,$$
$$\Theta(0,g) = 1 - \limsup_{r\to\infty} \frac{\sum_{k=1}^{n-2} \bar{N}(r,\beta^k;h) + \bar{N}(r,0;h)}{(n-1)T(r,h) + O(1)} = 0,$$

where $\beta = \exp\left(\frac{2\pi i}{n-1}\right)$. Clearly f and g share $(\infty;\infty)$ and $E_f(S,\infty) = E_g(S,\infty)$, because $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ but $f \not\equiv g$.

EXAMPLE 1.3. Let f and g be given as in Example 1.2, where $h = \frac{\alpha(\alpha e^z-1)}{e^z-1}$, $\alpha = exp(\frac{2\pi i}{n})$ and $n(\geqslant 3)$ is an integer.

Though the standard definitions and notations of the value distribution theory are available in [**7**], we explain some definitions and notations which are used in the paper.

DEFINITION 1.3. [**9**] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid \leqslant m)(N(r, a; f \mid \geqslant m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m, where each a-point is counted according to its multiplicity.

$\bar{N}(r, a; f \mid \leqslant m)(\bar{N}(r, a; f \mid \geqslant m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid < m)$, $N(r, a; f \mid > m)$, $\bar{N}(r, a; f \mid < m)$ and $\bar{N}(r, a; f \mid > m)$ are defined analogously.

DEFINITION 1.4. [**11**] Let $N_2(r, a; f)$ denote the sum $\bar{N}(r, a; f) + \bar{N}(r, a; f \mid \geqslant 2)$.

DEFINITION 1.5. We denote by $\bar{N}(r, a; f \mid = k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k \geqslant 2$ is an integer.

DEFINITION 1.6. [**10, 11**] Let f, g share a value a IM. We denote by $\bar{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\bar{N}_*(r, a; f, g) \equiv \bar{N}_*(r, a; g, f)$ and $\bar{N}_*(r, a; f, g) = \bar{N}_L(r, a; f) + \bar{N}_L(r, a; g)$, where by $\bar{N}_L(r, a; f)$ $(\bar{N}_L(r, a; g))$ we denote the reduced counting function of those a-points of f (g) which are greater than the a-points of g (f).

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined by

$$F = \frac{f^{n-1}(f+a)}{-b}, G = \frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.2}$$

LEMMA 2.1. [**15**] *Let f be a nonconstant meromorphic function and let $R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

LEMMA 2.2. [**12**, Lemma 5] *If f, g share $(\infty, 0)$, then for $n(\geqslant 2)$*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a,b are finite nonzero constants.

Lemma 2.3. [**3**, Lemma 2.13] *Let F, G share $(1,2)$, (∞,k) and $H \not\equiv 0$. Then*

i) $T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + \bar{N}(r,\infty;F) + \bar{N}(r,\infty;G)$
$+ \bar{N}_*(r,\infty;F,G) - m(r,1;G) - \bar{N}_E^{(3}(r,1;F) - \bar{N}_L(r,1;G)$
$+ S(r,F) + S(r,G)$

ii) $T(r,G) \leqslant N_2(r,0;F) + N_2(r,0;G) + \bar{N}(r,\infty;F) + \bar{N}(r,\infty;G)$
$+ \bar{N}_*(r,\infty;F,G) - m(r,1;F) - \bar{N}_E^{(3}(r,1;F) - \bar{N}_L(r,1;F)$
$+ S(r,F) + S(r,G)$

Lemma 2.4. *If f, g be two nonconstant meromorphic functions such that $\Theta(0;f) + \Theta(\infty;f) + \Theta(0;g) + \Theta(\infty;g) > \frac{4}{n-1}$, then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where n $(\geqslant 3)$ is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a) \tag{2.3}$$

and suppose $f \not\equiv g$. We consider two cases:

Case I Let $y = \frac{g}{f}$ be a constant. Then from (2.3) it follows that $y \neq 1$, $y^{n-1} \neq 1$, $y^n \neq 1$ and $f \equiv -a\frac{1-y^{n-1}}{1-y^n}$, a constant, which is impossible.

Case II Let $y = \frac{g}{f}$ be nonconstant. Then

$$f \equiv -a\frac{1-y^{n-1}}{1-y^n} \equiv a\Big(\frac{y^{n-1}}{1+y+y^2+\cdots+y^{n-1}} - 1\Big). \tag{2.4}$$

From (2.4) we see by Lemma 2.1 that

$$T(r,f) = T\left(r, \sum_{j=0}^{n-1} \frac{1}{y^j}\right) + O(1) = (n-1)\, T(r, \tfrac{1}{y}) + S(r,y) = (n-1)\, T(r,y) + S(r,y).$$

We first note that the zeros of $1 + y + y^2 + \cdots + y^{n-2}$ contributes to the zeros of both f and g. In addition to this the poles of y contributes to the zeros of f and since $g = fy$ the zeros of y contributes to the zeros of g. So from (2.4) we see that

$$\sum_{j=1}^{n-2} \bar{N}(r,v_j;y) + \bar{N}(r,\infty;y) \leqslant \bar{N}(r,0;f), \sum_{k=1}^{n-1} \bar{N}(r,u_k;y) \leqslant \bar{N}(r,\infty;f)$$

where $u_k = \exp(\frac{2k\pi i}{n})$ for $k = 1,2,\ldots,n-1$ and $v_j = \exp(\frac{2j\pi i}{n-1})$ for $j = 1,2,\ldots,n-2$.

By the second fundamental theorem we get

$$\begin{aligned}(2n-4)T(r,y) &\leqslant \bar{N}(r,\infty;y) + \sum_{j=1}^{n-2} \bar{N}(r,v_j;y) + \sum_{k=1}^{n-1} \bar{N}(r,u_k;y) + S(r,y)\\ &\leqslant \bar{N}(r,0;f) + \bar{N}(r,\infty;f) + S(r,y)\\ &\leqslant (2 - \Theta(0;f) - \Theta(\infty;f) + \varepsilon)\ T(r,f) + S(r,y)\\ &= (n-1)\ (2 - \Theta(0;f) - \Theta(\infty;f) + \varepsilon)\ T(r,y) + S(r,y)\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1}\,T(r,y) \leqslant (2-\Theta(0;f)-\Theta(\infty;f)+\varepsilon)\,T(r,y)+S(r,y), \tag{2.5}$$

where $0<2\varepsilon<\Theta(0;f)+\Theta(\infty;f)+\Theta(0;g)+\Theta(\infty;g)$.

Again noting that $\sum_{j=1}^{n-2}\bar{N}(r,v_j;y)+\bar{N}(r,0;y)\leqslant \bar{N}(r,0;g)$, by the second fundamental theorem we get

$$\begin{aligned}
&(2n-3)T(r,y)\\
&\qquad\leqslant \bar{N}(r,\infty;y)+\bar{N}(r,0;y)+\sum_{j=1}^{n-2}\bar{N}(r,v_j;y)+\sum_{k=1}^{n-1}\bar{N}(r,u_k;y)+S(r,y)\\
&\qquad\leqslant \bar{N}(r,\infty;y)+\bar{N}(r,0;g)+\bar{N}(r,\infty;g)+S(r,y)\\
&\qquad\leqslant \bar{N}(r,\infty;y)+(n-1)\,(2-\Theta(0;g)-\Theta(\infty;g)+\varepsilon)\;T(r,y)+S(r,y),
\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1}\,T(r,y) \leqslant (2-\Theta(0;g)-\Theta(\infty;g)+\varepsilon)\,T(r,y)+S(r,y), \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$\left(\frac{4n-8}{n-1}-4+\Theta(0;f)+\Theta(\infty;f)+\Theta(0;g)+\Theta(\infty;g)-2\varepsilon\right)T(r,y)\leqslant S(r,y),$$

which is a contradiction. Hence $f\equiv g$ and this proves the lemma. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F and G be given by (2.1) with $n=6$. Since $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},0)=E_g(\{\infty\},0)$ it follows that F, G share $(1,2)$ and $(\infty,0)$. So $\bar{N}_*(r,\infty;F,G)\leqslant \bar{N}(r,\infty;f)=\bar{N}(r,\infty;g)$.

Case 1. If possible let us suppose that $H\not\equiv 0$. Then from Lemmas 2.1–2.3 we obtain for $\varepsilon\ (>0)$

$$\begin{aligned}
6T(r,f) &\leqslant N_2(r,0;F)+N_2(r,0;G)+\bar{N}(r,\infty;F)+\bar{N}(r,\infty;G)\\
&\quad+\bar{N}(r,\infty;f)+S(r,F)+S(r,G)\\
&\leqslant 2\bar{N}(r,0;f)+N_2(r,0;f+a)+2\bar{N}(r,0;g)+N_2(r,0;g+a)\\
&\quad+\bar{N}(r,\infty;f)+\bar{N}(r,\infty;g)+\frac{1}{2}\{\bar{N}(r,\infty;f)+\bar{N}(r,\infty;g)\}\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant \frac{1}{2}\{\bar{N}(r,0;f)+\bar{N}(r,0;g)\}+N_2(r,0;f+a)+N_2(r,0;g+a)\\
&\quad+\frac{3}{2}\{\bar{N}(r,0;f)+\bar{N}(r,0;g)+\bar{N}(r,\infty;f)+\bar{N}(r,\infty;g)\}\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant 3T(r)+\frac{3}{2}(4-\Theta(0;f)-\Theta(\infty;f)-\Theta(0;g)-\Theta(\infty;g)+\varepsilon)T(r)+S(r).
\end{aligned} \tag{3.1}$$

Similarly we obtain

$$(3.2)\quad 6T(r,g) \leqslant 3T(r) + \frac{3}{2}(4 - \Theta(0;f) - \Theta(\infty;f) - \Theta(0;g) - \Theta(\infty;g) + \varepsilon)T(r) + S(r).$$

Combining (3.1) and (3.2) we obtain

$$(3.3)\qquad \left[\tfrac{3}{2}\left\{\Theta(0;f) + \Theta(\infty;f) + \Theta(0;g) + \Theta(\infty;g)\right\} - 3 - \varepsilon\right] T(r) \leqslant S(r).$$

Clearly (3.3) leads to a contradiction for $0 < \varepsilon < [\frac{3}{2}(\Theta_f + \Theta_g) - 3]$.

Case 2. $H \equiv 0$. On integration we get from (2.2)

$$(3.4)\qquad \frac{1}{F-1} \equiv \frac{A}{G-1} + B,$$

where A, B are constants and $A \neq 0$. From (3.4) we obtain

$$(3.5)\qquad F \equiv \frac{(B+1)G + A - B - 1}{BG + A - B}.$$

Clearly (3.5) together with Lemma 2.1 yields

$$(3.6)\qquad T(r,f) = T(r,g) + O(1).$$

Subcase 2.1. Suppose that $B \neq 0, -1$. If $A - B - 1 \neq 0$, from (3.5) we obtain

$$\bar{N}\Big(r, \frac{B+1-A}{B+1}; G\Big) = \bar{N}(r,0;F).$$

From above, Lemma 2.1 and the second fundamental theorem we obtain

$$\begin{aligned}6T(r,g) &< \bar{N}(r,\infty;G) + \bar{N}(r,0;G) + \bar{N}\Big(r, \frac{B+1-A}{B+1}; G\Big) + S(r,g)\\ &\leqslant \bar{N}(r,\infty;g) + \bar{N}(r,0;g) + \bar{N}(r,0;g+a) + \bar{N}(r,0;f) + \bar{N}(r,0;f+a) + S(r,g)\\ &\leqslant 2T(r,f) + 3T(r,g) + S(r,g),\end{aligned}$$

which in view of (3.6) implies a contradiction. Thus $A - B - 1 = 0$ and hence (3.6) reduces to $F \equiv \frac{(B+1)G}{BG+1}$. From this we have $\bar{N}(r, \frac{-1}{B}; G) = \bar{N}(r,\infty;f)$. Again by Lemma 2.1 and the second fundamental theorem we have

$$\begin{aligned}6T(r,g) &< \bar{N}(r,\infty;G) + \bar{N}(r,0;G) + \bar{N}(r, \tfrac{-1}{B}; G) + S(r,g)\\ &\leqslant \bar{N}(r,\infty;g) + \bar{N}(r,0;g) + \bar{N}(r,0;g+a) + \bar{N}(r,\infty;f) + S(r,g)\\ &\leqslant T(r,f) + 3T(r,g) + S(r,g),\end{aligned}$$

which in view of (3.6) leads to a contradiction.

Subcase 2.2. Suppose that $B = -1$. From (3.5) we obtain

$$(3.7)\qquad F \equiv \frac{A}{-G + A + 1}.$$

If $A + 1 \neq 0$, from (3.7) we obtain $\bar{N}(r, A+1; G) = \bar{N}(r,\infty;f)$. So using the same argument as in the above subcase we can again obtain a contradiction. Hence $A+1 = 0$ and we have from (3.7) that $FG \equiv 1$ that means $f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2$, which is impossible by Lemma 2.2.

Subcase 2.3. Suppose that $B = 0$. From (3.5) we obtain

$$F \equiv \frac{G + A - 1}{A}. \tag{3.8}$$

If $A - 1 \neq 0$, from (3.8) we obtain $\bar{N}(r, 1 - A; G) = \bar{N}(r, 0; F)$. So in the same manner as above we again get a contradiction. So $A = 1$ and hence $F \equiv G$ that is $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$. Now the theorem follows from Lemma 2.4. ☐

Proof of Theorem 1.2. Let F and G be given by (2.1) with $n = 7$. Since $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ it follows that F, G share $(1, 2)$ and (∞, ∞). So $\bar{N}_*(r, \infty; F, G) = 0$. We now omit the proof since the remaining part of the theorem can be proved in the line of proof of Theorem 1.1 ☐

Proof of Theorem 1.3. We omit the proof since the proof of the theorem can be carried out in the line of proof of Theorem 1.1. ☐

Proof of Theorem 1.4. Let F and G be given by (2.1). When $H \not\equiv 0$ we adopt the same procedure as done in the proof of Theorem 1.2 in [**3**]. When $H \equiv 0$, using Lemmas 2.8, 2.11, 2.12 of [**3**] and Lemmas 2.2, 2.3 and 2.4 we can easily get the desired result. So we omit it. ☐

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Department of Mathematics
West Bengal State University
Barasat, 24 Prgs. (North)
Kolkata-700126, West Bengal
India

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Current Address:
Department of Mathematics
University of Kalyani
Nadia, 741235, West Bengal
India
abanerjee_kal@yahoo.co.in
abanerjee_kal@rediffmail.com