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A TOUR THROUGH δ-INVARIANTS: FROM NASH'S EMBEDDING THEOREM TO IDEAL IMMERSIONS, BEST WAYS OF LIVING AND BEYOND

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To the memory of my esteemed friend and coworker,
Franki Dillen

Abstract. First I will explain my motivation to introduce the δ-invariants for Riemannian manifolds. I will also recall the notions of ideal immersions and best ways of living. Then I will present a few of the many applications of δ-invariants in several areas in mathematics. Finally, I will present two optimal inequalities involving δ-invariants for Lagrangian submanifolds obtained very recently in joint papers with F. Dillen, J. Van der Veken and L. Vrancken.

1. Why introduced δ-invariants ?

In this section I will explain my motivation to introduce the δ-invariants.

1.1. Nash's embedding theorem. In 1956, John F. Nash (1928–) proved in [**21**] the following famous embedding theorem.

Theorem 1.1. *Every Riemannian n-manifold can be isometrically embedded in a Euclidean m-space $\mathbb{E}^m$ with $m = \frac{n}{2}(n+1)(3n+11)$.*

For example, Nash's theorem implies that every 3-dimensional Riemannian manifold can be isometrically embedded in $\mathbb{E}^{120}$ with codimension 117.

The embedding problem had been around since Bernhard Riemann (1826–1866) and was posed explicitly by Ludwig Schläfli (1814–1895) in the 1870s. The problem has evolved naturally from a progression of other questions that had been posed and partly answered beginning in the middle of 19th century via the work of Jean F. Frenet (1816–1900), Joseph A. Serret (1819–1885), Louis S. X. Aoust (1814–1885); and then by Maurice Janet (1888–1984) and Élie Cartan (1869–1951) in the 1920s. First, mathematicians studied the embedding problem of ordinary curves, then of surfaces, and finally, of Riemannian manifolds of higher dimensions.

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For general references of this article, see [**4**, **7**, **8**].

John Conway (1937–), the Princeton mathematician who discovered surreal numbers and invented the game of life (known simply as Life), called Nash's result

> "*one of the most important pieces of mathematical analysis in the 20th century*".

Also, according to Shlomo Sternberg (1936–) of Harvard University,

> "*The embedding problem is a deep philosophical question concerning the foundations of geometry that virtually every mathematician – from Riemann and Hilbert to Élie Cartan and Hermann Weyl – working in the field of differential geometry for the past century has asked himself.*"

(Quoted from the book: "A Beautiful Mind; The life of mathematical genius and Nobel laureate John Nash" by Sylvia Nasar, 1998.)

1.2. Why is it so difficult to apply Nash's embedding theorem? Nash's embedding theorem served for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. Till when observed in [**19**] as such by Mikhail L. Gromov (1943–), this hope had not been materialized however.

There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces in general, as stated in Nash's theorem. On the other hand, submanifolds of higher codimension are very difficult to be understood, e.g., there are no general results for arbitrary Riemannian submanifolds, except the three fundamental equations of Gauss, Codazzi and Ricci.

Another reason, for instance, was explained in [**23**] as follows.

> *"What is lacking in the Nash theorem is the control of the extrinsic quantities in relation to the intrinsic quantities."*

In other words, another main difficulty to apply Nash's theorem is:

> "*There do not exist general optimal relationships between the known intrinsic invariants and the main extrinsic invariants for arbitrary Riemannian submanifolds of Euclidean spaces.*"

1.3. Obstructions to minimal immersions. Since there are no obstructions to isometric embeddings according to Nash's theorem, in order to study isometric embedding (or more generally, immersion) problems, it is natural to impose some suitable conditions on immersions.

For example, if one imposes the *minimality condition* it leads to the following problem proposed by Shiing-Shen Chern (1911–2004) during the 1960s.

Problem 1. *What are necessary conditions for a Riemannian manifold to admit a minimal isometric immersion into a Euclidean space?*

The equation of Gauss states that the Riemann curvature tensor R and the second fundamental form h of a Riemannian submanifold in a Euclidean space satisfy

$$R(X,Y;Z,W) = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle\,.$$

It follows immediately from the equation of Gauss that a necessary condition for a Riemannian manifold to admit a minimal immersion in any Euclidean space is

$$Ric \leqslant 0, \quad \text{(in particular, } \tau \leqslant 0\text{)},$$

where Ric and τ are the Ricci and scalar curvatures of the submanifold.

For many years, before the invention of δ-invariants, *this was the only known Riemannian obstruction* for a general Riemannian manifold to admit a minimal immersion into a Euclidean space with arbitrary codimension.

1.4. Obstructions to Lagrangian immersions. A submanifold M of a Kähler manifold $(\tilde{M}, J, g)$ is called *Lagrangian* if $J(T_pM) = T_p^{\perp}M$, $\forall p \in M$, where J and g are the complex structure and the Kähler metric of $\tilde{M}$.

A result of Gromov states that a compact n-manifold M admits a Lagrangian immersion into the complex Euclidean n-space $\mathbf{C}^n$ if and only if the complexification $TM \otimes \mathbf{C}$ of the tangent bundle of M is trivial [**18**]. Gromov's result implies that there are *no topological obstructions* to Lagrangian immersions for compact 3-manifolds in $\mathbf{C}^3$, because the tangent bundle of every 3-manifold is trivial.

On the other hand, if one imposes isometrical condition to the Lagrangian immersion problem, it leads to the following.

PROBLEM 2. *What are the Riemannian obstructions to Lagrangian isometric immersions of Riemannian n-manifolds into $\mathbf{C}^n$?*

1.5. What do we need to do to overcome the difficulties? We ask the following two basic questions:

(1) *What do we need to do to overcome the difficulties mentioned above?* (in order to apply Nash's embedding theorem)
(2) *How can we solve Problems* 1 *and* 2 *concerning minimal and Lagrangian immersions?*

1.6. My answers. My answers to these two basic questions are the following.

(a) *We need to introduce a new type of Riemannian invariants*, different in nature, *from the "classical" Riemannian invariants; namely, from scalar and Ricci curvatures* (which have been studied extensively for more than 150 years since Riemann).
(b) *We also need to establish general optimal relationships between the main extrinsic invariants of the submanifolds and the new type of intrinsic invariants.*

1.7. My motivation. These considerations provided me the motivation to introduce δ-invariants for Riemannian manifolds during the 1990s.

2. How I defined δ-invariants

Let M be a Riemannian n-manifold and $K(\pi)$ denote the sectional curvature of a plane section $\pi \subset T_pM$, $p \in M$. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, the *scalar curvature* τ at p is given by $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$.

Let L be a subspace of T_pM of dimension $r \geqslant 2$ and $\{e_1, \ldots, e_r\}$ be an orthonormal basis of L. The *scalar curvature* $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha < \beta \leqslant r.$$

2.1. The set $\mathcal{S}(n)$. For a given integer $n \geqslant 2$, denote by $\mathcal{S}(n)$ the finite set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers $2 \leqslant n_1, \ldots, n_k \leqslant n-1$ with $n_1 + \cdots + n_k \leqslant n$, where k is a non-negative integer.

In number theory and combinatorics, a partition of a positive integer n is a way of writing n as a sum of positive integers. The cardinal number $\#\mathcal{S}(n)$ of $\mathcal{S}(n)$ is equal to $p(n) - 1$, where $p(n)$ is the number of partitions of n. For instance, for

$$\begin{gathered} n = 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, \\ 50, \ldots, 100, \ldots, 200, \end{gathered}$$

the corresponding cardinal numbers are given respectively by

$$\begin{gathered} \#\mathcal{S}(n) = 1, 2, 4, 6, 10, 14, 21, 29, 41, \ldots, \\ 204\,225, \ldots, 190\,569\,291, \ldots, 3\,972\,999\,029\,387. \end{gathered}$$

The asymptotic behavior of $\#\mathcal{S}(n)$ is given by

$$\#\mathcal{S}(n) \approx \frac{1}{4n\sqrt{3}} \exp\left[\sqrt{\tfrac{2n}{3}}\,\pi\right] \quad \text{as} \quad n \to \infty.$$

2.2. Definition of δ-invariants. For each given k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, I defined the δ-invariant $\delta(n_1, \ldots, n_k)$ by

$$\delta(n_1, \ldots, n_k) = \tau - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$. In particular, we have

$\delta(\emptyset) = \tau \quad (k = 0$, the trivial δ-invariant),
$\delta(2) = \tau - \inf K$, where K is the sectional curvature,
$\delta(n-1)(p) = \max Ric(p)$.

Remark 2.1. The non-trivial δ-invariants are *very different in nature* from the "classical" scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are the "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the non-trivial δ-invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures. Consequently, the non-trivial δ-invariants are *very weak intrinsic invariants*.

Borrowing a term from biology, *δ-invariants are DNA of Riemannian manifolds*. Results in later sections illustrate that all of our δ-invariants do affect directly the behavior of Riemannian manifolds.

Remark 2.2. Some other invariants of similar nature, i.e., intrinsic invariants obtained from scalar curvature by throwing away certain amount of sectional curvature, are also called δ-invariants. For instance, we also have affine δ-invariants, Kählerian δ-invariants, submersion δ-invariants, contact δ-invariants, etc. Such δ-invariants were introduced while we investigated some special families of manifolds.

3. What can we do with δ-invariants?

To apply Nash's embedding theorem, we ask the following most fundamental question in the theory of Riemannian submanifolds.

FUNDAMENTAL QUESTION. *What can we conclude from an arbitrary isometric immersion of a Riemannian manifold in a Euclidean space with any codimension?*

That is, "$\forall$ *arbitrary isometric immersion* $\phi : M \to \mathbb{E}^m \implies ???$ "

Or, more generally,

What can we conclude from an arbitrary isometric immersion between Riemannian manifolds?

That is, " $\forall$ *arbitrary isometric immersion* $\phi : M \to \tilde{M} \implies ???$ "

REMARK 3.1. For surfaces in $\mathbb{E}^3$, the equation of Gauss was found in 1827 in principal, though not explicitly, by Carl F. Gauss (1777–1855); and the equation of Codazzi was given by Delfino Codazzi (1824–1875) in 1860, independently by Gaspare Mainardi (1800–1879) in 1856 and also by Karl M. Peterson (1828–1881) in his 1853 doctoral thesis [Über die Biegung der Flächen, Dorpat University]. In 1880, Aurel E. Voss (1845–1931) extended both equations to Riemannian submanifolds. The equation of Ricci was discovered by Gregorio Ricci (1853–1925) in 1899.

For so many years since then, the *only known solutions* to the Fundamental Question were the equations of Gauss, Codazzi and Ricci, as we have already said earlier.

3.1. A new general solution to Fundamental Question. By applying the δ-invariants, we are able to provide the following new optimal general solution to the Fundamental Question.

THEOREM 3.1. *For any isometric immersion of a Riemannian n-manifold M into another Riemannian manifold $\tilde{M}$, we have*

$$\delta(n_1,\dots,n_k) \leqslant \frac{n^2(n+k-1-\sum_{j=1}^k n_j)}{2(n+k-\sum_{j=1}^k n_j)}H^2 + \frac{1}{2}\left\{n(n-1)-\sum_{j=1}^k n_j(n_j-1)\right\}\max\tilde{K} \tag{3.1}$$

for every k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, where H^2 denotes the squared mean curvature and $\max\tilde{K}(p)$ *is the maximum of sectional curvatures of $\tilde{M}$ restricted to 2-plane sections of T_pM.*

The equality case of inequality (3.1) *holds at a point $p\in M$ if and only if the following two conditions hold:*

(a) *there exists an orthonormal basis* $\{e_1, \dots, e_m\}$ *of* T_pM *such that the shape operator* A *at* p *takes the form:*

$$\begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, m, \tag{3.2}$$

where I *is an identity matrix and* A_j^r *is a symmetric* $n_j \times n_j$ *submatrix satisfying* $\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \mu_r$.

(b) *for any* k *mutual orthogonal subspaces* $L_1, \dots, L_k$ *of* T_pM *satisfying*

$$\delta(n_1, \dots, n_k) = \tau - \sum\nolimits_{j=1}^{k} \tau(L_j)$$

at p, *we have* $\tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p)$ *for any* $\alpha_i \in \Delta_i, \alpha_j \in \Delta_j$ *with* $1 \leqslant i \neq j \leqslant k+1$, *where* $\Delta_1, \dots, \Delta_{k+1}$ *are given by*

$$\begin{aligned} \Delta_1 &= \{1, \dots, n_1\}, \dots \\ \Delta_k &= \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\}, \\ \Delta_{k+1} &= \{n_1 + \cdots + n_k + 1, \dots, n\}. \end{aligned}$$

Definition 3.1. For each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we define the *normalized δ-invariant* $\Delta(n_1, \dots, n_k)$ of a Riemannian n-manifold by

$$\Delta(n_1, \dots, n_k) = \frac{\delta(n_1, \dots, n_k)}{c(n_1, \dots, n_k)}, \quad c(n_1, \dots, n_k) = \frac{n^2\big(n+k-1-\sum_{j=1}^{k} n_j\big)}{2\big(n+k-\sum_{j=1}^{k} n_j\big)}.$$

When the ambient space $\tilde{M}$ is a Euclidean space, Theorem 3.1 becomes

Theorem 3.2. *For any isometric immersion of a Riemannian n-manifold M into a Euclidean space with arbitrary codimension, we have*

$$H^2 \geqslant \Delta(n_1, \dots, n_k) \tag{3.3}$$

for every $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

The equality case of (3.3) *holds at* $p \in M$ *if and only if there is an orthonormal basis* $\{e_1, \dots, e_m\}$ *of* T_pM *such that the shape operator* A *at* p *takes the form:*

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, m, \tag{3.4}$$

where I *is an identity matrix and* A_j^r *is a symmetric* $n_j \times n_j$ *submatrix which satisfies* $\operatorname{trace}(A_j^r) = \mu_r$ *for* $j = 1, \dots, k$.

Remark 3.2. For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$, inequality (3.3) is optimal, since there exists a non-minimal submanifold satisfying the equality case.

REMARK 3.3. Due to the well-known fact that the mean curvature vector field of an isometric immersion is exactly the tension field, Theorem 3.2 shows that the amount of tension a submanifold receives at each point from its ambient space is *predominated below by its δ-invariants.* Consequently, each δ-invariant does affect directly the behavior of Riemannian manifolds.

Theorem 3.2 implies immediately the following.

COROLLARY 3.1. *For every isometric immersion of a Riemannian n-manifold into a Euclidean space with arbitrary codimension, we have $H^2(p) \geqslant \hat{\Delta}_0(p)$, where $\hat{\Delta}_0 := \max\{\Delta(n_1, \ldots, n_k) : (n_1, \ldots, n_k) \in \mathcal{S}(n)\}$ is the maximal normalized δ-invariant.*

3.2. Maximum principle. In general, there do not exist direct relationships between δ-invariants. On the other hand, we have the following maximum principle, which follows immediately from Theorem 3.2.

MAXIMUM PRINCIPLE. *If an n-dimensional submanifold M of a Euclidean space satisfies the equality case of* (3.3) *for some k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, i.e., $H^2 = \Delta(n_1, \ldots, n_k)$, then for every $(m_1, \ldots, m_j) \in \mathcal{S}(n)$ and any j we have $\Delta(n_1, \ldots, n_k) = \hat{\Delta}_0 \geqslant \Delta(m_1, \ldots, m_j)$.*

4. Ideal immersions and best ways of living

What is a "*nice immersion*" of a Riemannian manifold? In my opinion, it is an isometric immersion which produces the least possible amount of tension at each point. For this reason I introduced the notion of ideal immersions in the 1990s.

DEFINITION 4.1. An isometric immersion of a Riemannian n-manifold into a Euclidean space is called *ideal* if $H^2 = \hat{\Delta}_0$ holds identically.

The Maximum Principle yields the following important fact.

THEOREM 4.1. *If an isometric immersion of a Riemannian n-manifold M in a Euclidean space satisfies $H^2 = \Delta(n_1, \ldots, n_k)$ identically for a k-tuple $(n_1, \ldots, n_k)$ in $\mathcal{S}(n)$, then it is an ideal immersion automatically.*

REMARK 4.1. Theorem 3.2 implies that ideal submanifolds are those which receive the *least amount of tension at each point.*

In the following, by a *best world* we mean a Riemannian space with the highest degree of homogeneity.

REMARK 4.2. According to the work of Sophus Lie (1842–1899), Felix Klein (1849–1925) and Wilhelm Killing (1847–1923), the family of best worlds consists of Euclidean spaces, spheres, real projective spaces, and real hyperbolic spaces, i.e., the family of real space forms. These spaces have the highest degree of homogeneity, since their groups of isometries have the maximal possible dimension.

DEFINITION 4.2. A *best way of living* is an ideal isometric embedding of a Riemannian manifold into a best world.

Remark 4.3. It follows from Definition 4.2 that a best way of living is a very comfortable way of living in a wonderful world which allows maximal degree of freedom (living in a best world), it preserves shape (isometric, no stretching), without self-cutting (embedding). Furthermore it receives the least possible amount of tension at each point from the outside world (ideal).

5. Some applications of δ-invariants

In this section we provide some of the many applications of δ-invariants. Many more applications can be found in my 2011 book [**8**].

5.1. Applications to minimal immersions. δ-invariants have many applications to minimal immersions. As an immediate application of Theorem 3.2, we have the following solution to Problem 1 concerning Riemannian obstructions to minimal immersions.

Theorem 5.1. *Let M be a Riemannian n-manifold. If there exist a point p and a k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$ with $\delta(n_1,\dots,n_k)(p)>0$, then M never admits a minimal immersion into any Riemannian manifold with non-positive sectional curvature. In particular, M never admits a minimal immersion into any Euclidean space for any codimension.*

5.2. Applications to spectral theory. By applying Theorem 3.2 and Nash's embedding theorem, we discovered the following intrinsic result.

Theorem 5.2. *Let M be a compact irreducible homogeneous Riemannian n-manifold. Then the first nonzero eigenvalue λ_1 of the Laplacian Δ satisfies*

$$\lambda_1 \geqslant n\,\Delta(n_1,\dots,n_k) \tag{5.1}$$

for every k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$.

If $k=0$, then (5.1) reduces to the following well-known result of Tadashi Nagano (1930–) obtained in [**20**]:

$$\lambda_1 \geqslant n\rho,\ \text{where}\ \rho=\frac{2\tau}{n(n-1)}\ \text{is the } \textit{normalized scalar curvature.}$$

Theorem 5.2 implies immediately the following.

Corollary 5.1. *For every compact irreducible homogeneous Riemannian n-manifold, we have $\lambda_1 \geqslant n\hat{\Delta}_0$.*

5.3. Who can live in the wonderland of best livings? Ordinary spheres in Euclidean spaces are the simplest examples of best ways of living. Besides spheres there are many other homogeneous spaces which admit best ways of living in some Euclidean spaces. For instance, the following three compact homogeneous spaces:

$$SU(3)/T^2,\ \ Sp(3)/Sp(1)^3,\ \ \text{and}\ \ F_4/\mathrm{Spin}(8)$$

admit best ways of living in $\mathbb{E}^8$, $\mathbb{E}^{14}$ and $\mathbb{E}^{26}$ associated with

$$(3,3)\in\mathcal{S}(6),\ (3,3,3,3)\in\mathcal{S}(12),\ \ (12,12)\in\mathcal{S}(24),$$

respectively. The best ways of living for $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/\mathrm{Spin}(8)$ in $\mathbb{E}^8$, $\mathbb{E}^{14}$ and $\mathbb{E}^{26}$ are given respectively by the corresponding minimal isoparametric hypersurfaces in S^7, S^{13} and S^{25}.

Although, it was shown in [**9**] that for any integer $n \geqslant 3$ there exists Riemnnian n-manifolds which admit more than one best way of living in a Euclidean space, not every Riemannian manifold is lucky enough to admit a best way of living in a Euclidean space. For example, although the unit n-sphere $S^n(1)$ admits a best way of living in $\mathbb{E}^{n+1}$; the real projective n-space $RP^n(1), n > 1$, never admits a best way of living in any Euclidean space no matter how hard it tries (see Corollary 5.2).

Consequently, I proposed in [**4**] the following two problems:

Existence Problem: *What are the necessary and sufficient conditions for a Riemannian manifold to admit a best way of living in a best world?*

Classification Problem: *If a Riemannian manifold admits best ways of living in a best world, what are its best ways of living?*

5.4. Two solutions to Existence Problem. By applying $\lambda_1 \geqslant n\hat{\Delta}_0$ given in Corollary 5.1 and a result from the theory of finite type submanifolds, we have the following solution for the Existence Problem.

Theorem 5.3. *A compact irreducible homogeneous Riemannian n-manifold admits a best way of living in some Euclidean space if and only if it satisfies the intrinsic condition: $\lambda_1 = n\hat{\Delta}_0$.*

Corollary 5.2. *$RP^n(1)$ admits no best ways of living in any Euclidean space.*

Proof. For $RP^n(1)$ we have $\lambda_1 = 2(n+1)$ and $\hat{\Delta}_0 = 1$. Thus $\lambda_1 \neq n\hat{\Delta}_0$. $\square$

Another simple solution of the Existence Problem is the following.

Theorem 5.4. *If a compact Riemannian n-manifold M satisfies*

$$\lambda_1 > \frac{n}{v(M)} \int_M \hat{\Delta}_0 * 1, \quad v(M) = \text{volume of } M,$$

then it never admits a best way of living in any Euclidean space.

5.5. Applications to rigidity theory: Uniqueness problem. If a Riemannian manifold admits a best way of living in a best world and if it is unique, it gives a rigidity theorem. Otherwise, it admits multiple ways of best living (lucky one!).

For instance, by applying δ-invariants we proved the following rigidity results.

Theorem 5.5. *If M is an open portion of $S^n(1)$, then for every isometric immersion of M in a Euclidean m-space with arbitrary codimension we have*

$$H^2 \geqslant 1. \tag{5.2}$$

The equality case of (5.2) *holds identically if and only if M is embedded as an open portion of an ordinary hypersphere in a totally geodesic $\mathbb{E}^{n+1} \subset \mathbb{E}^m$.*

Theorem 5.6. *Let M be an open part of $\mathbb{E}^{n_1-1} \times S^{n-n_1+1}(1)$. Then for every isometric immersion of M in $\mathbb{E}^m$ with arbitrary codimension we have*

$$H^2 \geqslant \Big(\frac{n-n_1+1}{n}\Big)^2. \tag{5.3}$$

The equality case of (5.3) *holds if and only if M is immersed as an open part of a spherical hypercylinder $\mathbb{E}^{n_1-1} \times S^{n-n_1+1}(1) \subset \mathbb{E}^{n_1-1} \times \mathbb{E}^{n-n_1+2} \subset \mathbb{E}^m$.*

Remark 5.1. Rigidity results discovered in this way are quite different from usual rigidity results, e.g., rigidity in the sense of Aleksandr D. Aleksandrov (1912–1999). Because rigidity theorems via ideal immersions do not require any assumption on topology or codimension of the submanifolds. On the other hand, (almost) all other rigidity results do require global conditions as well as assumptions of very small codimension (e.g., in the theory of convex surfaces à la Aleksandrov).

5.6. Applications to warped products. An application of δ-invariants to warped products is to obtain the following sharp result.

Theorem 5.7. *For every isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}$ of a warped product $N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}$, the warping function satisfies*

$$\frac{\Delta f}{f} \leqslant \frac{(n_1+n_2)^2}{4n_2} H^2 + n_1 \max \tilde{K}, \quad n_i = \dim N_i, \quad i = 1, 2,$$

where Δ is the Laplacian on N_1.

This theorem has many immediate consequences. E.g., it yields the following.

Corollary 5.3. *Let $N_1 \times_f N_2$ be a warped product of Riemannian manifolds. If the warping function f is harmonic, then we have:*

(1) *$N_1 \times_f N_2$ never admits minimal immersion into any Riemannian manifold of negative sectional curvature;*
(2) *every minimal immersion of $N_1 \times_f N_2$ into any Euclidean space is a warped product immersion regardless of codimension.*

Corollary 5.4. *If $f \in C^\infty(N_1)$ is an eigenfunction of Δ with eigenvalue $\lambda > 0$, then every warped product $N_1 \times_f N_2$ never admits a minimal immersion into any Riemannian manifold with non-positive sectional curvature.*

Corollary 5.5. *Let N_1 be a compact Riemannian manifold. Then we have:*

(1) *every warped product $N_1 \times_f N_2$ never admits a minimal immersion into any Riemannian manifold of negative sectional curvature;*
(2) *every warped product $N_1 \times_f N_2$ never admits any minimal immersion into any Euclidean space.*

5.7. Applications to theory of submersions. A Riemannian submersion $\pi : M \to B$ is called *trivial* if it is a direct product of a fiber and the base manifold B. Two applications of δ-invariants to the theory of submersions are the following.

Theorem 5.8. *If a Riemannian manifold M admits a non-trivial Riemannian submersion with totally geodesic fibers, it cannot be isometrically immersed into any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.*

THEOREM 5.9. *If a Riemannian manifold M admits a Riemannian submersion with totally geodesic fibers, then every minimal immersion of M in a Euclidean space is the direct product of a minimal immersion of the base manifold and a minimal immersion of a fiber in Euclidean spaces.*

5.8. Applications to affine geometry. δ-invariants can also be applied to affine geometry. For instance, by applying δ-invariants we have the following.

THEOREM 5.10. *If the Calabi metric of an improper affine hypersphere in an affine space is the Riemannian product metric of k Riemannian manifolds, then the improper affine hypersphere is the Calabi composition of k improper affine spheres.*

THEOREM 5.11. *If the warping function f of a warped product manifold $N_1 \times_f N_2$ satisfies $\Delta f < 0$ at some point on N_1, then $N_1 \times_f N_2$ cannot be realized as an improper affine hypersphere in an affine $(n+1)$-space $\mathbf{R}^{n+1}$.*

THEOREM 5.12. *Every warped product $N_1 \times_f N_2$ with harmonic warping function cannot be realized as an elliptic proper affine hypersphere in $\mathbf{R}^{n+1}$.*

5.9. Links between submersions and affine hypersurfaces. By applying an affine δ-invariant, we discovered the following links between theory of Riemannian submersions and affine differential geometry.

THEOREM 5.13. *If a Riemannian manifold M can be realized as an elliptic proper centroaffine hypersphere centered on the origin in some affine space, then every Riemannian submersion $\pi : M \to B$ with minimal fibers has a non-totally geodesic horizontal distribution.*

THEOREM 5.14. *If a Riemannian manifold M can be realized as an improper hypersphere in an affine space, then every Riemannian submersion $\pi : M \to B$ with non-totally geodesic minimal fibers has non-totally geodesic horizontal distribution.*

5.10. Applications to symplectic geometry. An application of δ-invariants to Lagrangian submanifolds is to provide a sharp solution to Problem 2.

THEOREM 5.15. *Let M be a compact Riemannian n-manifold with null first Betti number $b_1(M)$ or finite fundamental group $\pi_1(M)$. If there exists a k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ such that $\delta(n_1, \dots, n_k) > 0$, then M never admits a Lagrangian isometric immersion into $\mathbf{C}^n$.*

REMARK 5.2. The assumption on $\delta(n_1, \dots, n_k)$ in Theorem 5.15 is sharp. This can be seen as follows: Consider the *Whitney sphere* W^n defined by the Whitney immersion $w : S^n \to \mathbf{C}^n$:

$$w(y_0, \dots, y_n) = \frac{1 + iy_0}{1 + y_0^2}\,(y_1, \dots, y_n)$$

with $y_0^2 + y_1^2 + \cdots + y_n^2 = 1$. This immersion is Lagrangian with a unique self-intersection point at $w(-1, 0, \dots, 0) = w(1, 0, \dots, 0)$. For each k-tuple $(n_1, \dots, n_k)$, we have $\delta(n_1, \dots, n_k) \geqslant 0$ with respect to the induced metric via w. Moreover, we have $\delta(n_1, \dots, n_k) = 0$ only at the unique point of self-intersection. Furthermore, the assumption of $\#\pi_1(M) < \infty$ or $b_1(M) = 0$ is necessary for $n \geqslant 3$ (see [**8**]).

The proof of Theorem 5.15 is based on the following result.

THEOREM 5.16. *Let M be a Lagrangian submanifold of a complex space form $M^n(4c)$. Then for each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$ we have*

$$\delta(n_1,\dots,n_k)\leqslant \frac{n^2\big(n+k-1-\sum_{j=1}^k n_j\big)}{2\big(n+k-\sum_{j=1}^k n_j\big)}H^2 + \frac{1}{2}\Big\{n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\Big\}c \tag{5.4}$$

We have the following result from [**6**] for Lagrangian submanifolds satisfying the equality of (5.4); extending a result of [**12**] to $\delta(2)$-ideal Lagrangian submanifolds.

THEOREM 5.17. *If a Lagrangian submanifold of a complex space form satisfies the equality case of* (5.4) *at a point, then it is minimal at that point.*

6. Two recent optimal inequalities involving δ-invariants for Lagrangian submanifolds

In view of Theorem 5.17, it is very natural to look for optimal inequalities for non-minimal Lagrangian submanifolds in complex space forms.

6.1. Case I: $\sum_{i=1}^k n_i<n$. Recently, joint with F. Dillen, J. Van der Veken and L. Vrancken, we improve inequality (5.4) in [**10**,**13**] to the following inequality for $\sum_{i=1}^k n_i<n$ (see also [**11**]).

THEOREM 6.1. *If M is a Lagrangian submanifold of a complex space form $M^n(4c)$, then for a k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$ with $\sum_{i=1}^k n_i<n$ we have*

$$\delta(n_1,\dots,n_k)\leqslant \frac{n^2\big\{n-\sum_{i=1}^k n_i+3k-1-6\sum_{i=1}^k(2+n_i)^{-1}\big\}}{2\big\{n-\sum_{i=1}^k n_i+3k+2-6\sum_{i=1}^k(2+n_i)^{-1}\big\}}H^2 +\frac{1}{2}\Big\{n(n-1)-\sum_{i=1}^{k} n_i(n_i-1)\Big\}c. \tag{6.1}$$

The equality sign of (6.1) *holds at a point $p\in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ at p such that with respect to this basis the second fundamental form h takes the following form:*

$$\begin{aligned} &h(e_{\alpha_i},e_{\beta_i})=\sum_{\gamma_i} h^{\gamma_i}_{\alpha_i\beta_i}Je_{\gamma_i}+\frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{\mu+1},\ \sum_{\alpha_i=1}^{n_i} h^{\gamma_i}_{\alpha_i\alpha_i}=0,\\ &h(e_{\alpha_i},e_{\alpha_j})=0,\ i\neq j,\ h(e_{\alpha_i},e_{\mu+1})=\frac{3\lambda}{2+n_i}Je_{\alpha_i},\ h(e_{\alpha_i},e_u)=0,\\ &h(e_{\mu+1},e_{\mu+1})=3\lambda Je_{\mu+1},\ h(e_{\mu+1},e_u)=\lambda Je_u,\ h(e_u,e_v)=\lambda\delta_{uv}Je_{\mu+1}, \end{aligned} \tag{6.2}$$

for $1\leqslant i,j\leqslant k; \mu+2\leqslant u,v\leqslant n$ and $\lambda=\frac{1}{3}h^{\mu+1}_{\mu+1\mu+1}$, where $\mu=n_1+\cdots+n_k$.

REMARK 6.1. For $\delta(2)$, inequality (6.1) is due to T. Oprea [Chen's inequality in the Lagrangian case, Colloq. Math. **108** (2007), 163–169].

In [**13**] we also proved the following.

THEOREM 6.2. *For every k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, there exists a non-minimal Lagrangian submanifold which satisfies the equality case of* (6.1).

Theorem 6.2 shows that inequality (6.1) cannot be improved further.

6.2. Case II: $\sum_{i=1}^k n_i = n$. For this case we prove the following inequality with different coefficient in [**13**].

THEOREM 6.3. *Let M be a Lagrangian submanifold of a complex space form $M^n(4c)$. Then for each $(n_1,\dots,n_k)\in\mathcal{S}(n)$ with $\sum_{i=1}^k n_i = n$ we have*

$$\delta(n_1,\dots,n_k)\leqslant\frac{n^2\big\{k-1-2\sum_{i=2}^k(2+n_i)^{-1}\big\}}{2\big\{k-2\sum_{i=2}^k(2+n_i)^{-1}\big\}}H^2 + \frac{1}{2}\bigg\{n(n-1)-\sum_{i=1}^k n_i(n_i-1)\bigg\}c, \tag{6.3}$$

where we assume that $n_1=\min_{i=1}^n\{n_i\}$.

If the equality sign of (6.3) *holds at a point $p\in M$, the components of the second fundamental form with respect to some suitable orthonormal basis $\{e_1,\dots,e_n\}$ for T_pM satisfy the following conditions:*

(a) $h^A_{\alpha_i\alpha_j}=0$ *for $i\neq j$ and $A\neq\alpha_i,\alpha_j$;*
(b) *if $n_j\neq\min\{n_1,\dots,n_k\}$:*

$$h^{\beta_j}_{\alpha_i\alpha_i}=0 \text{ if } i\neq j \text{ and } \sum_{\alpha_j\in\Delta_j}h^{\beta_j}_{\alpha_j\alpha_j}=0,$$

(c) *if $n_j=\min\{n_1,\dots,n_k\}$:*

$$\sum_{\alpha_j\in\Delta_j}h^{\beta_j}_{\alpha_j\alpha_j}=(n_i+2)h^{\beta_j}_{\alpha_i\alpha_i} \text{ for any } i\neq j \text{ and any } \alpha_i\in\Delta_i.$$

REMARK 6.2. In the case of equality, we do not have information about $h^{\gamma_i}_{\alpha_i\beta_i}$, where α_i, β_i and γ_i are mutually different indices in the same block Δ_i.

The following theorem from [**13**] implies that inequality (6.3) is sharp.

THEOREM 6.4. *For each k-tuple $(n_1\dots,n_k)\in\mathcal{S}(n)$ with $\sum_{i=1}^k n_i=n$, there exists a Lagrangian submanifold in $M^n(4c)$ satisfying the equality of the improved inequality* (6.3) *identically.*

This result implies that inequality (6.3) cannot be improved further as well.

REMARK 6.3. Lagrangian submanifolds of complex space forms satisfying some special cases of the improved inequalities (6.1) and (6.3) have been investigated and classified recently in [**1, 2, 8, 10, 13–17**].

References

1. J. Bolton, F. Dillen, J. Fastenakels, L. Vrancken, *A best possible inequality for curvature-like tensor fields*, Math. Inequal. Appl. **12** (2009), 663–681.
2. J. Bolton, L. Vrancken, *Lagrangian submanifolds attaining equality in the improved Chen's inequality*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 311–315.
3. B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993), 568–578.
4. ———, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*; in: *Third Pacific Rim Geom. Conf.* Seoul 1996, 7–60, Intern. Press, MA (1998).
5. ———, *Some new obstruction to minimal and Lagrangian isometric immersions*, Japan. J. Math. **26** (2000), 105–127.
6. ———, *Ideal Lagrangian immersions in complex space forms*, Math. Proc. Cambridge Philos. Soc. **128** (2000), 511–533.
7. ———, *Riemannian submanifolds*; in F. Dillen and L. Verstraelen (eds.), *Handbook of Differential Geometry*, vol. I, 187–418, North Holland, 2000.
8. ———, *Pseudo-Riemannian Geometry, δ-invariants and Applications*, World Scientific, Hackensack, New Jersey, 2011.
9. ———, *On ideal hypersurfaces of Euclidean 4-space*, Arab J. Math. Sci. **19**(2) (2013), 129–144.
10. B.-Y. Chen, F. Dillen, *Optimal general inequalities for Lagrangian submanifolds in complex space forms*, J. Math. Anal. Appl. **379** (2011), 229–239.
11. ———, *δ-invariants for Lagrangian submanifolds of complex space forms*, in *Proceedings RIGA* 2011, pp. 75–94, Ed. Univ. Bucureşti, Bucharest, (2011).
12. B.-Y. Chen, F. Dillen, L. Verstraelen, L. Vrancken, *An exotic totally real minimal immersion of S^3 in CP^3 and its characterization*, Proc. Roy. Soc. Edinburgh Sec. A Math. **126** (1996), 153–165.
13. B.-Y. Chen, F. Dillen, J. Van der Veken, L. Vrancken, *Curvature inequalities for Lagrangian submanifolds: the final solution*, Diff. Geom. Appl. (to appear).
14. B.-Y. Chen, F. Dillen, L. Vrancken, *Lagrangian submanifolds in complex space forms attaining equality in a basic inequality*, J. Math. Anal. Appl. **386** (2012), 139–152.
15. B.-Y. Chen, A. Prieto-Martín, *Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving* $\delta(2,2)$, Differ. Geom. Appl. **30** (2012), 107–123.
16. B.-Y. Chen, A. Prieto-Martín, X. Wang, *Lagrangian submanifolds in complex space forms satisfying an improved equality involving* $\delta(2,2)$, Publ. Math. Debrecen **82** (2013), 193–217.
17. B.-Y. Chen, J. Van der Veken, L. Vrancken, *Lagrangian submanifolds with prescribed second fundamental form*, Proc. Pure Appl. Diff. Geom. PADGE 2012 (to appear).
18. M. Gromov, *A topological technique for the construction of solutions of differential equations and inequalities*, Intern. Congr. Math. (Nice 1970) **2** (1971), 221–225.
19. ———, *Isometric immersions of Riemannian manifolds*, The mathematical heritage of Élie Cartan (Lyon, 1984), Astérisque 1985, 129–133.
20. T. Nagano, *On the miniumum eigenvalues of the Laplacian in Riemannian manifolds*, Sci. Papers College Gen. Edu. Univ. Tokyo **11** (1961), 177–182.
21. J. F. Nash, *The imbedding problem for Riemannian manifolds*, Ann. Math. **63** (1956), 20–63.
22. S. Nasar, *A Beautiful Mind; The life of mathematical genius and Nobel laureate John Nash*, Simon and Schuster, New York, 1998.
23. S. T. Yau, *The current state and prospects of geometry and nonlinear differential equations*, in: *Mathematics research today and tomorrow: Viewpoints of Seven Fields Medalists*, Lect. Notes Math. 1525, Springer, 1991, pp. 29–39.

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