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MAGNETIC CURVES IN A EUCLIDEAN SPACE: ONE EXAMPLE, SEVERAL APPROACHES

Marian Ioan Munteanu

Abstract. This is a short review of different approaches in the study of magnetic curves for a certain magnetic field and on the fixed energy level. We emphasize them in the case when the magnetic trajectory corresponds to a Killing vector field associated to a screw motion in the Euclidean 3-space.

1. Introduction

The geodesic flow on a Riemannian manifold represents the extremals of the least action principle, namely it is determined by the motion of a certain physical system on the manifold. It is known that the geodesic equations are second order non-linear differential equations and they usually appear in the form of Euler-Lagrange equations of motion. Magnetic curves generalize geodesics. In physics, such a curve represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field.

Let (M, g) be an n-dimensional Riemannian manifold. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* of the magnetic field F on (M, g) is a $(1, 1)$ tensor field Φ given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{1.1}$$

The *magnetic trajectories* of F are curves γ on M that satisfy the *Lorentz equation* (sometimes called the *Newton equation*)

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'). \tag{1.2}$$

The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\gamma'}\gamma' = 0$. Therefore, from the point of view of the dynamical systems, a geodesic corresponds to a trajectory of a particle without an action of a magnetic field, while a magnetic trajectory is *a flowline of the dynamical system*, associated to the magnetic field. In contrast to geodesics, magnetic curves are not reversible and they cannot be rescaled, that is the trajectories depend on the energy $\|\gamma'\|$.

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The Lorentz force is skew symmetric and therefore the magnetic curves have a constant speed (and hence energy) $v(t) = \|\gamma'\| = v_0$. When they are parametrized by arc length ($v_0 = 1$), we use to call them *normal magnetic curves*.

An example of magnetic fields on a Riemannian surface can be obtained by multiplying the area element by a (smooth) function (usually called *strength*). When the surface is of constant Gaussian curvature K, trajectories of such magnetic fields are well known. More precisely, on the spheres, these trajectories are certain small circles, on the Euclidean plane they are arbitrary circles, while, on a hyperbolic plane, the trajectories can be either closed, or open curves (depending on the ratio of the strength and K) (see e.g., [**12, 18**]).

In the case of a 3-dimensional Riemannian manifold (M, g), 2-forms and vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of the manifold. Thus, magnetic fields mean divergence free vector fields (see e.g. [**11**]). In particular, Killing vector fields define an important class of magnetic fields, called *Killing magnetic fields*. Recall that a vector field V on M is *Killing* if and only if it satisfies the Killing equation:

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0 \tag{1.3}$$

for every vector fields Y, Z on M, where ∇ is the Levi Civita connection on M.

It is known that geodesics can be defined as extremal curves for the action or energy functional. A variational approach to describe Killing magnetic flows in spaces of constant curvature is given in [**8**].

Note that, one can define on M the *cross product* of two vector fields $X, Y \in \mathfrak{X}(M)$ as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \mathfrak{X}(M).$$

If V is a Killing vector field on M, let $F_V = \iota_V dv_g$ be the corresponding Killing magnetic field. By ι we denote the inner product. Then, the Lorentz force of F_V is (see [**11**])

$$\Phi(X) = V \times X.$$

Consequently, the Lorentz force equation (1.2) can be written as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{1.4}$$

If we consider the 3-dimensional Euclidian space $\mathbb{E}^3$ endowed with the usual scalar product $\langle\ ,\ \rangle$ we know the fundamental solutions of (1.3):

$$\{\partial_x, \partial_y, \partial_z, -y\partial_x + x\partial_y, -z\partial_y + y\partial_z, z\partial_x - x\partial_z\}$$

and they give a basis of Killing vector fields on $\mathbb{E}^3$. Here x, y, z denote the global coordinates on $\mathbb{E}^3$ and $\mathbb{R}^3 = \operatorname{span}\{\partial_x, \partial_y, \partial_z\}$ is regarded as a vector space. The easiest example is to consider the Killing vector field $\xi_0 = \partial_z$. (Similar discussions can be made for ∂_x and ∂_y, respectively.) Its magnetic trajectories are helices with axis ∂_z, namely $t \mapsto (x_0 + a\cos t, y_0 + a\sin t, z_0 + bt)$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $a, b \in \mathbb{R}$. An interesting fact is that Lancret curves (i.e. general helices) in $\mathbb{E}^3$ are characterized by the following property (in our framework): they are magnetic trajectories associated with magnetic fields parallel to their axis. A similar result, relating Killing magnetic fields and Lancret curves is provided on the 3-sphere (see e.g. [**8**]).

Magnetic curves determined by the Killing vector field $V = -y\partial_x + x\partial_y$ were classified (in terms of cylindrical coordinates ρ, ϕ, z) in [**13**] as follows:

THEOREM 1.1. *The normal magnetic trajectories of the Killing magnetic field F_V are: planar curves situated in a vertical strip, circular helices and curves parametrized by*

$$x(t) = \rho(t)\cos\phi(t),\ \ y(t) = \rho(t)\sin\phi(t),\ \ z(t) = -\frac{1}{2}\int^{t} \rho^2(\zeta)d\zeta,$$

where ρ and ϕ satisfy

$$\left(\frac{d\rho^2}{dt}\right)^2 + \mathcal{P}\big(\rho^2(t)\big) = 0, \quad \rho^2(t)\phi'(t) = \text{constant}$$

and $\mathcal{P}$ is a polynomial of degree 3*.*

In the last case, explicit solutions were obtained using elliptic integrals. This aspect is very important since the trajectories may be represented by using numerical approximations of the integrals involved.

The problem of studying magnetic curves was considered also for other ambient spaces. For example, Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ were classified in [**16**], and magnetic curves corresponding to translation Killing vector fields in $\mathbb{E}^3_1$ were described in [**14**].

If the ambient is a complex space form (of arbitrary dimension), Kähler magnetic fields are studied (see [**3**]); in particular, explicit trajectories for Kähler magnetic fields are found in the complex projective space $\mathbb{CP}^n$ (see [**2**]). On the other hand, if the ambient is a contact manifold, the fundamental 2-form defines the so-called *contact magnetic field*. In particular, when the manifold is Sasakian, it is proved that the angle between the velocity of a normal magnetic curve and the Reeb vector field is constant (see [**10**]). Moreover, explicit description for normal flowlines of the contact magnetic field on a 3-dimensional Sasakian manifold is given.

In this note we consider a Killing vector field associated to a screw motion in the Euclidean space and we present different approaches for studying Killing magnetic curves corresponding to it. In Section 2 we give a variational approach by considering a potential 1-form which determines the magnetic field. In Section 3 it is shown how magnetic curves can be found explicitly; we use similar techniques as in [**13**] and we point out the main differences. In Section 4 we sketch another approach related to dynamical systems; more precisely, the cotangent bundle of the manifold is considered as the phase space, namely the set of all possible values of position and momentum variables.

2. A variational approach

Let $M = \mathbb{E}^3 \setminus Ox$ and let x, y, z be the global coordinates on M. Consider on M the Killing vector field $V = a\partial_x - z\partial_y + y\partial_z$, $a \in (0, +\infty)$, whose integral curves are helices

$$\mathcal{H}: \quad x = x_0 + at, \quad y = \rho_0 \cos(t + t_0), \quad z = \rho_0 \sin(t + t_0) \tag{2.1}$$

The notations ∂_x, ∂_y and ∂_z have the usual meaning, and x_0, $\rho_0 > 0$ and t_0 are constants.

Let

$$\omega = -\tfrac{1}{3}\left(y^2 + z^2\right) dx + \left(\tfrac{1}{3}\ xy - \tfrac{1}{2}az\right) dy + \left(\tfrac{1}{3}\ xz + \tfrac{1}{2}ay\right) dz \tag{2.2}$$

be a 1-form on M having the property that $\omega(V)$ is constant on $\mathcal{H}$. Consider also the 2-form F on M defined by $F(X, Y) = \langle V \times X, Y\rangle$ for any X, Y tangent to M. Here $\times$ denotes the usual cross product on $\mathbb{R}^3$.

Notice that $F = d\omega$. The 2-form F is the magnetic field corresponding to V and hence ω is a potential 1-form. For a curve $\gamma : [t_0, t_1] \longrightarrow M$ consider the functional

$$LH(\gamma) = \int_{t_0}^{t_1} \left(\tfrac{1}{2}\langle \gamma'(t), \gamma'(t)\rangle - \omega(\gamma'(t))\right) dt. \tag{2.3}$$

It is sometimes called the *Landau–Hall* functional for the curve γ with the potential 1-form ω.

Consider now a variation of γ, namely let $\gamma_\epsilon : [t_0, t_1] \to M$, $\gamma_\epsilon(t) = \gamma(t) + \epsilon\eta(t)$, where $\eta : [t_0, t_1] \to M$ is the variation vector on γ, that is $\eta(t_0) = \eta(t_1) = 0$. In order to find the critical points of the functional LH, we have to compute $\frac{d}{d\epsilon} LH(\gamma_\epsilon)_{|\epsilon=0}$. If we put $\eta = (u, v, w)$, we have

$$\begin{aligned}
\frac{d}{d\epsilon} LH(\gamma_\epsilon)|_{\epsilon=0} = \int_{t_0}^{t_1} \Big\{ & x'u' + y'v' + z'w' + \frac{2}{3}(yv + zw)x' + \frac{1}{3}(y^2 + z^2)u' \\
& - \left[\frac{1}{3}(xv + yu) - \frac{aw}{2}\right] y' - \left[\frac{1}{3}xy - \frac{az}{2}\right] v' \\
& - \left[\frac{1}{3}(xw + zu) + \frac{av}{2}\right] z' - \left[\frac{1}{3}xz + \frac{ay}{2}\right] w' \Big\}\, dt \\
= \int_{t_0}^{t_1} & \left[\Big(x' + \frac{1}{3}(y^2 + z^2)\Big)u' + \Big(y' - \frac{1}{3}xy + \frac{az}{2}\Big)v' + \Big(z' - \frac{1}{3}xz - \frac{ay}{2}\Big)w'\right] dt \\
+ \int_{t_0}^{t_1} & \left[\Big(-\frac{yy'}{3} - \frac{zz'}{3}\Big)u + \Big(\frac{2}{3}x'y - \frac{1}{3}xy' - \frac{az'}{2}\Big)v + \Big(\frac{2}{3}x'z - \frac{1}{3}xz' + \frac{ay'}{2}\Big)w\right] dt.
\end{aligned}$$

Computing the first integral by parts, and taking into account that η is the variation vector, we obtain

$$\begin{aligned}\frac{d}{d\epsilon}LH(\gamma_\epsilon)|_{\epsilon=0} = &\int_{t_0}^{t_1}\left\{-\frac{d}{dt}\Big(x'+\frac{1}{3}(y^2+z^2)\Big)-\frac{yy'}{3}-\frac{zz'}{3}\right\}u\,dt\\ &+\int_{t_0}^{t_1}\left\{-\frac{d}{dt}\Big(y'-\frac{1}{3}xy+\frac{az}{2}\Big)+\frac{2}{3}x'y-\frac{1}{3}xy'-\frac{az'}{2}\right\}v\,dt\\ &+\int_{t_0}^{t_1}\left\{-\frac{d}{dt}\Big(z'-\frac{1}{3}xz-\frac{ay}{2}\Big)+\frac{2}{3}x'z-\frac{1}{3}xz'+\frac{ay'}{2}\right\}w\,dt.\end{aligned}$$

Since the variation vector η is arbitrary, the equation $\frac{d}{d\epsilon}LH(\gamma_\epsilon)_{|\epsilon=0}=0$ becomes

$$\begin{aligned}&x''+yy'+zz'=0,\\ &y''-x'y+az'=0,\\ &z''-x'z-ay'=0.\end{aligned}\tag{2.4}$$

This system of ordinary differential equations is nothing but the Lorentz equation (1.4), corresponding to the magnetic field F.

3. Direct approach

In this section we solve the Lorentz equation $\gamma''=V\times\gamma'$ to obtain the normal magnetic trajectories corresponding to V. In this approach, we study the differential equations system (2.4) in order to find explicit solutions. Similar techniques were used in [**13**] and [**16**], therefore here we only sketch the computations.

Let γ be parametrized by arc length and satisfying (2.4). The first equation yields

$$2\dot{x}+y^2+z^2=c_1,\quad c_1\in\mathbb{R}.$$

In what follows, we denote by $\dot{}$ the derivative with respect to the arc length parameter s. Combining the second and the third equations we get

$$y\dot{z}-\dot{y}z=\frac{a}{2}(y^2+z^2)+c_2,\quad c_2\in\mathbb{R}.$$

Let us consider cylindrical coordinates (x,ρ,ϕ) on M, that is $y=\rho\cos\phi$ and $z=\rho\sin\phi$ with $\rho>0$. We have

$$\begin{aligned}&\dot{x}^2+\dot{\rho}^2+\rho^2\dot{\phi}^2=1,\\ &2\dot{x}+\rho^2=c_1,\\ &\rho^2\dot{\phi}=\frac{a}{2}\,\rho^2+c_2.\end{aligned}\tag{3.1}$$

Denote $\mu=\rho^2$ which is a strictly positive function. From (3.1) we immediately obtain

$$\dot{\mu}^2+\mu^3+\big(a^2-2c_1\big)\,\mu^2+\big(c_1^2+4ac_2-4\big)\,\mu+4c_2^2=0.\tag{3.2}$$

This is an ordinary differential equation of the type $\dot{\mu}^2 + \mathcal{P}(\mu) = 0$, where $\mathcal{P}$ is a polynomial of degree 3. This equation has an obvious solution, that is $\mu = \alpha$, where α is a root of the polynomial $\mathcal{P}$. It follows (from (3.1)) that ϕ and x are affine functions and hence γ is a cylindrical helix with the axis Ox.

Let us show how to find a non-constant solution for (3.2). If we denote by Δ the discriminant of $\mathcal{P}$, the following situations appear:

- the equation $\mathcal{P} = 0$ has three distinct solutions iff $\Delta > 0$;
- the polynomial $\mathcal{P}$ has multiple roots iff $\Delta = 0$;
- the polynomial $\mathcal{P}$ has one real root and two complex conjugate roots iff $\Delta < 0$.

A detailed analysis of the above situations, leads us to conclude, after taking into account Viète's classical formulas, that equation (3.2) has solutions if and only if $\Delta \geqslant 0$.

To be more precise, if $\Delta < 0$, let $\alpha \in \mathbb{R}$ be the real root and $\beta, \bar{\beta} \in \mathbb{C} \setminus \mathbb{R}$ be the complex roots of $\mathcal{P}$. Then, ODE (3.2) can be rewritten as

$$\dot{\mu}^2 + \left(\mu^2 - 2\ Re(\beta)\ \mu + |\beta|^2\right)(\mu - \alpha) = 0.$$

where $Re(\beta)$ denotes the real part of the complex number β. From Viète's third formula, for $c_2 \neq 0$, we conclude that α should be negative, and consequently, ODE (3.2) has no solution. The case $c_2 = 0$ will be discussed separately.

If $\Delta = 0$, one can have

- either a triple root α, when
$$(a^2 - 2c_1)^2 = 3(c_1^2 + 4ac_2 - 4) \quad \text{and} \quad (a^2 - 2c_1)^3 = 108c_2^2;$$
- or one simple root $\alpha \in \mathbb{R}$ and one double root $\beta \in \mathbb{R}$.

Notice that, in contrast to [**13**], the first case should be discussed here since the polynomial could have a triple root; for example when $a = 19/4$, $c_1 = -157/16$ and $c_2 = 3375/128$. Equation (3.2) reads $\dot{\mu}^2 + (\mu - \alpha)^3 = 0$. In order to have a solution we should have $\mu \leqslant \alpha$. Hence α is positive. On the other hand, $\alpha^3 = -4c_2^2 < 0$ and this contradicts to $\alpha > 0$. In the second situation, equation (3.2) can be written as $\dot{\mu}^2 + (\mu - \alpha)(\mu - \beta)^2 = 0$. As before, no solution can be obtained.

Finally, if $\Delta > 0$, let $\alpha, \beta, \lambda \in \mathbb{R}$ be the three distinct roots of $\mathcal{P}$. Viète's third formula yields $\alpha\beta\lambda < 0$, and hence

- either α, β, λ are all negative,
- or two of them, say α and β, are positive and the third one, λ, is negative.

With a similar argument as before, the first situation cannot occur. In the second case, equation (3.2) reads $\dot{\mu}^2 + (\mu - \alpha)(\mu - \beta)(\mu - \lambda) = 0$, and it has a solution in the interval defined by the two positive roots, namely $\mu(s) = \mathcal{J}(s)$, where $\mathcal{J}$ is the inverse function of $\mathcal{I}(\mu) = \int^{\mu} \left((\xi - \alpha)(\beta - \xi)(\xi - \lambda)\right)^{-1/2} d\xi$. Thus $\rho(s) = \sqrt{\mathcal{J}(s)}$. Moreover, it can be expressed also in terms of elliptic functions. See e.g., [**13**]. Then, from the third equation of (3.1), by integration, we get ϕ. Hence we have obtained the coordinates y and z. The third coordinate x can be obtained, also by integration, from the second equation of (3.1). Therefore, the curve γ is completely determined.

Let us study the remained case $c_2 = 0$. We immediately get $\phi = \frac{as}{2} + \phi_0$ and hence, contrary to [**13**], our curve is no longer a planar curve. Then, we write the ODE in the form

$$\dot{\mu}^2 + \mu\left[\mu^2 + \left(a^2 - 2c_1\right)\mu + \left(c_1^2 - 4\right)\right] = 0.$$

If $c_1 \geqslant \frac{a^2}{4} + \frac{4}{a^2}$, then it has no solution. The case $c_1 < \frac{a^2}{4} + \frac{4}{a^2}$ is rather richer:

- if $|c_1| < 2$, there exist $\alpha < 0 < \beta$ such that $\mathcal{P} = \mu(\mu - \alpha)(\mu - \beta)$ and we get a solution inside the cylinder $y^2 + z^2 = \beta$, namely $\mu(s) = \mathcal{J}(s)$, where $\mathcal{J}$ is the inverse function of $\mathcal{I}(\mu) = \int^{\mu} \left(\xi(\xi - \alpha)(\beta - \xi)\right)^{-1/2} d\xi$;
- if $c_1 = -2$, the equation becomes $\dot{\mu}^2 + \mu^2(\mu + a^2 + 4) = 0$ and it has no solution;
- if $c_1 = 2$, the equation has a solution only when $|a| < 2$, and this is $\mu(s) = \frac{2A}{1+\cosh(\sqrt{A}(s-s_0))}$, where $A = 4 - a^2$ and s_0 depends on the initial conditions; for example, when $a = 1$, $s_0 = 0$ and $\phi_0 = 0$ we obtain
$$\rho = \frac{\sqrt{3}}{\cosh \frac{\sqrt{3}s}{2}}\ , \quad \phi = \frac{s}{2}\ , \quad x = s - \sqrt{3}\tanh\frac{\sqrt{3}s}{2};$$
- if $c_1 < -2$ the polynomial $\mathcal{P}$ has two negative roots, and hence the ODE has no solution;
- if $c_1 > 2$ the ODE has a solution only if $|a| < 2$, the case in which it is situated between two cylinders $\mu = \alpha$ and $\mu = \beta$, where $\alpha < \beta$ are the two positive roots of $\mathcal{P}$; the solution can be computed as in the case $|c_1| < 2$.

4. Hamiltonian approach

Let M be as in the previous sections and let $T^*M = M \times \mathbb{R}^3$ be its cotangent bundle. Denote by (ζ, p, q) the coordinates in the fiber $T^*_{(x,y,z)}M$. Hence, the canonical projection may be written as

$$\pi : M \times \mathbb{R}^3 \longrightarrow M, \quad (x, y, z; \zeta, p, q) \mapsto (x, y, z).$$

The 2-form

$$\Omega = d\zeta \wedge dx + dp \wedge dy + dq \wedge dz$$

is known as the canonical symplectic form on T^*M.

Consider the 2-form (on T^*M)

$$\Omega_F = \Omega - \pi^* F,$$

which defines also a symplectic structure on T^*M. This represents a deformation of the canonical form corresponding to the presence of the magnetic field F.

It is known that the geodesic flow can be described as the Hamiltonian flow of H, namely

$$\frac{d}{dt}(x, y, z) = \left(\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right) H, \quad \frac{d}{dt}(\zeta, p, q) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H, \tag{4.1}$$

where $H : T^*M \to \mathbb{R}$, $H(x, y, z; \zeta, p, q) = \frac{1}{2}\left(\zeta^2 + p^2 + q^2\right)$. These equations represent the motion of a particle under the action of gravity and they were written for an arbitrary Hamiltonian on M. For further reading on the Hamiltonian formulation see e.g., [**1, 17**].

System (4.1) can be expressed also in terms of the canonical Poisson bracket on $(M\times\mathbb{R}^3,\Omega)$

$$\{f,g\}=\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial \zeta}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial p}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial q}\right)-\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial \zeta}+\frac{\partial g}{\partial y}\frac{\partial f}{\partial p}+\frac{\partial g}{\partial z}\frac{\partial f}{\partial q}\right)$$

as follows

$$\frac{df}{dt}=\{f,H\},$$

which shows the evolution of an arbitrary function along the flow.

When the symplectic form Ω_F is considered, the corresponding Poisson bracket becomes (see e.g., [**15**])

$$\{f,g\}_F=\{f,g\}-y\left(\frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial \zeta}\frac{\partial f}{\partial p}\right)-a\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\right)+z\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial \zeta}-\frac{\partial g}{\partial q}\frac{\partial f}{\partial \zeta}\right).$$

We compute

$$\{x,H\}_F=\zeta,\;\;\{y,H\}_F=p,\;\;\{z,H\}_F=q,$$
$$\{\zeta,H\}_F=-yp-zq,\;\;\{p,H\}_F=y\zeta-aq,\;\;\{q,H\}_F=z\zeta+ap.$$

Then, the resulting Hamiltonian system $\frac{df}{dt}=\{f,H\}_F$ becomes

$$\begin{gathered}x'=\zeta,\;\;y'=p,\;\;z'=q,\\ \zeta'=-yp-zq,\;\;p'=y\zeta-aq,\;\;q'=z\zeta+ap,\end{gathered}\tag{4.2}$$

and (sometimes) it is called *magnetic geodesic flow* defined by F. This is a first order nonlinear differential equation system which represents the integral curve of the vector $\widetilde{V}=(\zeta,p,q,-yp-zq,y\zeta-aq,z\zeta+ap)$ on $M\times\mathbb{R}^3$, sometimes called *the Hamiltonian vector field* associated to H and Ω_F.

As we have already said in the Introduction, the trajectories of a magnetic field have a constant energy (constant speed). Moreover, unlike geodesics, a rescaling of a magnetic curve is no longer a magnetic curve. Therefore, we usually restrict the study to normal magnetic curves, namely parametrized by arc-length, which corresponds, from the mechanical point of view, to a restriction to a single level of energy. If we do this, the Hamiltonian is constant $\frac{1}{2}$. Therefore, we can parametrize the fibers of energy level as

$$p=\cos u\cos v,\quad q=\cos u\sin v,\quad \zeta=\sin u.$$

If ρ and ϕ are as in Section 3, system (4.2) reads

$$x'=\sin u,\;\;\rho'=\cos u\cos(\phi-v),\;\;\phi'=-\frac{1}{\rho}\cos u\sin(\phi-v),$$
$$u'+\rho\cos(\phi-v)=0,\;\;v'=a+\rho\tan u\sin(\phi-v).$$

Denote $\psi = \phi - v$. We get

$$\psi' = -\frac{1}{\rho}\ \cos u \sin\psi - a - \rho\tan u \sin\psi$$
$$u' + \rho\cos\psi = 0$$
$$\rho' - \cos u\cos\psi = 0.$$

The second and the third equations yield

$$\rho^2 + 2\sin u = \text{constant}$$

and this is precisely the constant c_1 from Section 3.

On the other hand, we have $(yq - pz)' = a\rho\cos u\cos\psi = a\rho\rho'$. Hence,

$$\rho\cos u\sin\psi + \frac{a}{2}\ \rho^2 = \text{constant}$$

and it is exactly $-c_2$ (from Section 3). We obtained the two first integrals as when we used the direct approach. Therefore, if one needs to find explicit expressions for the magnetic curve, the computations follow as in Section 3.

Notice that several conditions on the constants c_1 and c_2 may be obtained immediately in this approach, for example $c_1 > -2$.

5. Final remarks

Let $\{T = \gamma', N, B\}$ be the Frenet frame of a unit speed curve γ in M. The Frenet equations may be used to characterize when γ belongs to the magnetic flow associated to V. First of all, consider the *quasi-slope* of γ with respect to V, measured as $\alpha(s) = \langle V(s), \gamma'(s)\rangle$, where $V(s)$ is the restriction of V to γ, namely $V(s)$ is the value of V at the point $\gamma(s)$.

One can prove (see [**8**]) that the unit speed curve γ is a magnetic trajectory of V if and only if

$$V(s) = \alpha(s)\,T(s) + \kappa(s)\,B(s),$$

where κ is the curvature function of γ. Moreover, when V is Killing, then its magnetic curves have constant quasi-slope. Furthermore, the curvature and the torsion of γ satisfy some equations (see also [**8**]) which represent the field equations associated with the Kirchhoff elastic rod.

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Department of Mathematics
University Alexandru Ioan Cuza Iasi
700506
Romania
marian.ioan.munteanu@gmail.com