PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 94 (108) (2013), 3–15 DOI: 10.2298/PIM1308003I

GROMOV MINIMAL FILLINGS FOR FINITE METRIC SPACES

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ABSTRACT. The problem discussed in this paper was stated by Alexander O. Ivanov and Alexey A. Tuzhilin in 2009. It stands at the intersection of the theories of Gromov minimal fillings and Steiner minimal trees. Thus, it can be considered as one-dimensional stratified version of the Gromov minimal fillings problem. Here we state the problem; discuss various properties of one-dimensional minimal fillings, including a formula calculating their weights in terms of some special metrics characteristics of the metric spaces they join (it was obtained by A. Yu. Eremin after many fruitful discussions with participants of Ivanov–Tuzhilin seminar at Moscow State University); show various examples illustrating how one can apply the developed theory to get nontrivial results; discuss the connection with additive spaces appearing in bioinformatics and classical Steiner minimal trees having many applications, say, in transportation problem, chip design, evolution theory etc. In particular, we generalize the concept of Steiner ratio and get a few of its modifications defined by means of minimal fillings, which could give a new approach to attack the long standing Gilbert–Pollack Conjecture on the Steiner ratio of the Euclidean plane.

1. Introduction

The problem considered in this paper appears as a result of a synthesis of two classical problems: the Steiner problem on the shortest networks, and Gromov's problem on minimal fillings.

The classical Steiner problem asks how one can connect a finite set of n points of the Euclidean space to minimize the length of the obtained network. The first variant of the problem appeared in works of Fermat, who stated the question on finding the location of a point, such that the sum of the distances from it to the vertices of a given triangle is minimal (this can be considered as $n = 3$ case of the general Steiner problem). A few centuries later a complete answer was obtained by

2010 *Mathematics Subject Classification:* 51F99; 51K99.
Key words and phrases: finite metric spaces, Gromov minimal fillings, Steiner minimal trees, extreme networks, Steiner ratio, additive spaces.
Partially supported by RFBR grant 10-01-00748, the Programme for Support of Leading Scientific Schools (grant NSh-1410.2012.1), the Programme of Russian government, project 11.G34.31.0053.

Torricelli, Simpson, etc., see details in [**2**]. The next important case $n = 4$ appeared in Gauss' letters to Schumacher, where they discussed how to construct the shortest road network joining famous German cities Hamburg, Bremen, Hannover and Braunschweig. In 1934 Jarnik and Kössler [**3**] stated the general problem which is now known as Classical Steiner Problem (the priority misunderstanding appeared due to popular book of Courant and Robbins "What is Mathematics?" [**4**], where Fermat problem is referred as Steiner problem, and the Jarnik and Kössler problem is called just a generalization of the Steiner problem). As it concerns Steiner, he worked with another generalization of the Fermat problem: to find a point in the space such that the sum of the distances from it to the given ones is minimal.

Let us mention that the classical Steiner problem has numerous generalizations. First, one can change the ambient space, say, to a normed space, a Riemannian manifold, or some more general metric space, e.g., space of words; another possibility is to change the length functional to some other one, say, energy functional, or general Lagrangian-type functional (see a review in [**2**]). Another surge of interest in Steiner problem is related with Gilbert–Pollack Conjecture [**1**] on the Steiner ratio of the Euclidean plane. Numerous attempts to prove it have failed. The best known attempt belongs to Du and Hwang [**5**], but it turns out that their reasoning contains serious gaps, see [**6, 2, 7, 8, 9, 10**]. Ideas of Du and Hwang were advertised in the stage of announcing publications [**14**] that has led to popularization of the wrong construction. Several papers appeared (for example, [**15, 16**]) where the ideas of Du and Hwang were adopted to the case under consideration, and, as a result, some unfounded conclusions got the status of theorems. Notice that the validity of Gilbert–Pollack conjecture itself seems undoubted, therefore attempts to prove it appear again and again. In particular, numerous authors, including the authors of the present paper, have tried to improve the construction of Du and Hwang, but without succes. It might make sense to search for a completely different approach to the problem. The minimal fillings discussed in the present paper could be a base for such an approach.

The concept of a minimal filling appeared in papers of Gromov [**17**]. Let M be a manifold endowed with a distance function ρ. Consider all possible films W spanning M, i.e., compact manifolds with the boundary M. Consider on W a distance function d non-decreasing the distances between the points in M. Such a metric space $\mathcal{W} = (W, d)$ is called a *filling* of the metric space $\mathcal{M} = (M, \rho)$. The Gromov Problem consists in calculating the infimum of the volumes of the fillings and describing the spaces $\mathcal{W}$ where this infimum is achieved (such spaces are called *minimal fillings*).

An interest in minimal fillings is inspired, first of all, by the fact that many classical geometrical inequalities such as the Bezikovich one or the Pu one can be stated in terms of the fillings (see [**19**] and the dissertation of Ivanov [**20**]). Notice also that minimal fillings possess numerous applications in dynamic systems theory, asymptotic geometry, mathematical physics, etc.

In the scope of the Steiner problem, it is natural to consider M as a finite metric space. Then the possible fillings are metric spaces having the structure of one-dimensional stratified manifolds which can be considered as graphs whose edges have nonnegative weights. This leads to the following particular case of the generalized Gromov problem.

Let M be an arbitrary finite set, and $G = (V, E)$ be a connected graph. We say, that G *joins* M, if $M \subset V$. Now, let $\mathcal{M} = (M, \rho)$ be a finite metric space, $G = (V, E)$ be a connected graph joining M, and $\omega\colon E \to \mathbb{R}_+$ is a mapping into nonnegative numbers, which is usually referred to as a *weight function* and which generates the *weighted graph* $\mathcal{G} = (G, \omega)$. The function ω generates on V the pseudometric d_ω (some distances can be zero), namely, the distance between the vertices of the graph $\mathcal{G}$ is defined as the least possible weight of the paths in $\mathcal{G}$ joining these vertices. If for any two points p and q from M the inequality $\rho(p, q) \leqslant d_\omega(p, q)$ holds, then the weighted graph $\mathcal{G}$ is called a *filing* of the space $\mathcal{M}$, and the graph G is referred to as the *type* of this filing. The value $\mathrm{mf}(\mathcal{M}) = \inf \omega(\mathcal{G})$, where the infimum is taken over all the filings $\mathcal{G}$ of the space $\mathcal{M}$ is called the *weight of minimal filling*, and each filling $\mathcal{G}$ such that $\omega(\mathcal{G}) = \mathrm{mf}(\mathcal{M})$ is called a *minimal filling*.

A weighted graph $\mathcal{G}$ can be considered as a one-dimensional stratified manifold. Notice that stratified manifolds have appeared naturally in geometric problems; see for example [**21, 22, 24**], and in such applications as quantum physics [**25, 23**].

2. Preliminaries

In the present paragraph we discuss a few more optimization problems closely related with minimal fillings.

Let $\mathcal{X} = (X, d)$ be a metric space and $G = (V, E)$ an arbitrary connected graph. Any mapping $\Gamma\colon V \to X$ is called a *network in $\mathcal{X}$ parameterized by the graph $G = (V, E)$*, or a *network of the type G*. The *vertices* and *edges* of the network Γ are the restrictions of the mapping Γ onto the vertices and edges of the graph G, respectively. The *length of the edge* $\Gamma\colon vw \to X$ is the value $d\big(\Gamma(v), \Gamma(w)\big)$, and the *length $d(\Gamma)$ of the network* Γ is the sum of lengths of all its edges.

In what follows we shall consider various boundary value problems for graphs. To do that, we fix some subsets ∂G of the vertices sets V of our graphs $G = (V, E)$, and we call such ∂G the *boundaries*. We always suppose that in each graph under consideration a boundary, possibly, an empty one, is chosen. The *boundary $\partial\Gamma$ of a network* Γ is the restriction of Γ onto ∂G. If $M \subset X$ is finite and $M \subset \Gamma(V)$, then we say that the network Γ *joins* or *connects the set M*. The vertices of graphs and networks which are not boundary ones are called *interior* vertices.

The value $\mathrm{smt}(M) = \inf\{d(\Gamma) \mid \Gamma \text{ is a network joining } M\}$ is called the *length of shortest network for M*. Notice that the network Γ which joins M and satisfies $d(\Gamma) = \mathrm{smt}(M)$ may not exist, see [**6**] and [**27**] for nontrivial examples. If such a network exists, it is called a *shortest network joining M*, or *for M*. One variant of the Steiner problem is to describe the shortest networks joining finite subsets of metric spaces.[1]

Now let us define minimal parametric networks in a metric space $\mathcal{X} = (X, d)$. Let $G = (V, E)$ be a connected graph with some boundary ∂G, and let $\varphi\colon \partial G \to X$

[1]The denotation smt is an acronym for “Steiner Minimal Tree” which is a synonym for the shortest network whose edges are nondegenerate and, thus, it must be a tree.

be a mapping. By $[G,\varphi]$ we denote the set of all networks $\Gamma\colon V\to X$ of the type G such that $\partial\Gamma=\varphi$. We put $\operatorname{mpn}(G,\varphi)=\inf_{\Gamma\in[G,\varphi]}d(\Gamma)$ and we call this value the *length of minimal parametric network.* If there exists a network $\Gamma\in[G,\varphi]$ such that $d(\Gamma)=\operatorname{mpn}(G,\varphi)$, then Γ is called a *minimal parametric network of the type G with the boundary φ.*

PROPOSITION 2.1. *Let $\mathcal{X}=(X,d)$ be an arbitrary metric space and M be a finite subset of X. Then* $\operatorname{smt}(M)=\inf\{\operatorname{mpn}(G,\varphi)\mid\varphi(\partial G)=M\}$.

Thus, the problem of calculating the length of the shortest network is reduced to investigation of minimal parametric networks.

Let $\mathcal{M}=(M,\rho)$ be a finite metric space and $G=(V,E)$ an arbitrary connected graph joining M. In this case we always assume that the boundary of such G is fixed and equal to M. By $\Omega(\mathcal{M},G)$ we denote the set of all weight functions $\omega\colon E\to\mathbb{R}$ such that (G,ω) is a filling of the space $\mathcal{M}$. We put $\operatorname{mpf}(\mathcal{M},G)=\inf_{\omega\in\Omega(\mathcal{M},G)}\omega(G)$ and we call this value the *weight of minimal parametric filling of the type G for the space $\mathcal{M}$.* If there exists a weight function $\omega\in\Omega(\mathcal{M},G)$ such that $\omega(G)=\operatorname{mpf}(\mathcal{M},G)$, then (G,ω) is called a *minimal parametric filling of the type G for the space $\mathcal{M}$.*

PROPOSITION 2.2. *Let $\mathcal{M}=(M,\rho)$ be a finite metric space. Then* $\operatorname{mf}(\mathcal{M})=\inf\bigl\{\operatorname{mpf}(\mathcal{M},G)\bigr\}$.

It is not difficult to show that to investigate shortest networks and minimal fillings one can restrict the consideration to trees such that all their vertices of degree 1 and 2 belong to their boundaries. *In what follows, we always assume that this condition holds, providing the opposite is not declared.*

To be more precise, we recall the following definition. We say that a tree is a *binary* one if the degrees of its vertices can be 1 or 3 only, and the boundary consists just of all vertices of degree 1. Then each finite metric space has a binary minimal filling (possibly, with some degenerate edges), and a nondegenerate minimal filling (whose type is a tree and all whose vertices of degree 1 and 2 belong to its boundary in accordance with the above agreement).

3. Minimal realization

In this section we show that the problem on minimal filling can be reduced to the Steiner problem in special metric spaces and for special boundaries.

Consider a finite set $M=\{p_1,\dots,p_n\}$, and let $\mathcal{M}=(M,\rho)$ be a metric space. We put $\rho_{ij}=\rho(p_i,p_j)$. By ℓ_∞^n we denote the n-dimensional arithmetic space with the norm $\bigl\|(v^1,\dots,v^n)\bigr\|_\infty=\max\bigl\{|v^1|,\dots,|v^n|\bigr\}$, and by ρ_∞ the metric on ℓ_∞^n generated by $\|\cdot\|_\infty$, i.e., $\rho_\infty(v,w)=\|w-v\|_\infty$. Let us define a mapping $\varphi_{\mathcal{M}}\colon M\to\ell_\infty^n$ by the formula $\varphi_{\mathcal{M}}(p_i)=\bar p_i=(\rho_{i1},\dots,\rho_{in})$.

PROPOSITION 3.1. *The mapping $\varphi_{\mathcal{M}}$ is an isometry with its image.*

The mapping $\varphi_{\mathcal{M}}$ is called the *Kuratowski isometry.*

Let $\mathcal{G}=(G,\omega)$ be a filling of a space $\mathcal{M}=(M,\rho)$, where $G=(V,E)$, and d_ω be the pseudometric on V generated by the weight function ω. Denote by E_M the edges set of the complete graph on M and put $\bar{G} = (V, \bar{E} = E \cup E_M)$. Let $\bar{\omega}$ be the weight function on $\bar{E}$ coinciding with metric ρ on E_M and with ω on $\bar{E} \setminus E_M$. Recall that $d_{\bar{\omega}}$ denotes the pseudometric on V generated by $\bar{\omega}$.

We define the network $\Gamma_{\mathcal{G}}\colon V \to \ell_\infty^n$ of the type G as follows: $\Gamma_{\mathcal{G}}(v) = \big(d_{\bar{\omega}}(v, p_1), \dots, d_{\bar{\omega}}(v, p_n)\big)$. This network is called the *Kuratowski network for the filling $\mathcal{G}$*.

PROPOSITION 3.2. *We have $\partial\Gamma_{\mathcal{G}} = \varphi_{\mathcal{M}}$.*

For any network Γ in a metric space (X, d) by $\omega_{\mathcal{G}}amma$ we denote the *weight function on G induced by the network* Γ, i.e., $\omega_{\mathcal{G}}amma(vw) = d\big(\Gamma(v), \Gamma(w)\big)$.

COROLLARY 3.1. *Let $\mathcal{G} = (G, \omega)$ be a minimal parametric filling of a metric space (M, ρ) and $\Gamma = \Gamma_{\mathcal{G}}$ be the corresponding Kuratowski network. Then $\omega = \omega_{\mathcal{G}}amma$.*

Let Γ be a network in a metric space $\mathcal{X}$, let G be its parameterizing graph, and $\mathcal{H} = (H, \omega)$ be a weighted graph. We say that Γ *and $\mathcal{H}$ are isometric*, if there exists an isomorphism of the weighted graphs $\mathcal{H}$ and $\mathcal{G} = (G, \omega_{\mathcal{G}}amma)$.

Corollary 3.1 and the existence of minimal parametric and shortest networks in a finite-dimensional normed space [**26**] imply the following result.

COROLLARY 3.2. *Let $\mathcal{M} = (M, \rho)$ be a metric space consisting of n points, and $\varphi_{\mathcal{M}}\colon M \to \ell_\infty^n$ be the Kuratowski isometry. For any graph G joining M there exists a minimal parametric filling of the type G of the space $\mathcal{M}$. Each minimal parametric filling of the type G of the space $\mathcal{M}$ is isometric to the corresponding Kuratowski network, which is, in this case, a minimal parametric network of the type G with the boundary $\varphi_{\mathcal{M}}$. Conversely, each minimal parametric network of the type G on $\varphi_{\mathcal{M}}(M)$ is isometric to some minimal parametric filling of the type G of the space $\mathcal{M}$.*

COROLLARY 3.3. *Let $\mathcal{M} = (M, \rho)$ be a metric space consisting of n points, and $\varphi_{\mathcal{M}}\colon M \to \ell_\infty^n$ be the Kuratowski isometry. Then there exists a minimal filling $\mathcal{G}$ for $\mathcal{M}$, and the corresponding Kuratowski network $\Gamma_{\mathcal{G}}$ is a shortest network in the space ℓ_∞^n joining the set $\varphi_{\mathcal{M}}(M)$. Conversely, each shortest network on $\varphi_{\mathcal{M}}(M)$ is isometric to some minimal filling of the space $\mathcal{M}$.*

4. Minimal Parametric Fillings and Linear Programming

Let $\mathcal{M} = (M, \rho)$ be a finite metric space joined by a (connected) graph $G = (V, E)$. As above, by $\Omega(\mathcal{M}, G)$ we denote the set consisting of all the weight functions $\omega\colon E \to \mathbb{R}_+$ such that $\mathcal{G} = (G, \omega)$ is a filling of the space $\mathcal{M}$, and by $\Omega_m(\mathcal{M}, G)$ we denote its subset consisting of the weight functions such that $\mathcal{G}$ is a minimal parametric filling of the space $\mathcal{M}$.

PROPOSITION 4.1. *The set $\Omega(\mathcal{M}, G)$ is closed and convex in the linear space $\mathbb{R}^E$ of all the functions on E, and $\Omega_m(\mathcal{M}, G) \subset \Omega(\mathcal{M}, G)$ is a nonempty convex compact.*

Remark 4.1. The proof of Proposition 4.1 shows that the problem of searching for a minimal parametric filling of a metric space can be reduced to a linear programming.

5. Generalized Fillings and Formula for the Weight of Minimal Filling

In this section we give a review of the recent results obtained by our group, see [**11**] and [**12**].

5.1. Generalized fillings. Investigating the fillings of metric spaces, it turns out to be convenient to expand the class of weighted trees under consideration permitting arbitrary weights of the edges (not only nonnegative). The corresponding objects are called *generalized fillings*, *minimal generalized fillings* and *minimal parametric generalized fillings*. Their weights for a metric space $\mathcal{M}$ and a tree G are denoted by $\mathrm{mf}_-(\mathcal{M})$ and $\mathrm{mpf}_-(\mathcal{M},G)$, respectively.

For any finite metric space $\mathcal{M}=(M,\rho)$ and a tree G joining M, the next evident inequality is valid: $\mathrm{mpf}_-(\mathcal{M},G)\leqslant \mathrm{mpf}(\mathcal{M},G)$. And it is not difficult to construct an example, when this inequality becomes strict. However, for minimal generalized fillings the following result holds, see [**11**].

Theorem 5.1 (A. Ivanov, Z. Ovsyannikov, N. Strelkova, A. Tuzhilin). *The set of all minimal generalized fillings of an arbitrary finite metric space $\mathcal{M}$ contains its minimal filling, i.eȧ generalized minimal filling with nonnegative weight function. Hence,* $\mathrm{mf}_-(\mathcal{M})=\mathrm{mf}(\mathcal{M})$.

5.2. Multitours and the true formula for the weight of minimal filling. Let $\mathcal{M}=(M,\rho)$ be a finite metric space, and G be a tree joining M. Choose an arbitrary embedding G' of the tree G into the plane. Consider a walk around the tree G'. We draw the points of M consecutive with respect to this walk as a consecutive points of the circle S^1. Notice that each vertex p from M appears $\deg p$ times. For each vertex $p\in M$ of degree more than 1, we choose just one arbitrary point from the corresponding points of the circle. So, we construct an injection $\nu\colon M\to S^1$. Define a cyclic permutation π as follows: $\pi(p)=q$, where $\nu(q)$ follows after $\nu(p)$ on the circle S^1. We say that π *is generated by the embedding* G' (this procedure is not unique due to different possible choices of ν). Each π generated in this manner is called a *tour of M w.r.t. G*. The set of all tours on M w.r.t. G is denoted by $\mathcal{O}(M,G)$. For each tour $\pi\in\mathcal{O}(M,G)$ we put $p(\mathcal{M},G,\pi)=\frac12\sum_{x\in M}\rho\big(x,\pi(x)\big)$ and we call the value by the *half-perimeter of the space $\mathcal{M}$ w.r.t. the tour π*. The minimal value of $p(\mathcal{M},G,\pi)$ over all $\pi\in\mathcal{O}(M,G)$ for all possible G (in fact, over all possible cyclic permutations π on M) is called the *half-perimeter of the space $\mathcal{M}$*.

A. Ivanov and A. Tuzhilin proposed the following hypothesis.

Conjecture 5.1. *For an arbitrary metric space $\mathcal{M}=(M,\rho)$ the following formula is valid* $\mathrm{mf}(\mathcal{M})=\min_G\max_{\pi\in\mathcal{O}(M,G)}p(\mathcal{M},G,\pi)$, *where minimum is taken over all binary trees G joining M.*

Eremin [**12**] constructed a counter-example to the Conjecture 5.1 and showed that if one changes the concept of tour by the one of multitour, introduced by him, then the Conjecture 5.1 holds.

To define the multitours, let us consider the graph in which every edge of G is taken with the multiplicity $2k$, $k \geqslant 1$. The resulting graph possesses an Euler cycle consisting of *irreducible* boundary paths – the ones which do not contain properly other boundary paths. This Euler cycle generates a bijection $\pi\colon X \to X$, where $X = \sqcup_{i=1}^{k} M$, which is called *multitour of M w.r.t. G.* The set of all multitours on M w.r.t. G is denoted by $\mathcal{O}_\mu(M,G)$.

Let $\mathcal{M} = (M,\rho)$ be a finite metric space, and G be a tree joining M. As in the case of tours, for each multitour $\pi \in \mathcal{O}_\mu(M,G)$ we put $p(\mathcal{M},G,\pi) = \frac{1}{2k}\sum_{x\in X}\rho\bigl(x,\pi(x)\bigr)$.

Theorem 5.2 (A. Eremin [**12**]). *For an arbitrary finite metric space $\mathcal{M} = (M,\rho)$ and an arbitrary tree G joining M, the weight of minimal parametric generalized filling can be calculated as follows* $\mathrm{mpf}_-(\mathcal{M},G) = \max\bigl\{p(\mathcal{M},G,\pi) \mid \pi \in \mathcal{O}_\mu(M,G)\bigr\}$. *The weight of minimal filling can be calculated as follows* $\mathrm{mf}(\mathcal{M}) = \mathrm{mf}_-(\mathcal{M}) = \min_G \max\bigl\{p(\mathcal{M},G,\pi) \mid \pi \in \mathcal{O}_\mu(M,G)\bigr\}$, *where the minimum is taken over all binary trees G joining M.*

6. Minimal Fillings for Generic Metric Spaces

Theorem 5.2 gives an opportunity to get several interesting corollaries. To formulate one of them, we need to define what is a “generic” metric space. Notice that the set of all metric spaces consisting of n points can be naturally identified with a convex cone in $\mathbb{R}^{n(n-1)/2}$ (it suffices to enumerate the set of all two-elements subsets of these spaces and assign to each such space the vector of the distances between the pairs of points). This representation gives us an opportunity to speak about topological properties of families of metric spaces consisting of a fixed number of points.

We say, that some property holds for a *generic metric space*, if for any n this property is valid for an everywhere dense set of n-point metric spaces.

The following result can be found in [**12**].

Corollary 6.1 (A. Eremin). *Each general finite metric space has a minimal filling which is a nondegenerate binary tree.*

7. Additive Spaces

The additive spaces are very popular in bioinformatics, playing an important role in evolution theory. Recall that a finite metric space $\mathcal{M} = (M,\rho)$ is called *additive*, if M can be joined by a weighted tree $\mathcal{G} = (G,\omega)$ such that ρ coincides with the restriction of d_ω onto M. The tree $\mathcal{G}$ in this case is called *generating tree* for the space $\mathcal{M}$.

Not any metric space is additive. It turns out that an additivity criterion can be stated in terms of well-known 4 *points rule*: for any four points p_i, p_j, p_k, p_l, the values $\rho(p_i,p_j)+\rho(p_k,p_l)$, $\rho(p_i,p_k)+\rho(p_j,p_l)$, $\rho(p_i,p_l)+\rho(p_j,p_k)$ are the lengths of sides of an isosceles triangle whose base does not exceed its other sides.

Proposition 7.1. [**29, 30, 31, 32**] *A metric space is additive, if and only if it meets the* 4 *points rule. In the class of nondegenerate weighted trees, the generating tree of an additive metric space is unique.*

The next criterion solves completely the minimal filling problem for additive metric spaces.

Theorem 7.1. *Minimal fillings of an additive metric space are exactly its generating trees.*

The next additivity criterion is obtained by O. Rubleva, a student of mechanical and mathematical faculty of Moscow State University, see [**33**].

Proposition 7.2 (O. Rubleva). *The weight of a minimal filling of a finite metric space is equal to the half-perimeter of this space, if and only if this space is additive.*

In the scope of Proposition 7.2, we conjectured that if there exists a tree joining a metric space such that all the corresponding half-perimeters are equal to each other, then the space is additive. It turns out that it is not true. Z. Ovsyannikov suggested to consider a wider class of spaces, so called pseudo-additive spaces, for which our conjecture becomes true, see [**13**].

A finite metric space $\mathcal{M}=(M,\rho)$ is said to be *pseudo-additive*, if the metric ρ coincides with d_ω for a weighted tree (G,ω) (which is also called *generating*), where the weight function ω can take arbitrary (not necessary nonnegative) values. Z. Ovsyannikov shows that these spaces can be described in terms of the so-called *weak* 4*-points rule*: for any four points p_i, p_j, p_k, p_l, the values $\rho(p_i,p_j)+\rho(p_k,p_l)$, $\rho(p_i,p_k)+\rho(p_j,p_l)$, $\rho(p_i,p_l)+\rho(p_j,p_k)$ are the lengths of sides of an isosceles triangle. The generating tree is also unique in the class of nondegenerate trees. Moreover, the following result is valid, see [**13**].

Theorem 7.2 (Z. Ovsyannikov). *Let $\mathcal{M}=(M,\rho)$ be a finite metric space. Then the following statements are equivalent.*

- *There exists a tree G such that M coincides with the set of degree* 1 *vertices of G and all the half-perimeters $p(M,G,\pi)$ of M corresponding to the tours of G are equal to each other.*
- *The space $\mathcal{M}$ is pseudo-additive.*

Moreover, the tree G in this case is a generating tree for the space $\mathcal{M}$.

It would be interesting to see what role could play these pseudo-additive spaces in applications.

8. Rays of Metric Spaces and Minimal Fillings

Let $\mathcal{M}=(M,\rho)$ be a metric space and λ be a positive real number. By $\lambda\rho$ we denote the function on the pairs of points from M, defined as follows:

$(\lambda\rho)(x,y) = \lambda\rho(x,y)$. It is clear that $\lambda\rho$ is a metric for $\lambda > 0$, and that (G,ω) is a minimal (parametric) filling of the space (M,ρ), if and only if $(G,\lambda\omega)$ is a minimal (parametric) filling of the space $(M,\lambda\rho)$. The set of all metric spaces $(M,\lambda\rho)$, $\lambda > 0$, we call by an *open multiplicative ray* passing through $\mathcal{M}$.

By $\rho + a$ we denote the function on the pairs of points from M defined as follows: $(\rho + a)(x,y) = \rho(x,y) + a$. It is easy to see, that there exists a value $a_{\mathcal{M}} \leqslant 0$ such that the function $\rho + a$ is a metric for $a > a_{\mathcal{M}}$, and is not a metric for $a < a_{\mathcal{M}}$. The set of all the metric spaces $\mathcal{M} + a$, $a > a_{\mathcal{M}}$, we call by an *additive ray with the vertex at $\mathcal{M} + a_{\mathcal{M}}$, passing through $\mathcal{M}$.*

Theorem 8.1. *Let $\mathcal{G} = (G,\omega)$ be a minimal filling of a metric space $\mathcal{M} = (M,\rho)$, and M coincides with the set of degree 1 vertices of the tree G. Let b be the least weight of the boundary edges of $\mathcal{G}$. Then $-2b < a_{\mathcal{M}}$, and, hence, for any $a \geqslant a_{\mathcal{M}}$, the function ω_a obtained from ω by adding $a/2$ to all the weights of the boundary edges is nonnegative. Moreover, for any such a, the weighted tree $\mathcal{G}_a = (G,\omega_a)$ is a minimal filling of the space $\mathcal{M} + a$. In particular, all the spaces of the form $(M,\lambda\rho + a)$, $\lambda > 0$, $a > \lambda a_{\mathcal{M}}$, have the same set of types of minimal fillings.*

9. Examples of Minimal Fillings

In this section we give several examples of minimal filling and demonstrate how to use the technique elaborated above.

9.1. Triangle. Let $\mathcal{M} = (M,\rho)$ consist of three points p_1, p_2, and p_3. Put $\rho_{ij} = \rho(p_i,p_j)$. Consider the tree $G = (V,E)$ with $V = M \cup \{v\}$ and $E = \{vp_i\}_{i=1}^3$. Define the weight function ω on E by the following formula: $\omega(e_i) = (\rho_{ij} + \rho_{ik} - \rho_{jk})/2$, where $\{i,j,k\} = \{1,2,3\}$. Notice that d_ω restricted onto M coincides with ρ. Therefore, $\mathcal{M}$ is an additive space, $\mathcal{G} = (G,\omega)$ is a generating tree for $\mathcal{M}$, and, due to Theorem 7.1, $\mathcal{G}$ is a minimal filling of $\mathcal{M}$.

Recall that the value $(\rho_{ij}+\rho_{ik}-\rho_{jk})/2$ is called by the *Gromov product* $(p_j,p_k)_{p_i}$ of the points p_j and p_k of the space $\mathcal{M}$ with respect to the point p_i, see [**18**].

9.2. Regular Simplex. Let all the distances in the metric space $\mathcal{M}$ are the same and are equal to d, i.e., $\mathcal{M}$ is a regular simplex. Then the weighted tree $\mathcal{G} = (G,\omega)$, $G = (V,E)$, with the vertex set $V = M \cup \{v\}$ and edges vm, $m \in M$, of the weight $d/2$ is generating for $\mathcal{M}$. Therefore, the space $\mathcal{M}$ is additive, and, due to Theorem 7.1, $\mathcal{G}$ is its unique nondegenerate minimal filling. If n is the number of points in M, then the weight of the minimal filling is equal to $dn/2$.

9.3. Star. If a minimal filling $\mathcal{G} = (G,\omega)$ of a space $\mathcal{M} = (M,\rho)$ is a star whose single interior vertex v is joined with each point $p_i \in M$, $1 \leqslant i \leqslant n$, $n \geqslant 3$, then the metric space $\mathcal{M}$ is additive [**35, 36**]. In this case the weights of edges can be calculated easily. Indeed, put $e_i = vp_i$. Since a subspace of an additive space is additive itself, then we can use the results for three-points additive space, see above. So, we have $\omega(e_i) = (p_j,p_k)_{p_i}$, where p_i, p_j, and p_k are arbitrary distinct boundary vertices, and ρ_{ij} is the distance between the corresponding points.

9.4. Mustaches of Degree more than 2. Let $G=(V,E)$ be an arbitrary tree, and $v\in V$ be an interior vertex of degree $(k+1)\geqslant 3$ adjacent with k vertices $w_1,\ldots,w_k$ from ∂G. Then the set of the vertices $\{w_1,\ldots,w_k\}$, and also the set of the edges $\{vw_1,\ldots,vw_k\}$, are referred to as *mustaches*. The number k is called the *degree*, and the vertex v is called the *common vertex of the mustaches*. An edge incident to v and not belonging to $\{vw_1,\ldots,vw_k\}$ is called the *root edge* of the mustaches under consideration.

As shown in [**35, 36**], any mustaches of a minimal filling of a metric space forms an additive subspace. If the degree of such mustaches is more than 2, then we can calculate the weights of all the edges containing in the mustaches just in the same way as in the case of a star.

9.5. Four-Points Spaces. Here we give a complete description of minimal fillings for four-points spaces.

PROPOSITION 9.1. *Let $M=\{p_1,p_2,p_3,p_4\}$, and ρ be an arbitrary metric on M. Put $\rho_{ij}=\rho(p_i,p_j)$. Then the weight of a minimal filling $\mathcal{G}=(G,\omega)$ of the space $\mathcal{M}=(M,\rho)$ is given by the following formula*

$$\frac12\big(\min\{\rho_{12}+\rho_{34},\,\rho_{13}+\rho_{24},\,\rho_{14}+\rho_{23}\}+\max\{\rho_{12}+\rho_{34},\,\rho_{13}+\rho_{24},\,\rho_{14}+\rho_{23}\}\big).$$

If the minimum in this formula is equal to $\rho_{ij}+\rho_{rs}$, then the type of minimal filling is the binary tree with the mustaches $\{p_i,p_j\}$ and $\{p_r,p_s\}$.

We apply the obtained result to the vertex set of a planar convex quadrangle.

COROLLARY 9.1. *Let M be the vertex set of a convex quadrangle $p_1p_2p_3p_4\subset\mathbb{R}^2$ and $\rho(p_i,p_j)=\|p_i-p_j\|$. The weight of a minimal filling of the space (M,ρ) is equal to $\frac12\min(\rho_{12}+\rho_{34},\,\rho_{14}+\rho_{23})+\frac12(\rho_{13}+\rho_{24})$. The topology of minimal filling is a binary tree with mustaches corresponding to opposite sides of the less total length.*

10. Ratios

The Steiner ratio is an important characteristic in the Steiner minimal networks theory. Let M be a finite subset of a metric space $\mathcal{X}$ consisting of more than one point. Recall that the *Steiner ratio of M* is the ratio of the lengths of the Steiner minimal tree and minimal spanning tree constructed on M, i.e., the value $\mathrm{sr}(M)=\mathrm{smt}(M)/\mathrm{mst}(M)$. The infimum of the numbers $\mathrm{sr}(M)$ over all such subsets M of $\mathcal{X}$ is called the *Steiner ratio of the space $\mathcal{X}$* and is denoted by $\mathrm{sr}(\mathcal{X})$, see [**1**].

Notice that the exact values of the Steiner ratio are known for a very restricted class of spaces (see a review in [**6**], or in [**2**]). Below, we shall define other two ratios based on minimal fillings, which could be more available for calculating, and which could be useful to calculate the Steiner ratio, as we hope.

10.1. Steiner–Gromov Ratio. For convenience, the sets consisting of more than a single point are referred to as *nontrivial*. Let $\mathcal{X}=(X,\rho)$ be an arbitrary metric space, and let $M\subset X$ be some finite subset. Recall that by $\mathrm{mst}(M,\rho)$ we denote the length of minimal spanning tree of the space (M,ρ). Further, for nontrivial M, we define the value $\operatorname{sgr}(M) = \operatorname{mf}(M,\rho)/\operatorname{mst}(M,\rho)$ and call it the *Steiner–Gromov ratio* of the subset M. The value $\inf \operatorname{sgr}(M)$, where the infimum is taken over all nontrivial finite subsets of $\mathcal{X}$, consisting of at most $n > 1$ vertices is denoted by $\operatorname{sgr}_n(\mathcal{X})$ and is called the *degree n Steiner–Gromov ratio of the space $\mathcal{X}$*. At last, the value $\inf \operatorname{sgr}_n(\mathcal{X})$, where the infimum is taken over all positive integers $n > 1$ is called the *Steiner–Gromov ratio of the space $\mathcal{X}$* and is denoted by $\operatorname{sgr}(\mathcal{X})$, or by $\operatorname{sgr}(X)$, if it is clear what particular metric on X is considered. Notice that $\operatorname{sgr}_n(\mathcal{X})$ is nonincreasing function on n. Besides that, it is easy to see that $\operatorname{sgr}_2(\mathcal{X}) = 1$ for any nontrivial metric space $\mathcal{X}$.

Proposition 10.1. $\operatorname{sgr}_3(\mathcal{X}) = 3/4$.

Proposition 10.2. *The Steiner–Gromov ratio of an arbitrary metric space is not less than* $1/2$. *There exist metric spaces whose Steiner–Gromov ratio equals to* $1/2$.

Recently, A. Pakhomova, a student of the Mechanical and Mathematical Department of the Moscow State University, obtained an exact general estimate for the degree n Steiner–Gromov ratio, see [**37**].

Proposition 10.3 (A. Pakhomova). *For any metric space $\mathcal{X}$ the estimate $\operatorname{sgr}_n(\mathcal{X}) \geqslant n/(2n-2)$ is valid. Moreover, this estimate is exact, i.e., for any $n \geqslant 3$ there exists a metric space $\mathcal{X}_n$, such that $\operatorname{sgr}_n(\mathcal{X}_n) = n/(2n-2)$.*

Also recently, Ovsyannikov [**38**] investigated the metric space of all compact subsets of the Euclidean plane endowed with Hausdorff metric.

Proposition 10.4 (Z. Ovsyannikov). *The Steiner ratio and the Steiner–Gromov ratio of the metric space of all compact subsets of the Euclidean plane endowed with Hausdorff metric are equal to* $1/2$.

10.2. Steiner Subratio. Let $\mathcal{X} = (X,\rho)$ be an arbitrary metric space, and $M \subset \mathcal{X}$ its finite subset. Recall that by $\operatorname{smt}(M,\rho)$ we denote the length of Steiner minimal tree joining M. Further, for nontrivial subsets M, we define the value $\operatorname{ssr}(M) = \operatorname{mf}(M,\rho)/\operatorname{smt}(M,\rho)$ and call it the *Steiner subratio* of the set M. The value $\inf \operatorname{ssr}(M)$, where infimum is taken over all nontrivial finite subsets of $\mathcal{X}$ consisting of at most $n > 1$ points, is denoted by $\operatorname{ssr}_n(\mathcal{X})$ and is called the *degree n Steiner subratio of the space $\mathcal{X}$*. At last, the value $\inf \operatorname{ssr}_n(\mathcal{X})$, where the infimum is taken over all positive integers $n > 1$, is called the *Steiner subratio of the space $\mathcal{X}$* and is denoted by $\operatorname{ssr}(\mathcal{X})$, or by $\operatorname{ssr}(X)$, if it is clear what particular metric on X is considered. Notice that $\operatorname{ssr}_n(\mathcal{X})$ is a nonincreasing function on n. Besides that, it is easy to see that $\operatorname{ssr}_2(\mathcal{X}) = 1$ for any nontrivial metric space $\mathcal{X}$.

Proposition 10.5. $\operatorname{ssr}_3(\mathbb{R}^n) = \sqrt{3}/2$.

The next result is obtained by E. Filonenko, a student of the Mechanical and Mathematical department of the Moscow State University, see [**34**].

Proposition 10.6 (E. Filonenko). $\operatorname{ssr}_4(\mathbb{R}^2) = \sqrt{3}/2$.

Conjecture 10.1. *The Steiner subratio of the Euclidean plane is achieved at the regular triangle and, hence, is equal to $\sqrt{3}/2$.*

Recently, A. Pakhomova obtained an exact general estimate for the degree n Steiner subratio, see [**37**].

Proposition 10.7 (A. Pakhomova). *For any metric space $\mathcal{X}$ the estimate $\mathrm{ssr}_n(\mathcal{X}) \geqslant n/(2n-2)$ is valid. Moreover, this estimate is exact, i.e., for any $n \geqslant 3$ there exists a metric space $\mathcal{X}_n$ such that $\mathrm{ssr}_n(\mathcal{X}_n) = n/(2n-2)$.*

Also recently, Z. Ovsyannikov [**38**] investigated the metric space of all compact subsets of the Euclidean plane endowed with Hausdorff metric.

Proposition 10.8 (Z. Ovsyannikov). *Let $\mathcal{C}$ be the metric space of all compact subsets of the Euclidean plane endowed with Hausdorff metric. Then $\mathrm{ssr}_3(\mathcal{X}) = 3/4$ and $\mathrm{ssr}_4(\mathcal{X}) = 2/3$.*

References

1. E. N. Gilbert, H. O. Pollak, *Steiner minimal trees*, SIAM J. Appl. Math. **16**:1 (1968), 1–29.
2. A. O. Ivanov, A. A. Tuzhilin, *Extreme Networks Theory*, IKI, Moscow, Izhevsk, 2003 [in Russian].
3. V. Jarnik, M. Kössler, *O minimalnich grafeth obeahujicich n danijch bodu*, Čas. Pěstování Mat. Fys. **63** (1934), 223–235.
4. R. Courant, H. Robbins, *What is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford University Press, USA, 1996.
5. D. Z. Du, F. K. Hwang, *A proof of Gilbert–Pollak Conjecture on the Steiner ratio*, Algorithmica **7** (1992), 121–135.
6. A. O. Ivanov, A. A. Tuzhilin, *Steiner Ratio. The State of the Art*, Mat. Vopr. Kibern. **11** (2002), 27–48.
7. P. O. de Wet, *The Euclidean Steiner ratio: some recent results*, Proc. 5th Math. Conf., Malasya, 2009.
8. P. O. de Wet, *Geometric Steiner minimal trees*, PhD thesis, 2008.
9. N. Innami, B. H. Kim, Y. Mashiko, K. Shiohama, *The Steiner ratio conjecture of Gilbert-Pollak may still be open*, Algorithmica **57**:4 (2010), 869–872.
10. A. O. Ivanov, A. A. Tuzhilin, *The Steiner Ratio GilbertŰPollak Conjecture Is Still Open. Clarification Statement*, Algorithmica **62**:1–2 (2012), 630–632.
11. A. O. Ivanov, Z. N. Ovsyannikov, N. P. Strelkova, A. A. Tuzhilin *One-dimensional minimal fillings with edges of negative weight*, Vestn. Mosk. Univ., Ser I, Math., Mech. **5** (2012), 3–8 [Moscow Univ. Math. Bull. **67**:5/6, (2012), 189–194].
12. A. Yu. Eremin *Formula calculating the weight of minimal filling*, Mat. Sbornik **204**:9 (2013), 51-Ű72 [Sbornik: Math. (2013), to appear].
13. Z. N. Ovsyannikov, *Pseudo-additive metric spaces and minimal fillings*, Vestnik MGU (2013), to appear.
14. G. Kolata, *Solution to Old Puzzle: How Short a Shortcut?*, The New York Times, October 30, 1990.
15. J. H. Rubinstein, J. F. Weng, *Compression theorems and Steiner ratios of spheres*, J. Comb. Opt. **1** (1997), 67–78.
16. D. T. Lee, C. F. Shen, *The Steiner minimal tree problem in the λ-geometry plane*, Lect. Notes Comput. Sci. **1178** (1996), 247–255.
17. M. Gromov, *Filling Riemannian manifolds*, J. Diff. Geom. **18**:1 (1983), 1–147.
18. M. Gromov, *Hyperbolic groups*; in: S. M. Gersten (ed.), *Essays in Group Theory*, Springer, 1987.

19. D. Yu. Burago, Yu. D. Burago, S. V. Ivanov, *Lectures on Metric Geometry*, IKI, Moscow, Izhevsk, 2004.
20. S. V. Ivanov, *Volumes and Areas in Metric Geometry*, Dissert. na soisk. uch. step. d.f.-m.n., SPtb., 2009.
21. A. T. Fomenko, *Minimal compacts in Riemannian manifolds and Reifenberg Conjecture*, Izvest. AN SSSR, **36**:5 (1972), 1049–1080 [in Russian], translated in: Math. USSR, Izv. **6**:5 (1972), 1037Ű-1066.
22. A. T. Fomenko, *The multidimensional Plateau problem in Riemannian manifolds*, Mat. Sb. **89(131)**:3(11), 475-Ű519, (1972) (in Russian); translated in: Math. USSR, Sbornik **18**:3 (1972), 487Ű-527.
23. V. L. Chernyshev, A. I. Shafarevich, *The semiclassical spectrum of the Schro"dinger operator on a geometric graph*, Mat. Zametki **82**:4 (2007), 606–620 [in Russian], translated in: Math. Notes **82**:4 (2007), 542Ű-554.
24. N. S. Gusev, *Canonical decompositions of piecewise affine mappings, polyhedra-traces, and geometrical variational problems*, Fundam. Prikl. Mat. **12**:1 (2006), 57–94 [in Russian], translated in: J. Math. Sci. **149**:1 (2008), 896Ű-921.
25. A. O. Ivanov, A. A. Tuzhilin, *The Deformation of a Manifold, Decreasing the Volume with Maximal Velocity*, Vestnik MGU, Ser. Mat., Mekh. **3** (1989), 14–18 [in Russian].
26. A. O. Ivanov, A. A.Tuzhilin, *Minimal Networks: The Steiner Problem and Its Generalizations*, CRC Press, N.W., Boca Raton, Florida, 1994.
27. P A. Borodin, *An example of nonexistence of a steiner point in a Banach space*, Mat. Zametki, **87**:4 (2010), 514–518 (in Russian); translated in: Math. Notes **87**:4 (2010), 485Ű-488.
28. F. Preparata, M. Shamos *Computational Geometry: An Introduction*, Berlin, Heidelberg, Springer-Verlag, 1990.
29. K. A. Zareckii, *Constructing a tree on the basis of a set of distances between the hanging vertices*, Uspehi Mat. Nauk **20**:6 (1965), 90-Ű92 (in Russian).
30. J. M. S. Simões-Pereira, *A note on the tree realizability of a distance matrix*, J. Comb. Theory **6** (1969), 303–310.
31. E. A. Smolenskij, *About a linear denotation of graphs*, Zh. Vychisl. Mat. Mat. Fiz. **2**:2 (1962), 371–372 (in Russian); translated in: U.S.S.R. Comput. Math. Math. Phys. **2** (1962), 396–397.
32. S. L. Hakimi, S. S. Yau, *Distane matrix of a graph and its realizability*, Quart. Appl. Math. **12** (1975), 305–317.
33. O. V. Rubleva, *Additivity Criterion for finite metric spaces and minimal fillings*, Vestn. Mosk. Univ., Ser. I **2** (2012), 8–11.
34. E. I. Stepanova (Filonenko), *Degree* 4 *Steiner subratio of Euclidean plane*, Vestn. Mosk. Univ., Ser. I (2013), to appear.
35. A. O. Ivanov, A. A. Tuzhilin, *One-dimensional Gromov minimal filling problem*, Sbornik: Mathematics, **203**:5 (2012), 677–726.
36. A. O. Ivanov, A. A. Tuzhilin, *One-dimensional Gromov minimal filling*, ar*Xiv:1101.0106v2 [math.MG]* (2010) (http://arxiv.org).
37. A. S. Pakhomova, *Estimates for Steiner–Gromov ratio and Steiner subratio*, Vestn. Mosk. Univ., Ser. I (2013), to appear.
38. Z. N. Ovsyannikov, *Steiner ratio, Steiner–Gromov ratio and Steiner subratio for the metric space of all compact subsets of the Euclidean plane with Hausdorff metric*, Fundam. Prikl. Mat. (2013), to appear.

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