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# GEODESIC MAPPINGS ON COMPACT RIEMANNIAN MANIFOLDS WITH CONDITIONS ON SECTIONAL CURVATURE

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Abstract. We found new criteria for sectional curvatures on compact Riemannian manifolds for which geodesic mappings are affine, and, moreover, homothetic.

## 1. Introduction

To the theory of geodesic mappings and their transformations have been devoted many papers, these results are formulated in a large number of research papers and monographs [**2**, **4**–**12**, **16**–**19**, **21**–**26**, **30**, **33**], etc.

In 1953, Takeno and Ikeda [**31**] considered geodesic mappings of spherically symmetric spaces $V_4$, in 1954 Sinyukov [**26**, p. 88] studied the case of symmetric and recurrent spaces and, in 1976 Mikeš ( [**13**,**16**], [**21**, p. 206], [**26**, pp. 151–155]) proved that generalized recurrent (pseudo-) Riemannian spaces $V_n$ with nonconstant curvature do not admit nontrivial geodesic mappings. In this topic Prvanović [**23**] and Sobchuk [**20**, **29**] also have been interested. These results were obtained "locally" and they are contained in [**14**, **16**, **21**, **26**].

Global results for geodesic mappings of compact Riemannian manifolds were obtained by Vrançeanu [**33**], Sinyukova [**27**, **28**], Mikeš [**15**, **16**], etc.

The above results are related to questions of projective rigidity of (pseudo-) Riemannian manifolds and also of manifolds with affine connections.

In [**10**] and [**11**] we proved that these mappings preserve the smoothness class of metrics of geodesically equivalent (pseudo-) Riemannian manifolds. In [**10**] it was sufficient to suppose the metrics to be of differentiability class $C^2$, and in [**11**] to be of class $C^1$.

We present new results on geodesic mappings of compact Riemannian manifolds with certain conditions on the sectional curvature of the Ricci directions.

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## 2. Geodesic mapping theory

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be $n$-dimensional (pseudo-) Riemannian manifolds with metrics $g$ and $\bar{g}$, respectively.

DEFINITION 2.1. A diffeomorphism $f : V_n \to \bar{V}_n$ is called a *geodesic mapping* of $V_n$ onto $\bar{V}_n$ if $f$ maps any geodesic in $V_n$ onto a geodesic in $\bar{V}_n$.

We restricted ourselves to the study of a coordinate neighborhood $(U, x)$ of the points $x \in V_n$ and $f(x) \in \bar{V}_n$. The points $x$ and $f(x)$ have the same coordinates $x = (x^1, \dots, x^n)$. We assume that $V_n,\ \bar{V}_n \in C^1$ $(g, \bar{g} \in C^1)$ if their components $g_{ij}(x), \bar{g}_{ij}(x) \in C^1$ on $(U, x)$, respectively.

It is known [**12**], see [**6**, pp. 131–133], [**21**, p. 167], that $V_n$ admits a geodesic mapping onto $\bar{V}_n$ if and only if the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} \tag{2.1}$$

hold, where $\nabla$ is the Levi-Civita connection on $V_n$ and

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1} \ln \sqrt{|\det \bar{g} / \det g|}, \quad \partial_i = \partial / \partial x^i.$$

Sinyukov [**26**, p. 121], see [**21**, p. 167], proved that the Levi-Civita equations (2.1) are equivalent to

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{2.2}$$

where

$$\text{(a)}\ \ a_{ij} = e^{2\Psi}\, \bar{g}^{\alpha\beta} g_{i\alpha} g_{j\beta}; \quad \text{(b)}\ \ \lambda_i = -e^{2\Psi}\, \bar{g}^{\alpha\beta} \psi_\alpha g_{i\beta}, \tag{2.3}$$

and, moreover, $\lambda_i = \partial_i \Lambda$, $\Lambda = \frac{1}{2}\, a_{\alpha\beta} g^{\alpha\beta}$. Here $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ and $(g^{ij}) = (g_{ij})^{-1}$.

On the other hand:

$$\bar{g}_{ij} = e^{2\Psi} \hat{g}_{ij}, \quad \Psi = \ln \sqrt{|\det \hat{g} / \det g|}, \quad (\hat{g}_{ij}) = (a_{\alpha\beta} g^{i\alpha} g^{j\beta})^{-1}.$$

Furthermore, we assume that $V_n = (M, g) \in C^2$ and $\bar{V}_n = (M, \bar{g}) \in C^2$. In this case, the integrability conditions of the equations (2.2), due to the Ricci identity

$$\nabla_l \nabla_k a_{ij} - \nabla_k \nabla_l a_{ij} = a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl}, \tag{2.4}$$

have the following form

$$a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl} = g_{ik} \nabla_l \lambda_j + g_{jk} \nabla_l \lambda_i - g_{il} \nabla_k \lambda_j - g_{jl} \nabla_k \lambda_i, \tag{2.5}$$

where $R^h_{ijk}$ are components of the Riemannian tensor $R$ on $V_n$, and after contraction with $g^{ik}$ we get [**26**, p. 133]

$$n \nabla_l \lambda_j = \mu\, g_{jl} - a_{j\alpha} R^\alpha_l - a_{\alpha\beta} R_j{}^{\alpha\beta}{}_l, \tag{2.6}$$

where $\mu = \nabla_\alpha \lambda^\alpha$, $R^\alpha_i = g^{\alpha\beta} R_{\beta i}$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor Ric on $V_n$.

## 3. Integral formula

We introduce the vector field $\xi$ on $V_n \in C^2$ in the following way

$$\xi^i = a^\alpha_\beta \nabla_\alpha a^{i\beta} - a^i_\beta \nabla_\alpha a^{\alpha\beta}, \tag{3.1}$$

where $a^l_i = g^{l\alpha} a_{\alpha i}$, $a^{ij} = a_{\alpha\beta} g^{i\alpha} g^{j\beta}$. Using formula (3.1), the Ricci identity (2.4) and Sinyukov's equations (2.2) we obtained that the divergence of the vector $\xi$ has the following representation

$$\operatorname{div} \xi = \Phi(a) - (n-1)(n+2)\, \lambda_\alpha \lambda_\beta g^{\alpha\beta},$$

where $\Phi(a) = R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl}$.

Suppose that the Riemannian manifold $(M, g)$ is compact and without boundary, then on the basis of the Gauß theorem $\int_M \operatorname{div} \xi \, d\nu = 0$ we obtain the integral formula

$$\int_M \Phi(a)\, d\nu = (n-1)(n+2) \int_M \lambda_\alpha \lambda_\beta g^{\alpha\beta} d\nu. \tag{3.2}$$

For applying the Gauss theorem it is necessary to require the orientability of $M$, if $M$ is a non-orientable manifold, then we'll look at the oriented double cover.

Let $g(e_i, e_j) = \delta_{ij}$ and $a(e_i, e_j) = \alpha_i \delta_{ij}$ with the Kronecker symbol $\delta_{ij}$, i.e., $\{e_1, \dots, e_n\}$ is the orthonormal basis of eigenvectors to the eigenvalues $\alpha_1, \dots, \alpha_n$ of the tensor $a = (a_{ij})$ of $T_x M$ at any point $x \in M$. As we can see from direct calculation, $\Phi(a)$ has the following form (see [**3**, p. 592]):

$$\Phi(a) = \sum_{i<j} K(e_i, e_j)(\alpha_i - \alpha_j)^2, \tag{3.3}$$

where $K(e_i, e_j)$ are sectional curvatures in the two-directions $e_i \wedge e_j$.

It is easy to see:

$$\begin{aligned} \Phi(a) = R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl} &= \sum_{i,j} (\alpha_i)^2 R_{ijij} - \sum_{i,j} \alpha_i \alpha_j R_{ijij} \\ &= \sum_{i<j} ((\alpha_i)^2 + (\alpha_j)^2) \cdot R_{ijij} - 2 \sum_{i<j} \alpha_i \alpha_j R_{ijij} \\ &= \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot R_{ijij} = \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot K(e_i, e_j), \end{aligned}$$

where

$$K(e_i, e_j) = \frac{R(e_i, e_j, e_i, e_j)}{g(e_i, e_i) \cdot g(e_j, e_j) - (g(e_i, e_j))^2} = R_{ijij}.$$

## 4. Principal orthonormal basis

Eisenhart [**6**, pp. 113–114] introduced a *principal direction* in a Riemannian manifold $(M, g)$, as an eigenvector of the Ricci tensor. He showed that at any point $x \in M$ there exists the orthonormal basis $\{e_1, \dots, e_n\}$ in which

$$g_{ij} = \delta_{ij} \quad \text{and} \quad R_{ij} = \rho_i \delta_{ij},$$

i.e., $e_1, \dots, e_n$ are the vectors of the principal directions and $\rho_1, \dots, \rho_n$ are their eigenvalues. This basis is called the *principal orthonormal basis*.

This means that the existence of this basis is a property only of the Riemannian manifold $(M, g)$, independent of the solution $a_{ij}$ of equation (2.2). Generally the set of principal orthonormal bases is a proper subset of the set of orthonormal bases. Because the vector field $\lambda_i$ is gradient-like, formula (2.6) implies [**26**, p. 138]

$$a_{i\alpha} R^{\alpha}_{j} = a_{j\alpha} R^{\alpha}_{i}.$$

So the tensors $a_{ij}$ and $R_{ij}$ commute and have common eigenvectors. From this fact it follows that there exist a principal orthonormal basis in which $g_{ij} = \delta_{ij}$ and $a_{ij} = \alpha_i \delta_{ij}$ hold. This basis is called a *joint principal orthonormal basis*. Note that we do not restrict the signature of the Ricci tensor and the tensor $a_{ij}$. In the following we restrict ourselves to the study of formulas (3.2) and (3.3) on joint principal orthonormal bases.

## 5. Main Theorems

For the following we recall that a compact Riemannian manifold $V_n$ admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$.

If we assume that at each point $x \in M$ all sectional curvatures $K(e_i, e_j)$ are non-positive in the two-directions $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor, then from integral formula (3.2) it follows

$$\text{(5.1)} \qquad \text{(a)} \int_M \Phi(a)\, d\nu = 0 \quad \text{and} \quad \text{(b)} \int_M \lambda_\alpha \lambda_\beta g^{\alpha\beta}\, d\nu = 0.$$

From integral (5.1b) follows $\lambda_\alpha \lambda_\beta g^{\alpha\beta} = 0$ and this fact implies that $\lambda_i$ is vanishing on $M$, i.e., $\lambda_1 = \dots = \lambda_n = 0$. In this case, the geodesic mapping is *affine* (see [**21**, p. 150]). We proved the following theorem:

THEOREM 5.1. *Assume a compact Riemannian manifold $(M, g)$ without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive for any two-direction $e_i \wedge e_j$ from all the principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main direction of the Ricci tensor, then any geodesic mapping of $(M, g)$ is affine.*

Moreover, we suppose at each point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and that there is a certain point $x_0 \in M$ where the sectional curvature $K(e_i, e_j)$ in any two-direction $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor is negative. Then from integral (3.2) follows equation (5.1). On the basis of Theorem 5.1 it follows $\lambda_1 = \dots = \lambda_n = 0$ and the geodesic mapping is affine.

Further, from integral (5.1a) follows $\Phi(a) = 0$ on $M$. Then from formula (3.3) at the point $x_0 \in M$ we obtain $\alpha_1 = \dots = \alpha_n = \alpha$. Hence $a_{ij} = \alpha \delta_{ij}$, i.e., $a_{ij}(x_0) = \alpha g_{ij}(x_0)$.

In this case, the affine mapping is homothetic, i.e., $\bar{g} = \alpha' g$, where $\alpha' = \text{const}$. This fact follows from the uniqueness of solutions of the fundamental equations of affine mappings $V_n \to \bar{V}_n : \nabla_k \bar{g}_{ij} = 0$ with initial values $\bar{g}_{ij}(x_0) = \alpha' \, g_{ij}(x_0)$. This is equivalent to $a_{ij}(x_0) = \alpha g_{ij}(x_0)$, this fact follows from equation (2.3).

We proved the following theorem:

THEOREM 5.2. *Assume a compact Riemannian manifold $(M, g)$ without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and if there is a certain point $x_0 \in M$, where the sectional curvature $K(e_i, e_j)$ is negative in any two-direction $e_i \wedge e_j$ of all the principal orthonormal basis $\{e_1, \ldots, e_n\}$ of vectors of the main directions of the Ricci tensor, then any geodesic mapping of $(M, g)$ is homothetic.*

These Theorems generalize the results of Mikeš [**15**] (see [**16**]), which were obtained by means of modifications of integral inequalities obtained by Švec [**1**, p. 10].

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