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$\mathcal{D}$-HOMOTHETIC WARPING

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The goal of this lecture will be to introduce the notion of $\mathcal{D}$-homothetic warping, give a few rudimentary properties and a couple of applications. As with the usual warped product it is hoped that this idea will prove useful for generating further results and examples of various structures. Details of the proofs will appear in [**3**].

For this purpose we must first review the geometry of contact metric and almost contact metric manifolds. By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. The vector field ξ is known as the *characteristic vector field* or *Reeb vector field* of the contact structure η.

Denote by $\mathcal{D}$ the *contact subbundle* defined by $\{X \in T_mM : \eta(X) = 0\}$.

A Riemannian metric g is an *associated metric* for a contact form η if, first of all, $\eta(X) = g(X, \xi)$ and secondly, there exists a field of endomorphisms, ϕ, such that $\phi^2 = -I + \eta \otimes \xi, \quad d\eta(X, Y) = g(X, \phi Y)$. We refer to (ϕ, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold*.

By an *almost contact manifold* we mean a C^∞ manifold M^{2n+1} together with a field of endomorphisms ϕ, a 1-form η and a vector field ξ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

A Riemannian metric is said to be *compatible* if $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and we refer to an *almost contact metric structure* (ϕ, ξ, η, g). Again we denote by $\mathcal{D}$ the subbundle defined by $\eta = 0$. The fundamental 2-form, Φ, of an almost contact metric structure is the 2-form defined by $\Phi(X, Y) = g(X, \phi Y)$.

The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by

$$J\Big(X, a\frac{d}{dt}\Big) = \Big(\phi X - a\xi, \eta(X)\frac{d}{dt}\Big)$$

where a is a function on the product manifold. The underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be

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expressed as

$$[\phi,\phi](X,Y)+2d\eta(X,Y)\xi=0,$$

$[\phi,\phi]$ being the Nijenhuis tensor of ϕ (see e.g. [**2**, Chapter 6]).

Some special cases are worthy of attention. A contact metric structure is *K-contact* if ξ is a Killing vector field and a *Sasakian manifold* is a normal contact metric manfiold, equivalently, if $(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X$. Sasakian manifolds are K-contact and in dimension 3 the converse is also true (again see [**2**, Chapter 6]).

An almost contact metric structure is said to be *almost cosymplectic* if both η and Φ are closed. If in addition the structure is normal (see e.g. [**2**, Chapter 6]), the structure is said to be *cosymplectic*.

An almost contact metric manifold is a *Kenmotsu manifold* ([**6**] or [**2**, p. 98]) if

$$(\nabla_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X.$$

The notion of a $\mathcal{D}$-homothetic deformation on a contact metric manifold was introduced by Tanno [**12**]. For a contact metric structure (ϕ,ξ,η,g) and positive constant a, the structure

$$\bar\eta=a\eta,\quad \bar\xi=\frac{1}{a}\xi,\quad \bar\phi=\phi,\quad \bar g=ag+a(a-1)\eta\otimes\eta$$

is again a contact metric structure.

The idea works equally well for almost contact metric structures; the deformation

$$\bar\eta=c\eta,\quad \bar\xi=\frac{1}{c}\xi,\quad \bar\phi=\phi,\quad \bar g=ag+b\eta\otimes\eta,\quad a>0,\quad a+b>0$$

is again an almost contact metric structure if $c^2=a+b$. In particular, if $c=a$, then the deformed structure satisfies $\bar\eta(X)=\bar g(X,\bar\xi)$ if and only if $b=a(a-1)$.

The notion of warped product is very well known: Given two Riemannian manifolds (M_1,g_1) and (M_2,g_2), and a positive function f on M_1, the Riemannian metric $g=g_1+fg_2$ on $M_1\times M_2$ is known as a *warped product metric*.

Now consider the product of a Riemannian manifold (M_1,g_1) and an almost contact metric manifold $(M_2,\phi_2,\xi_2,\eta_2,g_2)$. On $M_1\times M_2$ define a metric g by

$$g=g_1+fg_2+f(f-1)\eta_2\otimes\eta_2$$

for a positive function f on M_1. We refer to this construction as *$\mathcal{D}$-homothetic warping*.

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric,

$$\begin{aligned}2g(\nabla_XY,Z)&=Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)\\&\quad+g([X,Y],Z)+g([Z,X],Y)-g([Y,Z],X),\end{aligned}$$

one can compute the Levi-Civita connection of the $\mathcal{D}$-homothetically warped metric. Denote by ∇^1 and ∇^2 the Levi-Civita connections of g_1 and g_2 respectively and choose vector fields X_i,Y_i,Z_i, $i=1,2$, such that they are tangent to the manifold with the corresponding index and that their component functions are functions on that manifold. In particular $[X_i, Y_i]$ is tangent to M_i and $[X_1, Y_2] = 0$. Then we have

$$\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1, \quad \nabla_{X_1} Y_2 = \nabla_{Y_2} X_1 = \frac{X_1 f}{2f}(Y_2 + \eta_2(Y_2)\,\xi_2)$$

which in turn can be used to find $g(\nabla_{X_2} Y_2, Z_1) = -g(\nabla_{X_2} Z_1, Y_2)$. Finally

$$\begin{aligned} 2g(\nabla_{X_2} Y_2, Z_2) = 2g(\nabla^2_{X_2} Y_2, Z_2) \\ + f(f-1)\big\{\big(g_2(\nabla^2_{X_2}\xi_2, Y_2) + g_2(\nabla^2_{Y_2}\xi_2, X_2)\big)\eta_2(Z_2) \\ + 2d\eta_2(X_2, Z_2)\,\eta_2(Y_2) + 2d\eta_2(Y_2, Z_2)\,\eta_2(X_2)\big\}. \end{aligned}$$

Let σ denote the second fundamental form of M_2 in $M_1 \times M_2$ and while f is a function on M_1, for emphasis we denote its gradient by $\operatorname{grad}_1 f$. We then have the following Theorem.

Theorem 1. *For an almost contact metric manifold $(M_2, \phi_2, \xi_2, \eta_2, g_2)$ and a $\mathcal{D}$-homothetically warped metric on $M_1 \times M_2$ we have the following:*

(1) *M_1 is a totally geodesic submanifold.*
(2) *M_2 is a quasi-umbilical submanifold and its second fundamental form is given by*

$$\sigma(X_2, Y_2) = -\tfrac{1}{2}\big(g_2(X_2, Y_2) + (2f-1)\,\eta_2(X_2)\,\eta_2(Y_2)\big)\operatorname{grad}_1 f.$$

(3) *The mean curvature vector of M_2 in $M_1 \times M_2$ is*

$$H = -\frac{n+f}{2n+1}\operatorname{grad}_1 f.$$

(4) *If f is nowhere constant, then a geodesic initially tangent to a copy of M_2 cannot remain tangent.*
(5) *If in addition, the integral curves of ξ_2 are geodesics, then ξ_2 is g-Killing if and only if it is g_2-Killing.*

We remark that on a contact metric manifold, as well as on almost cosymplectic manifolds and Kenmotsu manifolds, one has that $\nabla^2_{\xi_2}\xi_2 = 0$ and hence the integral curves of ξ_2 are geodesics. With respect to the metric g on $M_1 \times M_2$,

$$\nabla_{\xi_2}\xi_2 = -f\operatorname{grad}_1 f.$$

For our first application of the idea of $\mathcal{D}$-homothetic warping we consider the case where $M_1 = \mathbb{R}$, M_2 is an almost contact metric manifold and the metric $g = dt^2 + fg_2 + f(f-1)\eta_2 \otimes \eta_2$. For brevity we denote the unit tangent field to M_1 by ∂_t and for notational emphasis we will sometimes write η_1 for dt. Vector fields on $\mathbb{R} \times M_2$ will be denoted by either $(a\partial_t, X_2)$ or $\tilde{X}$ depending on convenience.

Define an almost complex structure J on the product manifold by

$$J(a\partial_t, X_2) = \Big(f\eta_2(X_2)\,\partial_t, \phi_2 X_2 - \frac{1}{f}\eta_1(a\partial_t)\,\xi_2\Big)$$

where a is in general a function, but for the definition it would suffice to take a to be 0 or 1 and then keep track of when $\eta_1 = dt$ is to be used. That $J^2 = -I$ and

$$g\big(J(a\partial_t, X_2), J(b\partial_t, Y_2)\big) = g\big((a\partial_t, X_2), (b\partial_t, Y_2)\big)$$

are easily verified. The fundamental 2-form of this almost Hermitian structure is

$$\Omega\big((a\partial_t, X_2), (b\partial_t, Y_2)\big) = g\big((a\partial_t, X_2), J(b\partial_t, Y_2)\big)$$

or, upon expansion, simply $\Omega = f(\Phi_2 + 2dt \wedge \eta_2)$ where $\Phi_2(X_2, Y_2) = g_2(X_2, \phi_2 Y_2)$ is the fundamental 2-form of the almost contact metric structure. We have immediately that

$$d\Omega = f' dt \wedge \Phi_2 + f d\Phi_2 - 2f dt \wedge d\eta_2.$$

For the special cases we have the following:

(1) contact metric: $d\Omega = (f' - 2f)dt \wedge d\eta_2$.
(2) almost cosymplectic: $d\Omega = f' dt \wedge \Phi_2$.
(3) Kenmotsu: $d\Omega = (f' dt + f\eta_2) \wedge \Phi_2$.

We note that Ω is closed in the contact metric case if and only if $f = Ae^{2t}$ and in the almost cosymplectic case if and only if f is constant. In the Kenmotsu case Ω cannot be closed; it would force f to be zero.

Theorem 2. (1) *The almost contact metric structure on M_2 is a contact metric structure if and only if the almost Hermitian structure (g, J) satisfies $d\Omega = (f' - 2f)\, dt \wedge d\eta_2$ in which case the structure is conformally almost Kähler. If the structure on M_2 is Sasakian, then the structure (g, J) is Hermitian and conformally Kähler.*

(2) *The almost contact metric structure on M_2 is almost cosymplectic if and only if the almost Hermitian structure (g, J) satisfies $d\Omega = f' dt \wedge \Phi_2$ in which case the structure is conformally almost Kähler.*

For a second application of the idea of $\mathcal{D}$-homothetic warping we consider a semi-Riemannian setting with $M_1 = \mathbb{R}$. Recall that in general relativity one has the notion of a synchronous space-time as a 4-dimensional manifold equipped with a metric of the form

$$ds^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 1, ..., 3$$

where t is the time and the $g_{\alpha\beta}$ depend on all four variables $x^0 = t, x^1, x^2, x^3$. The most well known of these are the classical Friedman–Lemaître–Robertson–Walker metrics which are warped product metrics of the form

$$ds^2 = -dt^2 + f(t) g_2$$

where g_2 is of constant curvature $+1$, 0 or -1.

The Wilkinson Microwave Anisotropy Probe (WMAP) revealed some anomalies in the cosmic microwave background (CMB). In particular, the analysis of Oliveira-Costa et al. [**11**] suggests that there is a preferred direction in space in the direction $(b, l) \approx (60°, -110°)$ in galactic coordinates, see also [**7**]. The universe may be expanding faster in such a direction than in orthogonal directions. The question of the opposite direction was taken up in [**8**] and the work suggests a symmetry with respect to planar reflections. (For a more popular discussion see [**9**] and/or [**10**].) All of this suggests investigating a manifold $\mathbb{R} \times M_2$ where M_2 has a preferred direction.

Consider a 3-dimensional almost contact metric manifold $(M_2, \phi_2, \xi_2, \eta_2, g_2)$. On the product $\mathbb{R} \times M_2$ we have the synchronous metric

$$ds^2 = -dt^2 + fg_2 + f(f-1)\eta_2 \otimes \eta_2. \tag{†}$$

where f is a positive function of time. Of course, $\nabla_{\partial_t}\partial_t = 0$ and from our earlier work we have

$$\nabla_{X_2}\partial_t = \frac{f'}{2f}(X_2 + \eta_2(X_2)\xi_2).$$

Then a direct computation yields the following curvature term.

$$R_{X_2\,\partial_t}\partial_t = -\Big(\frac{f''}{2f} + \frac{(f')^2}{4f^2}\Big)(X_2 + \eta_2(X_2)\xi_2) + \frac{(f')^2}{2f^2}X_2$$

and hence for the Ricci tensor, ρ, we have

$$\rho(\partial_t, \partial_t) = -\frac{2f''}{f} + \frac{(f')^2}{2f^2}.$$

For the metric (†) we proceed with the following considerations. Let e be a local unit vector field in the subbundle $\mathcal{D}$ on M_2. Then we have the local orthonormal basis $\{e, \phi e, \xi\}$ on M_2. Now for the metric (†), form the local orthonormal basis

$$E_0 = \partial_t, \quad E_1 = \frac{e}{\sqrt{f}}, \quad E_2 = \frac{\phi e}{\sqrt{f}}, \quad E_3 = \frac{\xi}{f}.$$

For the time direction ∂_t, or more generally a unit time-like vector field V, the *strain* Θ and *vorticity* Ω are defined by

$$\Theta(X,Y) = \tfrac{1}{2}\big(g(\nabla_X V, Y) + g(\nabla_Y V, X)\big),$$
$$\Omega(X,Y) = \tfrac{1}{2}\big(g(\nabla_X V, Y) - g(\nabla_Y V, X)\big)$$

where X and Y are g-orthogonal to V. For ∂_t there is no vorticity. One defines the *expansion* θ as the divergence of V which for us becomes

$$\theta = \operatorname{div}\partial_t = \sum_{i=0}^{3}\epsilon_i g(\nabla_{E_i}\partial_t, E_i) = \sum_{\alpha=1}^{3} g(\nabla_{E_\alpha}\partial_t, E_\alpha)$$

where $\epsilon_0 = -1$ and $\epsilon_\alpha = +1$. The trace free part of Θ, $\sigma_{\alpha\beta} = \Theta_{\alpha\beta} - \frac{1}{3}\theta\delta_{\alpha\beta}$, is called *shear*. For further reference see [**4**] p. 409 or [**5**] pp. 74-79.

For our metric (†) we have readily that

$$\Theta(E_1, E_1) = \Theta(E_2, E_2) = \frac{f'}{2f}, \quad \Theta(E_3, E_3) = \frac{f'}{f}$$

and hence that the expansion $\theta = \frac{2f'}{f}$. For the shear we have

$$\sigma_{11} = \sigma_{22} = -\frac{f'}{6f}, \quad \sigma_{33} = \frac{f'}{3f}.$$

In a universe with a preferred direction one might expect to have some shear and from the above calculations we have the reassuring result that the Raychaudhuri equation,

$$\frac{d\theta}{dt} = -\rho(\partial_t, \partial_t) - \tfrac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta},$$

is satisfied. For a discussion of the Raychaudhuri equation, see e.g. [**4**, p. 411] or [**5**, pp. 151–152].

Now one might also expect that the shear is something that is oblique to the subbundle $\mathcal{D}$. To proceed, first note that $E_1 = \frac{e}{\sqrt{f}}$ is an arbitrary direction in $\mathcal{D}$ and introduce a basis for the time direction and a plane field $\mathcal{P}$ by

$$F_0 = \partial_t, \quad F_1 = E_1, \quad F_2 = AE_2 + BE_3,\ A^2 + B^2 = 1.$$

Define an expansion and a Ricci tensor relative to $\mathcal{P}$ by

$$\theta^{\mathcal{P}} = \sum_{i=0}^{2} \epsilon_i g(\nabla_{F_i}\partial_t, F_i), \quad \rho^{\mathcal{P}}(\partial_t, \partial_t) = \sum_{i=0}^{2} \epsilon_i g(R_{F_i\partial_t}\partial_t, F_i).$$

Clearly the first term on the right in each case vanishes but conceptually the term should be present. The corresponding shear is

$$\sigma^{\mathcal{P}}(F_\alpha, F_\beta) = \Theta(F_\alpha, F_\beta) - \tfrac{1}{2}\theta^{\mathcal{P}}\delta_{\alpha\beta},\ \alpha, \beta = 1, 2.$$

Computing explicitly we have

$$\theta^{\mathcal{P}} = \frac{f'}{f}\Big(1 + \frac{B^2}{2}\Big), \quad \Theta_{11} = \frac{f'}{2f}, \quad \Theta_{22} = \frac{f'}{2f}(1 + B^2)$$

and hence for the shear we have

$$\sigma^{\mathcal{P}}_{11} = -\frac{B^2 f'}{4f}, \quad \sigma^{\mathcal{P}}_{22} = \frac{B^2 f'}{4f}.$$

Thus for the contact plane field $\mathcal{D}$ $(B = 0)$ there is no shear and the expansion is $\theta^{\mathcal{D}} = \frac{f'}{f}$. At the other extreme when $B = 1$, the expansion is $\theta^{\mathcal{P}} = \frac{3f'}{2f}$ and for the shear we have $(\sigma^{\mathcal{P}}_{11})^2 + (\sigma^{\mathcal{P}}_{22})^2 = \frac{(f')^2}{8f^2}$. In these two cases the Raychaudhuri equation

$$\frac{d\theta^{\mathcal{P}}}{dt} = -\rho^{\mathcal{P}}(\partial_t, \partial_t) - \tfrac{1}{2}(\theta^{\mathcal{P}})^2 - (\sigma^{\mathcal{P}}_{11})^2 - (\sigma^{\mathcal{P}}_{22})^2$$

is satisfied. In general the plane field $\mathcal{P}$ may evolve with time and the Raychaudhuri equation then imposes a relation between B and f, viz.

$$B' = \frac{f'}{4f}(B - B^3) \ \text{ or integrating } \ \frac{B^4}{(B^2 - 1)^2} = (\text{const.})f.$$

For the $\mathcal{D}$-homothetically warped metric (†) there are a number possibilities for the almost contact metric structure on M_2. For example the 3-dimensional unit sphere is a very well known Sasakian manifold.

Many years ago in [**1**], the author proved that in dimensions $\geqslant 5$ there are no flat contact metric manifolds. However in dimension 3 the torus carries a flat contact metric manifold as does $\mathbb{R}^3(x, y, z)$ with the following structure. The contact form and characteristic vector field are

$$\eta = \frac{1}{2}(dz - y\,dx) \text{ (the standard Darboux form) and } \xi = 2\frac{\partial}{\partial z},$$

and the associated metric

$$g = \frac{1}{4}\begin{pmatrix} 1+y^2+z^2 & z & -y \\ z & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is flat.

On the other hand we can consider a couple of almost contact metric structures which are not contact metric.

Again for the flat case, one could chose the very simple cosymplectic structure on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ given by the Euclidean metric and

$$\eta = dz, \quad \xi = \frac{\partial}{\partial z}, \quad \Phi = dx \wedge dy.$$

The standard example of a Kenmotsu manifold is hyperbolic space. In particular in dimension 3 we have the following structure on the $\mathbb{R}^3$ model of hyperbolic space:

$$\eta = dz, \quad \xi = \frac{\partial}{\partial z}, \quad g = e^{2z}(dx^2 + dy^2) + dz^2, \quad \Phi = -2e^{2z} dx \wedge dy.$$

For the Sasakian structure on the sphere $S^3(1)$, the flat cosymplectic structure on $\mathbb{R}^3$ and the Kenmotsu structure on 3-dimensional hyperbolic space, the metric (†) can be considered as an alternative to the Friedman–Lemaître–Robertson–Walker metrics which incorporates a preferred direction.

The existence of a preferred axis in space is somewhat controversial and isn't overly apparent; thus one might expect that for the above metrics the function (or constant) f should be close to 1; this would then possibly entail weighting g_2 by a different function (cf. our earlier brief discussion of $\mathcal{D}$-homothetic deformation in the almost contact metric case). There is considerable literature raising questions of the robustness of the data from WMAP and its statisical analysis. On the other hand one has a recent remark by Moyer [**10**] in favor of keeping an open mind on the topic. The Planck satellite has recently completed a new mapping of the CMB and analysis of the data is expected later this year or next; so we must wait and see.

Finally recall the notion of a *doubly warped product metric*, namely

$$g = Fg_1 + fg_2$$

where f is a positive function on M_1 and F is a positive function on M_2. If now both M_1 and M_2 are almost contact metric manifolds we can define a *doubly $\mathcal{D}$-homothetically warped metric* by

$$g = Fg_1 + F(F-1)\eta_1 \otimes \eta_1 + fg_2 + f(f-1)\eta_2 \otimes \eta_2.$$

While this is an area of possible future research we mention briefly that one easily has the following:

(1) Both M_1 and M_2 are quasi-umbilical submanifolds and e.g. the second fundamental form of M_1 is

$$\sigma_1(X_1,Y_1) = -\frac{1}{2f}\big(g_1(X_1,Y_1)+(2F-1)\eta_1(X_1)\eta_1(Y_1)\big)\Big(\operatorname{grad}_2 F+\frac{(1-f)(\xi_2 F)}{f}\xi_2\Big).$$

(2) M_1 is minimal if and only if F is constant in which case it is totally geodesic.

(3) If $\nabla^1_{\xi_1}\xi_1 = 0$,

$$\nabla_{\xi_1}\xi_1 = -\frac{F}{f}\Big(\operatorname{grad}_2 F + \frac{(1-f)(\xi_2 F)}{f}\xi_2\Big).$$

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