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ON THE CLASSIFICATION OF LORENTZIAN SASAKI SPACE FORMS

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Abstract. Sasaki manifolds admit a nowhere vanishing vector field and it is always possible to consider a Lorentz metric on them. Then we are able to obtain a classification result for compact Lorentz–Sasaki space forms.

1. Introduction

Sasaki manifolds, as all parts of contact geometry, were studied by several authors for their large interest in mechanics and physics [**1**,**6**]. Anyway, these special contact manifolds have been studied even in the Lorentz context for their properties [**4**] and physical applications. Although the Lorentz manifolds are the main topic in physics, there may be some obstructions to the existence of a Lorentz metric on a manifold. A condition under which it is possible to construct a Lorentz metric from a Riemannian one is the existence of a globally defined nowhere vanishing vector field. Such a condition is clearly verified by a Sasaki manifold hence using O'Neill's construction we realize a Lorentz metric on Sasaki manifold and we study its properties.

Furthermore, using Tanno's classification [**6**] we obtain a model for a compact Lorentz–Sasaki manifold with constant φ-sectional curvature.

All manifolds, tensor fields and maps are assumed to be smooth, and all manifolds are supposed to be connected. We shall use the Einstein convention, omitting the sum symbol for repeated indexes. Moreover, we shall use the symbol $\mathfrak{X}(M)$ to denote the Lie algebra of vector fields on a manifold M.

2. Preliminaries

An almost contact manifold is an odd-dimensional smooth manifold M endowed with a tensor field φ of endomorphisms of the tangent spaces, a vector field ξ and a 1-form η satisfying $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. In this case such a manifold will be denoted by (M, φ, ξ, η) [**1**].

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A semi-Riemannian metric g on an almost contact manifold (M,φ,ξ,η) is said to be compatible with the almost contact structure (φ,ξ,η) if, for any $X,Y\in\mathfrak{X}(M)$,

$$g(\varphi X,\varphi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y), \tag{2.1}$$

where $\varepsilon=\pm1$ according to the casual character of the Reeb vector field ξ. An almost contact manifold (M,φ,ξ,η) with a compatible semi-Riemannian metric g is called an indefinite almost contact metric manifold and denoted by (M,φ,ξ,η,g). We recall that TM orthogonally decomposes as $\mathfrak{D}\oplus\mathrm{span}(\xi)$, where $\mathfrak{D}=\mathrm{Im}\,\varphi=\ker\eta$. From (2.1) it follows that the bilinear form $\Phi:=g(\cdot,\varphi\cdot)$ is a 2-form, called the Sasaki 2-form of the almost contact metric manifold. An indefinite almost contact metric manifold (M,φ,ξ,η,g) is said to be an indefinite contact metric manifold if $d\eta(X,Y)=\Phi(X,Y)$, for any $X,Y\in\mathfrak{X}(M)$. If in addition the structure is normal, i.e., $N=[\varphi,\varphi]+2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, then (M,φ,ξ,η,g) is said to be an indefinite Sasaki manifold [**4**]. An easy computation shows that in such a manifold the Reeb characteristic vector field is Killing.

In the context of contact metric manifolds we consider a $\mathfrak{D}$-homothetic deformation, that is a change of the structure tensors (φ,ξ,η,g) as follows

$$\varphi':=\varphi,\quad \xi':=\frac{1}{\alpha}\xi,\quad \eta':=\alpha\eta,\quad g':=\alpha g+\alpha(\alpha-1)\eta\otimes\eta$$

where $\alpha>0$. This notion was introduced by Tanno [**5**] in the contact metric case, but it can be easily extended to the more general context of almost contact metric structures. In particular, it can be proved that the class of Sasaki structures is preserved by $\mathfrak{D}$-homothetic deformations.

3. Lorentz–Sasaki manifolds

By contrast with the Riemannian case, not every smooth manifold can admit a Lorentz structure, in fact, this is possible if and only if there exists a nowhere vanishing vector field [**3**, page 149]. Moreover, the following well-known result states how to obtain such a metric.

Proposition 3.1. *Let (M,g) be a Riemannian manifold, U a unit vector field and U^* its dual 1-form. Then $\tilde g=g-2U^*\otimes U^*$ is a Lorentz metric on M. Furthermore, U becomes timelike so the resulting Lorentz manifold is time-orientable.*

Now, we consider a Sasaki manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ and we obtain a Lorentz metric putting

$$\tilde g=g-2\eta\otimes\eta. \tag{3.1}$$

Proposition 3.2. *The manifold $(M^{2n+1},\varphi,\xi,\eta,\tilde g)$ is a Lorentz–Sasaki manifold.*

Proof. Obviously, ξ is timelike with respect to the metric $\tilde g$, whereas, for any $X,Y\in\ker\eta$, $\tilde g(X,Y)=g(X,Y)$, therefore $\tilde g$ has index $\nu=1$. Denoting by $\tilde\eta$ the $\tilde g$-dual 1-form of ξ, one has $\tilde\eta(X)=\varepsilon\tilde g(X,\xi)=-(g(X,\xi)-2\eta(X)\eta(\xi))=\eta(X)$, for any $X \in \mathfrak{X}(M)$ and so $\tilde{\eta} = \eta$. Clearly $\varphi^2(X) = -X + \eta(X)\xi$, for any $X \in \mathfrak{X}(M)$. Now we check the compatibility of $\tilde{g}$ with the structure: for any $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned}\tilde{g}(\varphi X, \varphi Y) &= g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\\ &= g(X, Y) - 2\eta(X)\eta(Y) + \eta(X)\eta(Y) = \tilde{g}(X, Y) - \varepsilon\eta(X)\eta(Y),\end{aligned}$$

being $\varepsilon = \tilde{g}(\xi, \xi) = -1$. Finally, the normality condition holds since it does not depend on the metric. Furthermore, being $\tilde{\Phi}(X, Y) = \Phi(X, Y)$ for any vector fields X, Y, we have $\tilde{\Phi} = d\tilde{\eta}$ and we obtain a Lorentz–Sasaki structure on M. □

From now on, such a Lorentz–Sasaki manifold is said to be the *associated Lorentz–Sasaki manifold*.

Let $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ be a Lorentz–Sasaki manifold, considering the transformation

$$\bar{g} = \tilde{g} + 2\eta \otimes \eta, \tag{3.2}$$

one obtains that $(M^{2n+1}, \varphi, \xi, \eta, \bar{g})$ turns out to be a Sasaki manifold.

To compare the Levi-Civita connections ∇ and $\tilde{\nabla}$ with respect to the Riemannian metric g and the Lorentz one $\tilde{g}$, we prove the following proposition.

Proposition 3.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasaki manifold and consider $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ the associated Lorentz–Sasaki manifold. Then, the Levi-Civita connections ∇ and $\tilde{\nabla}$ are related by $\tilde{\nabla}_X Y = \nabla_X Y + 2\eta(Y)\varphi X + 2\eta(X)\varphi Y$.*

Proof. By Koszul's formula for $\tilde{\nabla}$ we have

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X(\tilde{g}(Y, Z)) + Y(\tilde{g}(Z, X)) - Z(\tilde{g}(X, Y))\\ &\quad + \tilde{g}(Z, [X, Y]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(X, [Z, Y]),\end{aligned}$$

and, applying the definition of $\tilde{g}$, the above formula becomes

$$2(g(\tilde{\nabla}_X Y, Z) - 2\eta(\tilde{\nabla}_X Y)\eta(Z)) = 2g(\nabla_X Y, Z) + 2H'(X, Y, Z), \tag{3.3}$$

where

$$\begin{aligned}H'(X, Y, Z) &= \eta(Z)(-X(\eta(Y)) - Y(\eta(X)) - \eta[X, Y])\\ &\quad + \eta(Y)(-X(\eta(Z)) + Z(\eta(X)) - \eta[Z, X])\\ &\quad + \eta(X)(-Y(\eta(Z)) + Z(\eta(Y)) - \eta[Z, Y]).\end{aligned}$$

It is easy to check that

$$H'(X, Y, Z) = \eta(Z)(2d\eta(X, Y) - 2X(\eta(Y))) + 2\eta(Y)d\eta(Z, X) + 2\eta(X)d\eta(Z, Y),$$

then, replacing $H'(X, Y, Z)$ in (3.3), using $\eta(\tilde{\nabla}_X Y) = X(\eta(Y)) - d\eta(X, Y)$, we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + 2\eta(Y)d\eta(Z, X) + 2\eta(X)d\eta(Z, Y), \tag{3.4}$$

Then, (3.4) gives $g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + 2\eta(Y)g(Z, \varphi X) + 2\eta(X)g(Z, \varphi Y)$, which completes the proof. □

As a consequence of the relation between the Levi-Civita connections, we find the relation between the φ-sectional curvature of $(M, \varphi, \xi, \eta, \tilde{g})$ and $(M, \varphi, \xi, \eta, g)$, denoted by $\tilde{K}(X, \varphi X)$ and $K(X, \varphi X)$ respectively, for any $X \in \mathrm{Im}(\varphi)$.

PROPOSITION 3.4. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasaki manifold and consider the Lorentz–Sasaki manifold $(M^{2n+1},\varphi,\xi,\eta,\tilde g)$. Then the φ-sectional curvatures are related by $\tilde K(X,\varphi X)=K(X,\varphi X)+6$.*

PROOF. By the above proposition we know that

$$\tilde\nabla_X Y=\nabla_X Y+2\eta(Y)\varphi X+2\eta(X)\varphi Y, \tag{3.5}$$

which implies that, for any $X,Y\in\ker\eta$, $\tilde\nabla_X Y=\nabla_X Y$. For $X\in\ker\eta$ such that $1=\|X\|_g=\|X\|_{\tilde g}$, we prove that

$$\tilde R_{X\varphi X}\varphi X=R_{X\varphi X}\varphi X+6X, \tag{3.6}$$

and this yields

$$\begin{aligned}\tilde K(X,\varphi X)&=\tilde R(X,\varphi X,X,\varphi X)=\tilde g(X,\tilde R_{X\varphi X}\varphi X)\\&=g(X,\tilde R_{X\varphi X}\varphi X)=g(X,R_{X\varphi X}\varphi X)+6=K(X,\varphi X)+6.\end{aligned}$$

To show (3.6), using (3.5) we compute

$$\begin{aligned}\tilde\nabla_X\tilde\nabla_{\varphi X}\varphi X&=\tilde\nabla_X\nabla_{\varphi X}\varphi X=\nabla_X\nabla_{\varphi X}\varphi X+2\eta(\nabla_{\varphi X}\varphi X)\varphi X\\&=\nabla_X\nabla_{\varphi X}\varphi X,\end{aligned}\tag{3.7}$$

since $\eta(\nabla_{\varphi X}\varphi X)=g(\nabla_{\varphi X}\varphi X,\xi)=-g(\varphi X,\nabla_{\varphi X}\xi)=g(\varphi X,\varphi^2X)=0$.
Analogously

$$\begin{aligned}\tilde\nabla_{\varphi X}\tilde\nabla_X\varphi X&=\tilde\nabla_{\varphi X}\nabla_X\varphi X=\nabla_{\varphi X}\nabla_X\varphi X+2\eta(\nabla_X\varphi X)\varphi^2X\\&=\nabla_{\varphi X}\nabla_X\varphi X+2\varphi^2X=\nabla_{\varphi X}\nabla_X\varphi X-2X,\end{aligned}\tag{3.8}$$

since $\eta(\nabla_X\varphi X)=g(\nabla_X\varphi X,\xi)=-g(\varphi X,\nabla_X\xi)=g(\varphi X,\varphi X)=1$.
Furthermore,

$$\begin{aligned}\tilde\nabla_{[X,\varphi X]}\varphi X&=\nabla_{[X,\varphi X]}\varphi X+2\eta([X,\varphi X])\varphi^2X\\&=\nabla_{[X,\varphi X]}\varphi X+4d\eta(X,\varphi X)X=\nabla_{[X,\varphi X]}\varphi X-4X.\end{aligned}\tag{3.9}$$

From (3.7–3.9) we get $\tilde R_{X\varphi X}\varphi X=R_{X\varphi X}\varphi X+2X+4X=R_{X\varphi X}\varphi X+6X$. □

Looking at the above relation between the φ-sectional curvatures, it is clear that the behaviors of a Sasaki manifold and its associated Lorentz–Sasaki manifold are strictly related as the following theorem shows

THEOREM 3.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasaki manifold, $(M^{2n+1},\varphi,\xi,\eta,\tilde g)$ the associated Lorentz–Sasaki manifold. Then the Sasaki manifold has constant φ-sectional curvature $c\in\mathbb{R}$ if and only if the Lorentz–Sasaki manifold has constant φ-sectional curvature $\tilde c=c+6$ and we have*

$$c>-3\Leftrightarrow\tilde c>3,\quad c=-3\Leftrightarrow\tilde c=3,\quad c<-3\Leftrightarrow\tilde c<3.$$

In [**6**] Tanno defined three model spaces of Sasaki manifolds with constant φ-sectional curvature. On the left-hand side of the above equivalence we find exactly the three values for the φ-sectional curvature characterizing the three model spaces of Sasaki manifolds.

We want to find a model for a compact Lorentz–Sasaki manifold, using Tanno's classification cited previously and Theorem 3.1. We begin defining a Lorentz–Sasaki structure on spheres. As Tanno did in [**6**], regarding the sphere $\mathbb{S}^{2n+1}$ as an hypersurface of the complex manifold $(\mathbb{R}^{2n+2}, J, <,>)$, one obtains a standard Sasaki structure $(\varphi^*, \xi^*, \eta^*, g^*)$ such that $\xi^* = -JN, \quad JX = \varphi^* X + \eta^*(X)N$, for any $X \in \mathfrak{X}(\mathbb{S}^{2n+1})$, where J is the standard complex structure on $\mathbb{R}^{2n+2}$ and N is a unit normal vector field to $\mathbb{S}^{2n+1}$. Deforming the above structure by a $\mathfrak{D}$-homothetic transformation with $\alpha > 0$ we get the following structure

$$\varphi = \varphi^*, \quad \xi = \alpha^{-1}\xi^*, \quad \eta = \alpha\eta^* \quad g = \alpha g^* + (\alpha^2 - \alpha)\eta^* \otimes \eta^*$$

and $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasaki manifold of constant φ-sectional curvature $c > -3$, denoted by $\mathbb{S}^{2n+1}[c]$. Now, applying the metric transformation (3.1) we realize a Lorentz–Sasaki structure $(\varphi, \xi, \eta, \tilde{g})$ on the sphere. We denote the Lorentz–Sasaki manifold $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ by $\mathbb{S}_1^{2n+1}[\tilde{c}]$ and, from Theorem 3.1, $\tilde{c} > 3$.

Since the metric on $\mathbb{S}_1^{2n+1}[\tilde{c}]$ has index 1 and ξ is a nowhere vanishing time-like Killing vector field, $\mathbb{S}_1^{2n+1}[\tilde{c}]$ is complete according to a result of Guediri and Lafontaine which we cite here for the convenience of the reader.

Theorem 3.2. [**2**] *Let (M, g) be an n-dimentional compact pseudo-Riemannian manifold with signature $(n-p, p)$ such that $2p \leqslant n$. We suppose that there exist p linear independent timelike Killing vector fields on M. Then (M, g) is geodesically complete.*

By Theorem 3.1 and the following two theorems, we obtain that $\mathbb{S}_1^{2n+1}[\tilde{c}]$ is the model of compact, simply connected Lorentz–Sasaki manifold with constant φ-sectional curvature greater than 3.

Theorem 3.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, \tilde{g})$ be a compact, simply connected Lorentz–Sasaki manifolds with constant φ-sectional curvature $\tilde{c} > 3$. Then M is isomorphic to $\mathbb{S}_1^{2n+1}[\tilde{c}]$.*

Proof. Considering the Lorentz metric $\tilde{g}$ and regard of (3.2), $g = \tilde{g} + 2\eta \otimes \eta$ is a Riemannian metric and $(M^{2n+1}, \varphi, \xi, \eta, g)$ turns out to be a compact, simply connected Sasaki manifold with constant φ-sectional curvature $c = \tilde{c} - 6 > -3$. Therefore, on the basis of Tanno's classification $(M^{2n+1}, \varphi, \xi, \eta, g)$ is isomorphic to $\mathbb{S}^{2n+1}[c]$ and going back to (3.1) we get the statement. □

Tanno in [**6**] also proved that complete, simply connected Sasaki manifolds, having the same constant φ-sectional curvature c, are isomorphic that is there exists a diffeomorphism f which preserves the Sasaki structures. At least in the compact case we prove an analogous result.

Theorem 3.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ and $(M'^{2n+1}, \varphi', \xi', \eta', g')$ be compact and simply connected Lorentz–Sasaki manifolds with the same constant φ-sectional curvature $\tilde{c} > 3$. Then they are isomorphic.*

Proof. We consider the following changes of the metrics: $\bar{g} = g + 2\eta \otimes \eta$ and $\bar{g}' = g' + 2\eta' \otimes \eta'$. Then the manifolds $(M^{2n+1}, \varphi, \xi, \eta, \bar{g})$ and $(M'^{2n+1}, \varphi', \xi', \eta', \bar{g}')$

are complete and simply connected Sasaki manifolds with the same constant φ-sectional curvature $\bar{c} = \tilde{c} - 6$. Let f be the isomorphism provided by Tanno's theorem. To end the proof we only need to check if f preserves the Lorentz metrics g and g'. For any $X, Y \in \mathfrak{X}(M^{2n+1})$ we get
$g'(f_*X, f_*Y) = \bar{g}'(f_*X, f_*Y) - 2\eta'(f_*X)\eta'(f_*Y) = \bar{g}(X,Y) - 2\eta(X)\eta(Y) = g(X,Y)$. □

Moreover, in [**5**] the author proved that every complete Sasaki manifold with constant φ-sectional curvature $c > -3$ is obtained by a $\mathcal{D}$-homothetic deformation from a complete Sasaki manifold of constant curvature 1. Using this result, we obtain a similar statement.

Proposition 3.5. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Lorentz–Sasaki manifold. Suppose that the φ-sectional curvature of M is a constant c such that $c > 3$. Then the structure (φ, ξ, η, g) is obtained by a $\mathfrak{D}$-homothetic deformation of a Lorentz–Sasaki structure of constant φ-sectional curvature 7 on M.*

Proof. Being $(M, \varphi, \xi, \eta, g)$ a Lorentz–Sasaki manifold, following (3.2) we consider the Riemannian metric $\bar{g}$ on M, therefore $(M, \varphi, \xi, \eta, \bar{g})$ is a Sasaki manifold such that $\bar{c} > -3$.

Now, applying Tanno's result, it is possible to find a real number α such that $\varphi' = \varphi$, $\quad \xi' = \frac{1}{\alpha}\xi$, $\quad \eta' = \alpha\eta$, $\quad \bar{g}' = \alpha\bar{g} + (\alpha^2 - \alpha)\eta \otimes \eta$ define a Sasaki structure on M and the φ'-sectional curvature $\bar{c}'$ of $(M, \varphi', \xi', \eta', \bar{g}')$ is 1. Changing the metric $\bar{g}'$ according to (3.1) we have $\tilde{g}' = \bar{g}' - 2\eta' \otimes \eta'$ and a Lorentz–Sasaki manifold $(M, \varphi', \xi', \eta', \tilde{g}')$. An easy computation shows that the φ'-sectional curvature of $(M, \varphi', \xi', \eta', \tilde{g}')$ is $\tilde{c}' = 7$. It is easy to see that the two structures (φ, ξ, η, g) and $(\varphi', \xi', \eta', \tilde{g}')$ are $\mathfrak{D}$-homothetic, in fact we have: $\varphi' = \varphi$, $\xi' = \frac{1}{\alpha}\xi$, $\eta' = \alpha\eta$ and
$\tilde{g}' = \bar{g}' - 2\eta' \otimes \eta' = \alpha\bar{g} + (\alpha^2 - \alpha)\eta \otimes \eta - 2\alpha^2\eta \otimes \eta = \alpha(g + 2\eta \otimes \eta) + (-\alpha^2 - \alpha)\eta \otimes \eta = \alpha g - (\alpha^2 - \alpha)\eta \otimes \eta$. This completes the proof. □

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