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GROUP CLASSIFICATION OF VARIABLE COEFFICIENT QUASILINEAR REACTION-DIFFUSION EQUATIONS

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Abstract. The group classification of variable coefficient quasilinear reaction-diffusion equations $u_t = u_{xx} + h(x)B(u)$ is carried out exhaustively. This became possible due to usage of a conditional equivalence group found in the course of the study of admissible point transformations within the class.

1. Introduction

Geometrical study of differential equations (DEs) has a long and distinguished history dating back to the second part of the XIX century when the pioneering works of Gaston Darboux, Sophus Lie and Élie Cartan were published. Their ideas became a source for a number of developments including the theory of completely integrable systems and the study of conservation laws. A brief but very nice review of both classical and modern treatments of geometrical study of differential equations is presented in [**1**]. One of the classical topics of such studies is Lie (point) symmetries of differential equations. Lie proved that knowledge of continuous group of nondegenerate point transformations that leave an equation invariant allows one to reduce this equation to one with fewer independent variables. In many cases, this reduction procedure results in construction of a group-invariant solution in closed form. It is worthy to say that Lie symmetries give a powerful tool for finding exact solutions for partial differential equations (PDEs), and this is one of the most successful applications of geometrical studies of DEs [**2**, **3**]. Another feature of Lie symmetries is that they reveal equations which are important for applications among wide set of admissible ones. Indeed, all basic equations of mathematical physics, e.g., the equations of Newton, Laplace, Euler–Lagrange, d'Alembert, Lamé, Hamilton–Jacobi, Maxwell, Schrödinger etc., have rich symmetry properties [**4**]. This property distinguishes these equations from other PDEs. Therefore, an important problem arises to single out from a given class of PDEs those admitting Lie symmetry algebra of the maximally possible dimension. This

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problem is called the *group classification problem* and is formulated as follows [**3**,**5**]: given a class of PDEs, to classify all possible cases of extension of Lie invariance algebras of such equations with respect to the equivalence group of the class.

At this stage the modern group analysis provides us with two main approaches for solving group classification problems. The first is algebraic one, based on subgroup analysis of the corresponding equivalence group [**6**,**7**]. It results in complete group classification only if the class under study is normalized [**8**]. Roughly speaking, the class is normalized if any point transformation between two fixed equations from the class is induced by a transformation from its equivalence group. The second approach is based on a direct integration of determining equations implied by the infinitesimal invariance criterion. It is usually efficient only for classes of simple structure parameterized by a few arbitrary elements that are constants or functions of a single variable. Obviously, normalized classes or ones of simple structure do not exhaust a set of classes of PDEs or their systems that are important for applications. To solve more group classification problems, a number of notions was introduced recently as well as new approaches were developed. These are, e.g., notions of admissible [**8**] (synonym: form-preserving [**9**]) transformations, generalized [**10**] and extended [**11**] equivalence groups, normalized class of DEs [**8**], equivalence groupoid [**12**], contractions of equations and conservation laws [**13**,**14**]. Among new approaches it is worthy to mention the method of furcate split [**15**], the method of mapping between classes [**16**], the partition of a class into normalized subclasses [**7**,**8**], etc.

In this paper we solve the group classification problem for the class of variable coefficient semilinear reaction-diffusion equations of the form

$$u_t = u_{xx} + h(x)B(u), \tag{1.1}$$

where $h = h(x)$ and $B = B(u)$ are arbitrary smooth functions of their variables, $hB_{uu} \neq 0$. Linear equations singled out from class (1.1) by the condition $B_{uu} = 0$ are excluded from consideration since group classification of all second-order linear PDEs in two dimensions was performed by Lie (see [**17**]).

Equations from this class are used to model various phenomena such as microwave heating, problems in population genetics, etc. (see, e.g., [**18**] and references therein). Theorems on existence and uniqueness of bounded solutions for equations of a more general form $u_t = u_{xx} + F(t, x, u)$ were proved in [**19**].

Lie symmetries of certain subclasses of (1.1) are known. The group classification of constant coefficient equations from class (1.1) was carried out by Dorodnitsyn [**20**] (the results are adduced in handbook [**5**]). Class (1.1) includes the generalized Huxley equations

$$u_t = u_{xx} + h(x)u^2(1-u),$$

whose Lie symmetries were studied in [**21**,**22**]. There exists also a certain intersection with the results on group classification of the classes

$$u_t = u_{xx} + H(x)u^m + F(x)u, \quad m \neq 0, 1,\ H \neq 0, \quad \text{and} \tag{1.2}$$

$$u_t = u_{xx} + H(x)u^2 + G(x), \quad H \neq 0, \tag{1.3}$$

which were obtained in [**16**]. Note that the group classification for the general class of (1+1)-dimensional second-order quasilinear evolution equations

$$u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x), \quad F \neq 0,$$

was carried out in [**6**]. Nevertheless those results obtained up to a very wide equivalence group seem to be inconvenient to derive a group classification for class (1.1).

The structure of this paper is as follows. In the next section we investigate equivalence transformations in class (1.1). Lie symmetries are classified in Section 3.

2. Equivalence transformations

To solve a group classification problem for a class of differential equations it is important to describe point transformations that preserve the differential structure of the class and transform only its arbitrary elements. Such transformations are called *equivalence transformations* and form a group [**3**]. According to Ovsiannikov, the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables do not involve arbitrary elements, i.e., they are projectible on the space of independent and dependent variables. If this restriction is neglected, then the corresponding equivalence group is called the generalized equivalence group [**10**]. If new arbitrary elements appear to depend on old ones in a nonpoint (possibly, nonlocal) way, then the corresponding equivalence group is called extended [**11**]. An admissible transformation is a triple consisting of two fixed equations from a class and a point transformation that links these equations. The set of admissible transformations of a class naturally possesses the groupoid structure with respect to the composition of transformations, and hence it is called the *equivalence groupoid* of the class [**12**]. We look for admissible transformations for class (1.1) using the direct method [**9**, **14**].

Consider a pair of equations from the class under consideration, i.e., equation (1.1) and the equation

$$\tilde u_{\tilde t} = \tilde u_{\tilde x\tilde x} + \tilde h(\tilde x)\tilde B(\tilde u), \tag{2.1}$$

and assume that they are connected via a point transformation $\mathcal{T}$ of the general form

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u), \tag{2.2}$$

where $T_tX_xU_u \neq 0$. We can restrict ourselves by the transformations of this form instead of general transformations $\tilde t = \hat T(t,x,u)$, $\tilde x = \hat X(t,x,u)$, $\tilde u = \hat U(t,x,u)$. It is due to the fact that class (1.1) is a subclass of a more general class of (1+1)-dimensional quasi-linear evolution equations $u_t = F(t,x,u)u_{xx} + G(t,x,u,u_x)$ with $F \neq 0$, for which admissible transformations are proved to be of the form (2.2) [**15**].

We have to derive the determining equations for the functions T, X and U and then to solve them. Simultaneously we have to find a connection between arbitrary elements of equations (1.1) and (2.1). Substituting the expressions for the new (tilded) variables into (2.1), we obtain an equation in the old (untilded) variables. It should be an identity on the manifold $\mathcal{L}$ determined by (1.1) in the second-order jet space J^2 with the independent variables (t,x) and the dependent variable u. To involve the constraint between variables of J^2 on the manifold $\mathcal{L}$, we substitute the expression of u_t implied by equation (1.1). The splitting of this identity with respect to the derivatives u_{xx} and u_x gives the determining equations for the functions T, X and U:

$$U_{uu}=0,\quad {X_x}^2=T_t,\quad 2\frac{U_{xu}}{U_u}=-\frac{X_tX_x}{T_t}+\frac{X_{xx}}{X_x}, \tag{2.3}$$

$$T_t\tilde h\tilde B-U_uhB=U_t-U_{xx}-\frac{X_t}{X_x}U_x. \tag{2.4}$$

Solving at first equations (2.3) we get that $T_t>0$ and

$$X=\varepsilon\sqrt{T_t}\,x+\sigma(t),\quad U=U^1(t,x)u+U^0(t,x),$$
$$U^1=\zeta(t)\exp\left(-\frac{1}{8}\frac{T_{tt}}{T_t}x^2-\frac{\varepsilon}{2}\frac{\sigma_t}{\sqrt{T_t}}x\right),$$

where U^0, σ and ζ are arbitrary smooth functions of their variables; $\varepsilon=\pm1$. Then equation (2.4) can be written as

$$T_t\tilde h\tilde B-U^1hB=\sum_{i=0}^{1}\left(U^i_t-U^i_{xx}-\frac{1}{2}\frac{T_{tt}}{T_t}U^i_xx-\varepsilon\frac{\sigma_t}{\sqrt{T_t}}U^i_x\right)u^i, \tag{2.5}$$

where $u^1=u$, $u^0=1$. Investigating (2.5) when the arbitrary elements h and B vary, we derive the usual equivalence group of class (1.1).

Theorem 2.1. *The usual equivalence group $G^\sim$ of class* (1.1) *consists of the transformations*

$$\tilde t=\delta_1^2t+\delta_2,\quad \tilde x=\delta_1x+\delta_3,\quad \tilde u=\delta_4u+\delta_5,\quad \tilde h=\frac{\delta_4}{\delta_1^2\delta_0}h,\quad \tilde B=\delta_0B,$$

where δ_j, $j=0,\dots,5$, are arbitrary constants with $\delta_0\delta_1\delta_4\neq0$.

It appears that there exist point transformations between equations from (1.1) which do not belong to $G^\sim$ and form a conditional equivalence group. Moreover, this group is not usual but a generalized extended one.

Theorem 2.2. *The generalized extended equivalence group $\hat G^\sim_{\exp}$ of the subclass*

$$u_t=u_{xx}+h(x)(e^{nu}+r) \tag{2.6}$$

of class (1.1) *is formed by the transformations*

$$\tilde t=\delta_1^2t+\delta_2,\quad \tilde x=\delta_1x+\delta_3,\quad \tilde u=\delta_4u+\varphi(x),$$
$$\tilde h=\frac{\delta_4}{\delta_1^2}\exp\Big(-\frac{n}{\delta_4}\varphi\Big)h,\quad \tilde n=\frac{n}{\delta_4},\quad \tilde r=\exp\Big(\frac{n}{\delta_4}\varphi\Big)\Big(r-\frac{\varphi_{xx}}{\delta_4h}\Big),$$

where r and δ_j, $j=1,\dots,4$, are arbitrary constants with $\delta_1\delta_4\neq0$. The transformation component for r can be interpreted as the constraint for φ,

$$\varphi_{xx}=\delta_4h\Big(r-\tilde r\exp\Big(-\frac{n}{\delta_4}\varphi\Big)\Big).$$

Theorem 2.2 implies that class (2.6) reduces to the class $\tilde u_t = \tilde u_{xx} + \tilde h(x)e^{n\tilde u}$ by the transformation $\tilde t = t$, $\tilde x = x$, $\tilde u = u + \varphi(x)$, where $\tilde h(\tilde x) = e^{-\varphi(x)}h(x)$ and $\varphi_{xx} = rh(x)$. Class (2.6) is normalized. Therefore, the equivalence group of class (2.6) with $r = 0$ can be found by setting $\tilde r = r = 0$ in transformations from the group $\hat G^{\sim}_{\rm exp}$.

Corollary 2.1. *The usual equivalence group $G^{\sim}_{\rm exp}$ of the class*

$$u_t = u_{xx} + h(x)e^{nu}$$

consists of the transformations

$$\tilde t = \delta_1^2 t + \delta_2, \quad \tilde x = \delta_1 x + \delta_3, \quad \tilde u = \delta_4 u + \delta_5 x + \delta_6,$$
$$\tilde h = \frac{\delta_4}{\delta_1^2}\exp\Big(-\frac{n}{\delta_4}(\delta_5 x + \delta_6)\Big)h, \quad \tilde n = \frac{n}{\delta_4},$$

where δ_j, $j = 1,\dots,6$, are arbitrary constants with $\delta_1\delta_4 \neq 0$.

In the course of the study of Lie symmetries we will use the derived equivalence transformations for the simplification of calculations and for presenting the final results in a concise form.

3. Lie symmetries

We study Lie symmetries of equations from class (1.1) using the classical approach [**3**] in a combination with the method of furcate split [**15**]. We search for vector fields of the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

that generate one-parameter Lie symmetry groups of a fixed equation $\mathcal L$ from class (1.1). These vector fields form the maximal Lie invariance algebra $A^{\max} = A^{\max}(\mathcal L)$ of the equation $\mathcal L$. Any Lie symmetry generator Q satisfies the infinitesimal invariance criterion, i.e., the action of the second prolongation $Q^{(2)}$ of Q on the equation $\mathcal L$ results in the condition identically satisfied for all solutions of $\mathcal L$. Namely, we require

$$Q^{(2)}\big(u_t - u_{xx} - h(x)B(u)\big)\big|_{\mathcal L} = 0. \tag{3.1}$$

After elimination of u_t by means of (1.1), equation (3.1) can be regarded as a polynomial in the variables u_x, u_{xx} and u_{tx}. The coefficients of different powers of these variables should be zeros. This results in the determining equations for the coefficients τ, ξ and η. Solving these equations implies that $\tau = \tau(t)$ and $\xi = \xi(t,x)$, which agrees with the general results on point transformations between evolution equations [**9**]. The remaining determining equations have the form

$$2\xi_x = \tau_t, \quad \eta_{uu} = 0, \quad 2\eta_{xu} = \xi_{xx} - \xi_t, \tag{3.2}$$

$$\eta h B_u = (-\xi h_x + (\eta_u - \tau_t)\,h)\,B + \eta_t - \eta_{xx}. \tag{3.3}$$

Integrating equations (3.2) we get the following expressions for ξ and η:

$$\xi = \tfrac12\tau_t\,x + \sigma(t), \quad \eta = \big(-\tfrac18\tau_{tt}x^2 - \tfrac12\sigma_t x + \zeta(t)\big)\,u + \eta^0(t,x),$$

where σ, ζ and η^0 are arbitrary smooth functions of their variables. Then equation (3.3) becomes

$$\begin{aligned}(3.4)\quad \left(\left(\tfrac{1}{8}\tau_{tt}x^2+\tfrac{1}{2}\sigma_t x-\zeta\right)u-\eta^0\right)hB_u &= \left(\left(\tfrac{1}{8}\tau_{tt}x^2+\tfrac{1}{2}\sigma_t x-\zeta+\tau_t\right)h\right.\\ &+\left.\left(\tfrac{1}{2}\tau_t x+\sigma\right)h_x\right)B+\left(\tfrac{1}{8}\tau_{ttt}x^2+\tfrac{1}{2}\sigma_{tt}x-\zeta_t-\tfrac{1}{4}\tau_{tt}\right)u-\eta^0_t+\eta^0_{xx}.\end{aligned}$$

It is called the classifying equation and should be solved simultaneously with respect to remaining uncertainties in the coefficients of infinitesimal generator Q, i.e., the functions τ, σ, ζ and η^0, and arbitrary elements of the class, namely, h and B.

In order to find the common part of Lie symmetries for all equations from class (1.1), we split it with respect to the arbitrary elements in equation (3.4). This results in $\tau_t=\xi=\eta=0$.

Proposition 3.1. *The intersection of the maximal Lie invariance algebras of equations from class* (1.1) *(called the kernel algebra) is the one-dimensional algebra* $A^\cap=\langle\partial_t\rangle$.

The next step is to classify possible extensions of $A^\cap$ using the method of furcate split [**14**, **15**]. For any operator Q from $A^{\max}$ the substitution of its coefficients into (3.4) gives an equation on B of the general form

$$(3.5)\qquad (au+b)B_u=pB+qu+r,$$

where a, b, p, q and r are constants which are defined up to a nonzero multiplier. The set $\mathcal{V}$ of values of the coefficient tuple (a,b,p,q,r) obtained by varying of an operator from $A^{\max}$ is a linear space. Note that $(a,b)\neq(0,0)$ and the dimension $k=k(A^{\max})$ of the space $\mathcal{V}$ is not greater than 2. Otherwise the corresponding equations imply that either B is linear in u or the system is incompatible. The value of k is an invariant of the transformations from $G^\sim$. Therefore, there exist three $G^\sim$-inequivalent cases for the value of k: $k=0$, $k=1$ and $k=2$. We consider these possibilities separately.

I. The condition $k=0$ means that (3.4) is not an equation with respect to B but an identity. Therefore, B is not constrained and $\tau_{tt}=\sigma_t=\zeta=\eta^0=0$. We obtain that $\tau=c_1t+c_2$, $\sigma=c_3$ and the classifying equation on h has the form

$$(3.6)\qquad \left(\tfrac{1}{2}c_1x+c_3\right)h_x+c_1h=0,$$

where c_1, c_2, and c_3 are arbitrary constants. If h is arbitrary we get the kernel algebra $A^\cap$ presented by Case 0 of Table 1. It follows from (3.6) that the extensions are possible in two cases: either $h=\delta(x+\beta)^{-2}$ or $h=\delta$, where β and δ are arbitrary constants, $\delta\neq0$. Up to $G^\sim$-equivalence β can be set to zero value and δ to ±1 depending on its sign. These two cases are presented by Cases 1 and 2 of Table 1.

II. If $k=1$, then we have, up to a nonzero multiplier, exactly one equation of the form (3.5) with respect to the function B. The integration of this equation up to $G^\sim$-equivalence gives three cases

$$1.\ B=u^m+\beta_1u+\beta_2,\quad 2.\ B=e^u+\beta_1u+\beta_2,\quad 3.\ B=u\ln u+\beta_2,$$

where β_1,β_2 and m are arbitrary constants, $m\neq0,1$. The next step is to substitute each of the three derived forms of B into equation (3.4) and subsequently to split it with respect to linearly independent functions of the variable u. Let us consider these cases separately.

II.1. Let $B = u^m + \beta_1 u + \beta_2$. If $m \neq 2$ then splitting equation (3.4) leads to the condition $\eta^0 = 0$ and the three classifying equations on the function h

$$\left(\tfrac{1}{2}\tau_t x + \sigma\right) h_x + \left((1-n)W + \tau_t\right) h = 0, \quad \beta_2 \left(\left(\tfrac{1}{2}\tau_t x + \sigma\right) h_x + (W + \tau_t)\, h\right) = 0,$$
$$(1-\beta_1)\left(\tfrac{1}{2}\tau_t x + \sigma\right) h_x + \left(W + (1-\beta_1)\tau_t\right) h - W_t + \tfrac{1}{4}\tau_{tt} = 0,$$

where $W = \frac{1}{8}\tau_{tt}x^2 + \frac{1}{2}\sigma_t x - \zeta$, $W_t = \frac{\partial W}{\partial t}$. The study of compatibility of this system shows that if $\beta_2 \neq 0$, then $\tau = c_1 t + c_2$, $\sigma = c_3$, $\zeta = 0$ and the classifying equation on h is of the form (3.6). Therefore, this is a subcase of the case $k = 0$ considered above.

If $m \neq 2$ and $\beta_2 = 0$, then class (1.1) takes the form (1.2) with $H(x) = h(x)$ and $F(x) = \beta_1 h(x)$. Using the results of [**16**] we derive that extension of $A^{\cap}$ is possible if $\beta_1 = 0$ and the function h takes one of the forms presented by Cases 3, 4 and 5 of Table 1. If $\beta_1 \neq 0$, then the cases of extension of $A^{\cap}$ are subcases of Cases 1 and 2 of Table 1.

If $B = u^2 + \beta_1 u + \beta_2$, then β_1 can always be set to zero by the translation $u \mapsto u - \beta_1/2$ and then class (1.1) coincides with (1.3), where $H(x) = h(x)$ and $G(x) = \beta_2 h(x)$. Using the results derived in [**16**] we obtain that the cases of Lie symmetry extension are either presented by Cases 1 and 2 of Table 1 if $\beta_2 \neq 0$ or by Cases 3–5 of Table 1 if $\beta_2 = 0$.

II.2. If $B = e^u + \beta_1 u + \beta_2$, then the classifying equations imply $\tau_{tt} = \sigma_t = \zeta = 0$. We obtain $\tau = c_1 t + c_2$, $\sigma = c_3$ and the system of equations on h

$$\left(\left(\tfrac{1}{2}c_1 x + c_3\right) h_x + c_1 h\right)\beta_1 = 0, \quad \eta^0 h + \left(\tfrac{1}{2}c_1 x + c_3\right) h_x + c_1 h = 0,$$
$$\left(\left(\tfrac{1}{2}c_1 x + c_3\right) h_x + c_1 h\right)\beta_2 + \eta^0 h\, \beta_1 + \eta^0_{xx} - \eta^0_t = 0.$$

If $\beta^1 \neq 0$, then $\eta^0 = 0$ and the classifying condition on h is exactly equation (3.6). This case can be included in the general case with arbitrary B. If $\beta^1 = 0$, then β_2 can be set to zero using the transformation from the conditional equivalence group $\hat{G}^{\sim}_{\exp}$. Therefore, $\eta^0 = c_4 x + c_5$, where c_4 and c_5 are arbitrary constants and the remaining classifying equation on h takes the form

$$\left(\tfrac{1}{2}c_1 x + c_3\right) h_x + (c_4 x + c_5 + c_1) h = 0. \tag{3.7}$$

Combined with the multiplication by a nonzero constant, each transformation from the equivalence group $G^{\sim}_{\exp}$ is extended to the coefficient tuple of the above equation in the following way

$$\tilde{c}_1 = \kappa\, c_1, \quad \tilde{c}_3 = \kappa\left(c_3\delta_1 - \tfrac{1}{2}c_1\delta_3\right), \quad \tilde{c}_4 = \frac{\kappa}{\delta_1}\left(c_4 + \tfrac{1}{2}c_1\delta_5\right),$$
$$\tilde{c}_5 = \frac{\kappa}{\delta_1}\left(c_5\delta_1 + c_3\delta_1\delta_5 - c_4\delta_3 - \tfrac{1}{2}c_1\delta_3\delta_5\right).$$

Here κ is an arbitrary nonzero constant. Using these transformations we derive the following statement.

Table 1. The group classification of the class $u_t = u_{xx} + h(x)B(u)$, $hB_{uu} \neq 0$.

no.	$B(u)$	$h(x)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t
1	$\forall$	δx^{-2}	∂_t, $2t\partial_t + x\partial_x$
2	$\forall$	δ	∂_t, ∂_x
3	u^m	δx^s	∂_t, $2(m-1)t\partial_t + (m-1)x\partial_x - (s+2)u\partial_u$
4	u^m	δe^x	∂_t, $(1-m)\partial_x + u\partial_u$
5	u^m	δ	∂_t, ∂_x, $2(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
6	e^u	δx^s	∂_t, $2t\partial_t + x\partial_x - (s+2)\partial_u$
7	e^u	$\delta e^{\pm x^2}$	∂_t, $\partial_x \mp 2x\partial_u$
8	e^u	δ	∂_t, ∂_x, $2t\partial_t + x\partial_x - 2\partial_u$
9	$u \ln u$	δ	∂_t, ∂_x, $e^{\delta t}u\partial_u$, $e^{\delta t}(\partial_x - \frac{\delta}{2}xu\partial_u)$

Here δ, m and s are arbitrary constants, $m \neq 0, 1$, $s \neq 0$, $\delta = \pm 1 \bmod G^{\sim}$.

Lemma 3.1. *Up to $G^{\sim}_{\exp}$-equivalence the parameter tuple (c_1, c_3, c_4, c_5) can be assumed to belong to the set* $\{(1,0,0,\bar{c}_5),\ (0,1,\pm 2,0),\ (0,1,0,0),\}$*, where $\bar{c}_5$ is an arbitrary constant.*

The integration of equation (3.7) up to $G^{\sim}_{\exp}$-equivalence leads to the forms of h presented in Cases 6–8 of Table 1. In Case 6 we used the notation $s = -2(\bar{c}_5 + 1)$.
II.3. If $B = u \ln u + \beta_2$, then $\eta^0 = 0$ and the remaining classifying conditions are

$$\left(\tfrac{1}{2}\tau_t x + \sigma\right) h_x + \tau_t h = 0, \quad \left(\tfrac{1}{8}\tau_{tt}x^2 + \tfrac{1}{2}\sigma_t x - \zeta\right) h = \tfrac{1}{8}\tau_{ttt}x^2 + \tfrac{1}{2}\sigma_{tt}x - \zeta_t - \tfrac{1}{4}\tau_{tt},$$
$$\beta_2 \left(\left(\tfrac{1}{2}\tau_t x + \sigma\right) h_x + \left(\tfrac{1}{8}\tau_{tt}x^2 + \tfrac{1}{2}\sigma_t x - \zeta + \tau_t\right) h\right) = 0.$$

Investigation of this system implies that if $\beta_2 \neq 0$ or $\beta_2 = 0$, but $h_x \neq 0$, then $\tau_{tt} = \sigma_t = \zeta = 0$. Therefore $\tau = c_1 t + c_2$, $\sigma = c_3$ and the classifying equation on h is (3.6). In other words, we get nothing but subcases of Cases 1 and 2 of Table 1. At the same time, if $\beta_2 = 0$ and $h = \delta = \text{const}$, then the classifying equations lead to the conditions $\tau_t = \sigma_{tt} - \delta\sigma_t = \zeta_t - \delta\zeta = 0$. Therefore $\tau = c_1$, $\xi = c_2 + c_3 e^{\delta t}$, $\eta = \left(-\frac{\delta}{2}c_3 x + c_4\right) e^{\delta t} u$ and we have an extension of $A^{\cap}$ on three Lie symmetry operators (Case 9 of Table 1).
III. Let $k = 2$. We choose a basis $\{(a_i, b_i, p_i, q_i, r_i),\ i = 1, 2\}$ of the space $\mathcal{V}$ of the tuples (a, b, p, q, r) associated with $A^{\max}$. The system on B is of the form

$$(a_1 u + b_1)B_u = p_1 B + q_1 u + r_1,$$
$$(a_2 u + b_2)B_u = p_2 B + q_2 u + r_2.$$

The determinant of the matrix $M = \left(\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right)$ is nonzero, since otherwise B is linear in u. Therefore, the system can be rewritten in the form

$$uB_u = p'_1 B + q'_1 u + r'_1,$$
$$B_u = p'_2 B + q'_2 u + r'_2,$$

where

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = M^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = M^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \begin{pmatrix} r_1' \\ r_2' \end{pmatrix} = M^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix};$$

M^{-1} is the inverse of the matrix M. Studying the latter system we get that it is compatible only if B is either linear or quadratic. The case B is quadratic in u is considered already in the course of the study of the case $B = u^m + \beta_1 u + \beta_2$. So, the case $k = 2$ does not lead to a new case of Lie symmetry extension for equations (1.1) with $B_{uu} \neq 0$.

The group classification problem for class (1.1) is solved exhaustively. The results are summarized in Table 1. It is important to note that group classification of subclass (2.6) is carried out up to the $\hat{G}^{\sim}_{\rm exp}$-equivalence, whereas all other cases are classified up to the usual $G^{\sim}$-equivalence.

4. Conclusion

In this paper we solve the group classification problem for the class of (1+1)-dimensional quasilinear diffusion equations with a variable coefficient nonlinear source (1.1) which arise as mathematical models in problems of mathematical biology and other applied areas [**18**]. The results of group classification can be applied for searching closed form solutions via the classical reduction method. The knowledge of Lie symmetries is also necessary for finding nonclassical symmetries (called also Q-conditional symmetries or reduction operators) of equations (1.1). This will be the subject of a forthcoming paper.

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