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A NEW MODEL OF NONLOCAL MODIFIED GRAVITY

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Abstract. We consider a new modified gravity model with nonlocal term of the form $R^{-1}\mathcal{F}(\Box)R$. This kind of nonlocality is motivated by investigation of applicability of a few unusual ansätze to obtain some exact cosmological solutions. In particular, we find attractive and useful quadratic ansatz $\Box R = qR^2$.

1. Introduction

In spite of the great successes of General Relativity (GR) it has not got the status of a complete theory of gravity. To modify GR there are motivations coming from its quantum aspects, string theory, astrophysics and cosmology. For example, cosmological solutions of GR contain Big Bang singularity, and Dark Energy as a cause for accelerated expansion of the Universe. This initial cosmological singularity is an evident signature that GR is not appropriate theory of the Universe at cosmic time $t = 0$. Also, GR has not been verified at the very large cosmic scale and dark energy has not been discovered in the laboratory experiments. This situation gives rise to research for an adequate modification of GR among numerous possibilities (for a recent review, see [**1**]).

Recently it has been shown that nonlocal modified gravity with action

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + CR\mathcal{F}(\Box)R\Big), \tag{1.1}$$

where R is scalar curvature, Λ–cosmological constant, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ is an analytic function of the d'Alembert–Beltrami operator

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu, \quad g = \det(g_{\mu\nu})$$

and C is a constant, has nonsingular bounce cosmological solutions, see [**2, 3, 4, 5**]. To solve equations of motion the ansatz $\Box R = rR+s$ was used. In [**6**] we introduced

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some new ansätze, which gave trivial solutions for the above nonlocal model (1.1). Here we consider some modification of the above action in the nonlocal sector, i.e.,

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big) \tag{1.2}$$

and look for nontrivial cosmological solutions for the new ansätze (see [**6**]). Note that the cosmological constant Λ in (1.2) is hidden in the term f_0, i.e., $\Lambda = -8\pi G f_0$. To the best of our knowledge action (1.2) has not been considered so far. However, there are investigations of gravity modified by $1/R$ term (see, e.g. [**7**] and references therein), but it is without nonlocality.

2. Equations of motion

By variation of action (1.2) with respect to metric $g^{\mu\nu}$ one obtains the equations of motion for $g_{\mu\nu}$

$$\begin{aligned} R_{\mu\nu}V - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)V - \frac{1}{2}g_{\mu\nu}R^{-1}\mathcal{F}(\Box)R \\ + \sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\Big(g_{\mu\nu}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\right) \\ - 2\partial_\mu\Box^l(R^{-1})\partial_\nu\Box^{n-1-l}R\Big) = -\frac{G_{\mu\nu}}{16\pi G}, \\ V = \mathcal{F}(\Box)R^{-1} - R^{-2}\mathcal{F}(\Box)R. \end{aligned} \tag{2.1}$$

The trace of the equation (2.1) is

$$\begin{aligned} RV + 3\Box V + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + 2\Box^l(R^{-1})\Box^{n-l}R\right) \\ - 2R^{-1}\mathcal{F}(\Box)R = \frac{R}{16\pi G}. \end{aligned} \tag{2.2}$$

The 00 component of (2.1) is

$$\begin{aligned} R_{00}V - (\nabla_0\nabla_0 - g_{00}\Box)V - \frac{1}{2}g_{00}R^{-1}\mathcal{F}(\Box)R \\ + \sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\Big(g_{00}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\right) \\ - 2\partial_0\Box^l(R^{-1})\partial_0\Box^{n-1-l}R\Big) = -\frac{G_{00}}{16\pi G}. \end{aligned} \tag{2.3}$$

We use Friedmann–Lemaître–Robertson–Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\big)$ and investigate all three possibilities for curvature parameter k $(0, \pm 1)$. In the FLRW metric scalar curvature is $R = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$ and $\Box h(t) = -\partial_t^2 h(t) - 3H\partial_t h(t)$, where $H = \frac{\dot a}{a}$ is the Hubble parameter. In the sequel we shall use three kinds of ansätze (two of them introduced in [**6**]) and solve equations of motions (2.2) and (2.3) for cosmological scale factor in the form $a(t) = a_0|t - t_0|^\alpha$.

3. Quadratic ansatz: $\Box R = qR^2$

Looking for solutions in the form $a(t) = a_0|t - t_0|^\alpha$ this ansatz becomes

$$\begin{aligned}(3.1)\qquad &\alpha(2\alpha-1)(q\alpha(2\alpha-1)-(\alpha-1))(t-t_0)^{-4}\\ &+\frac{\alpha k}{3a_0^2}(1-\alpha+6q(2\alpha-1))(t-t_0)^{-2\alpha-2}+\frac{qk^2}{a_0^4}(t-t_0)^{-4\alpha}=0.\end{aligned}$$

Equation (3.1) is satisfied for all values of time t in six cases:

(1) $k = 0,\ \ \alpha = 0,\ \ q \in \mathbb{R}$,
(2) $k = 0,\ \ \alpha = \frac{1}{2},\ \ q \in \mathbb{R}$,
(3) $k = 0,\ \ \alpha \neq 0, \frac{1}{2},\ \ q = \frac{\alpha-1}{\alpha(2\alpha-1)}$,
(4) $k = -1,\ \ \alpha = 1,\ \ q \neq 0,\ \ a_0 = 1$,
(5) $k \neq 0,\ \ \alpha = 0,\ \ q = 0$,
(6) $k \neq 0,\ \ \alpha = 1,\ \ q = 0$.

In cases (1), (2) and (4) we have $R = 0$ and therefore R^{-1} is not defined. Case (5) yields a solution which does not satisfy the equations of motion. Hence there remain two cases for further consideration.

3.1. Case $k = 0$, $q = \frac{\alpha-1}{\alpha(2\alpha-1)}$.

For this case, we have the following expressions depending on the parameter α:

$$\begin{aligned} q &= \frac{\alpha-1}{\alpha(2\alpha-1)}, & R &= 6\alpha(2\alpha-1)(t-t_0)^{-2},\\ a &= a_0|t-t_0|^\alpha, & H &= \alpha(t-t_0)^{-1},\\ R_{00} &= 3\alpha(1-\alpha)(t-t_0)^{-2}, & G_{00} &= 3\alpha^2(t-t_0)^{-2}.\end{aligned}$$

We now express $\Box^n R$ and $\Box^n R^{-1}$ in the following way:

$$\begin{aligned} &\Box^n R = B(n,1)(t-t_0)^{-2n-2}, \quad \Box^n R^{-1} = B(n,-1)(t-t_0)^{2-2n},\\ &B(n,1) = 6\alpha(2\alpha-1)(-2)^n n!\prod_{l=1}^{n}(1-3\alpha+2l),\ n \geqslant 1,\\ &B(n,-1) = (6\alpha(2\alpha-1))^{-1}2^n\prod_{l=1}^{n}(2-l)(-3-3\alpha+2l),\ n\geqslant 1,\\ &B(0,1) = 6\alpha(2\alpha-1), \quad B(0,-1) = B(0,1)^{-1}.\end{aligned}$$

Note that $B(1,-1) = -\frac{3\alpha+1}{3\alpha(2\alpha-1)} = -2(3\alpha+1)B(0,1)^{-1}$ and $B(n,-1) = 0$ if $n \geqslant 2$. Also, we obtain

$$\begin{aligned}\mathcal{F}(\Box)R &= \sum_{n=0}^{\infty} f_n B(n,1)(t-t_0)^{-2n-2},\\ \mathcal{F}(\Box)R^{-1} &= f_0 B(0,-1)(t-t_0)^2 + f_1 B(1,-1).\end{aligned}$$

Substituting these equations into trace and 00 component of the EOM one has

$$\begin{aligned}
&r^{-1}\sum_{n=0}^{\infty} f_n B(n,1)\left(-3r+6(1-n)(1-2n+3\alpha)\right)(t-t_0)^{-2n} \\
&+r\sum_{n=0}^{1} f_n\left(rB(n,-1)+3B(n+1,-1)\right)(t-t_0)^{-2n} \\
&+2r\sum_{n=1}^{\infty} f_n\gamma_n(t-t_0)^{-2n} = \frac{r^2}{16\pi G}(t-t_0)^{-2}, \\
&\sum_{n=0}^{\infty} f_n r^{-1}B(n,1)\left(\frac{r}{2}-A_n\right)(t-t_0)^{-2n} \\
&+\sum_{n=0}^{1} f_n r B(n,-1)\,A_n\,(t-t_0)^{-2n}+\frac{r}{2}\sum_{n=1}^{\infty} f_n\delta_n(t-t_0)^{-2n} \\
&\qquad = \frac{-r^2}{32\pi G}\frac{\alpha}{2\alpha-1}(t-t_0)^{-2},
\end{aligned} \tag{3.2}$$

where $r = B(0,1)$ and

$$\begin{aligned}
\gamma_n &= \sum_{l=0}^{n-1} B(l,-1)(B(n-l,1)+2(1-l)(n-l)B(n-l-1,1)), \\
\delta_n &= \sum_{l=0}^{n-1} B(l,-1)(-B(n-l,1)+4(1-l)(n-l)B(n-l-1,1)), \\
A_n &= 6\alpha(1-n) - r\frac{\alpha-1}{2(2\alpha-1)} = \frac{r}{2}\,\frac{3-2n-\alpha}{2\alpha-1}.
\end{aligned}$$

Equations (3.2) can be split into system of pairs of equations with respect to each coefficient f_n. In the case $n > 1$, there are the following pairs:

$$f_n\big(B(n,1)\left(-3r+6(1-n)(1-2n+3\alpha)\right)+2r^2\gamma_n\big)=0,$$

$$f_n\Big(B(n,1)\left(\frac{r}{2}-A_n\right)+\frac{r^2}{2}\delta_n\Big)=0.$$

Taking $\frac{3\alpha-1}{2}$ to be a natural number one obtains:

$$B(n,1) = 6\alpha(2\alpha-1)4^n n!\frac{(\frac{3}{2}(\alpha-1))!}{(\frac{3}{2}(\alpha-1)-n)!}, \quad n < \frac{3\alpha-1}{2},$$

$$B(n,1) = 0, \quad n \geqslant \frac{3\alpha-1}{2},$$

$$\gamma_n = 2B(0,-1)B(n-1,1)(3n\alpha-2n^2-3\alpha-1), \quad n \leqslant \frac{3\alpha-1}{2},$$

$$\delta_n = 2B(0,-1)B(n-1,1)(2n^2+3n+3\alpha-3\alpha n+1), \quad n \leqslant \frac{3\alpha-1}{2},$$

$$\gamma_n = \delta_n = 0, \quad n > \frac{3\alpha-1}{2}.$$

If $n > \frac{3\alpha-1}{2}$, then $B(n,1) = \gamma_n = \delta_n = 0$ and hence the system is trivially satisfied for arbitrary values of coefficients f_n. On the other hand for $2 \leqslant n \leqslant \frac{3\alpha-1}{2}$ the system has only the trivial solution $f_n = 0$.

When $n = 0$, the pair becomes

$$f_0\big(-2r + 6(1+3\alpha) + 3rB(1,-1)\big) = 0, \quad f_0 = 0$$

and its solution is $f_0 = 0$. The remaining case $n = 1$ reads

$$\begin{aligned} f_1\big(-3r^{-1}B(1,1) + rB(1,-1) + 2\gamma_1\big) &= \frac{r}{16\pi G}, \\ f_1\Big(A_1(rB(1,-1) - r^{-1}B(1,1)) + \frac{1}{2}(B(1,1) + r\delta_1)\Big) &= \frac{-r^2}{32\pi G}\,\frac{\alpha}{2\alpha-1}, \end{aligned}$$

and it gives $f_1 = -\frac{3\alpha(2\alpha-1)}{32\pi G(3\alpha-2)}$.

3.2. Case $k \neq 0$, $\alpha = 1$, $q = 0$.

In this case

$$\begin{gathered} a = a_0|t-t_0|, \quad H = (t-t_0)^{-1}, \quad R = s(t-t_0)^{-2}, \\ s = 6\Big(1 + \frac{k}{a_0^2}\Big), \quad \Box R = 0, \quad R_{00} = 0, \\ \Box^n R^{-1} = D(n,-1)(t-t_0)^{2-2n}, \\ D(0,-1) = s^{-1}, \quad D(1,-1) = -8s^{-1}, \quad D(n,-1) = 0,\ n \geqslant 2. \end{gathered}$$

Substitution of the above expressions in the trace and 00 component of the EOM yields

$$3f_0 + \sum_{n=0}^{1} f_n s D(n,-1)(t-t_0)^{-2n} + 4f_1(t-t_0)^{-2} = \frac{s}{16\pi G}(t-t_0)^{-2},$$

$$\begin{aligned} -6f_0 s^{-1} + \frac{1}{2}f_0 + 6\sum_{n=0}^{1} f_n D(n,-1)(1-n)(t-t_0)^{-2n} & \\ + 2f_1(t-t_0)^{-2} &= -\frac{s}{32\pi G}(t-t_0)^{-2}. \end{aligned}$$

This system leads to conditions for f_0 and f_1:

$$\begin{aligned} -2f_0 - 4f_1(t-t_0)^{-2} &= \frac{s}{16\pi G}(t-t_0)^{-2}, \\ \frac{1}{2}f_0 + 2f_1(t-t_0)^{-2} &= -\frac{s}{32\pi G}(t-t_0)^{-2}. \end{aligned}$$

The corresponding solution is

$$f_0 = 0, \quad f_1 = \frac{-s}{64\pi G}, \quad f_n \in \mathbb{R}, \quad n \geqslant 2.$$

4. Ansatz $\Box^n R = c_n R^{n+1}$, $n \geqslant 1$

Presenting $\Box^{n+1} R$ in two ways:

$$\begin{aligned}\Box^{n+1} R = \Box c_n R^{n+1} &= c_n((n+1)R^n \Box R - n(n+1)R^{n-1}\dot{R}^2)\\ &= c_n(n+1)(c_1 R^{n+2} - nR^{n-1}\dot{R}^2) = c_{n+1}R^{n+2}\end{aligned}$$

it follows

$$\dot{R}^2 = R^3, \tag{4.1}$$

$$c_{n+1} = c_n(n+1)(c_1 - n), \tag{4.2}$$

where $\dot{R}^2$ means $(\dot{R})^2$.

The general solution of equation (4.1) is

$$R = \frac{4}{(t-t_0)^2}, \quad t_0 \in \mathbb{R}. \tag{4.3}$$

Taking $n = 1$ in the ansatz yields

$$\Box R = c_1 R^2. \tag{4.4}$$

The substitution of (4.3) in (4.4) gives $H = \frac{2c_1+3}{3(t-t_0)}$. This implies

$$a(t) = a_0|t-t_0|^{\frac{2c_1+3}{3}}, \quad a_0 > 0. \tag{4.5}$$

Using (4.3) in the equation

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

gives

$$(t-t_0)^2\ddot{y} - \frac{4}{3}y = -2k(t-t_0)^2, \text{ where } y = a^2(t).$$

It can be shown that the general solution of the last equation is

$$a^2(t) = \tilde{C_1}|t-t_0|^{\frac{3+\sqrt{57}}{6}} + \tilde{C_2}|t-t_0|^{\frac{3-\sqrt{57}}{6}} - 3k|t-t_0|^2, \quad \tilde{C_1}, \tilde{C_2} \in \mathbb{R}.$$

By comparison of the last equation with (4.5) one can conclude:

(1) If $c_1 = 0$, then k must be equal to -1. In this case $\Box^n R = 0, \quad n \geqslant 1$.
(2) If $c_1 \neq 0$, then k must be equal to 0. In this case $c_1 = \frac{1}{8}\big(-9 \pm \sqrt{57}\big)$.

4.1. Case $\Box^n R = c_n R^{n+1}$, $c_1 = 0$.

From the previous analysis, it follows:

$$\begin{aligned} &k = -1, && a(t) = \sqrt{3}|t-t_0|, && H(t) = \frac{1}{t-t_0},\\ &R = \frac{4}{(t-t_0)^2}, && \Box^n R = 0, \quad n \geqslant 1, && \mathcal{F}(\Box)R = f_0 R. \end{aligned} \tag{4.6}$$

It can be shown that

$$\Box^n R^{-1} = (-1)^n 4^{n-1} \prod_{l=0}^{n-1} (1-l)(2-l)(t-t_0)^{2-2n}. \tag{4.7}$$

From (4.7) it follows $\Box^n R^{-1} = 0$, $n > 1$. Then

$$\mathcal{F}(\Box)(R^{-1}) = f_0 R^{-1} + f_1 \Box R^{-1}. \tag{4.8}$$

Substituting (4.6) and (4.8) in the 00 component of the EOM one obtains

$$\frac{f_0}{2}(t-t_0)^2 + 2f_1 + \frac{1}{8\pi G} = 0$$

and it follows

$$f_0 = 0, \quad f_1 = \frac{-1}{16\pi G}, \quad f_n \in \mathbb{R}, \quad n \geqslant 2. \tag{4.9}$$

Substituting (4.6) and (4.8) in the trace equation one has

$$-2f_0(t-t_0)^2 - 4f_1 - \frac{1}{4\pi G} = 0$$

and it gives the same result (4.9).

4.2. Case $\Box^n R = c_n R^{n+1}$, $c_1 = \frac{1}{8}\big(-9 \pm \sqrt{57}\big)$. In this case:

$$k = 0, \quad R = \frac{4}{(t-t_0)^2}, \quad H = \frac{2c_1+3}{3(t-t_0)}, \quad a = a_0|t-t_0|^{\frac{1}{3}(2c_1+3)}, \ a_0 > 0,$$

$$R_{00} = 3\alpha(1-\alpha)(t-t_0)^{-2}, \quad G_{00} = (3\alpha(1-\alpha)+2)(t-t_0)^{-2}, \quad \alpha = \frac{2c_1+3}{3},$$

$$\Box^n R = 4^{n+1} c_n (t-t_0)^{-2n-2}, \quad c_0 = 1.$$

One can show that $\Box^n R^{-1} = M(n,-1)(t-t_0)^{2-2n}$, where

$$M(0,-1) = \frac{1}{4}, \quad M(1,-1) = -(c_1+2), \quad M(n,-1) = 0, \quad n > 1.$$

Also one obtains

$$\begin{aligned} \mathcal{F}(\Box)R &= \sum_{n=0}^{\infty} 4^{n+1} f_n c_n (t-t_0)^{-2n-2}, \\ \mathcal{F}(\Box)R^{-1} &= f_0 M(0,-1)(t-t_0)^2 + f_1 M(1,-1). \end{aligned} \tag{4.10}$$

Substituting (4.10) in the trace equation it becomes

$$\begin{aligned} &-\frac{1}{4\pi G}(t-t_0)^{-2} - 2f_0 - 3\sum_{n=1}^{\infty} 4^n f_n c_n (t-t_0)^{-2n} \\ &+ \sum_{n=1}^{\infty} f_n \Bigg(\sum_{l=0}^{n-1} M(l,-1) 4^{n-l+1}((1-l)(n-l)c_{n-1-l} + 2c_{n-l}) \Bigg)(t-t_0)^{-2n} \\ &\qquad + f_1(4M(1,-1) + 3M(2,-1))(t-t_0)^{-2} = 0. \end{aligned} \tag{4.11}$$

To satisfy equation (4.11) for all values of time t one obtains:

$$f_0 = 0, \quad f_1(2c_1+1) = -\frac{1}{16\pi G},$$

$$f_n\Bigg(-3c_n + \sum_{l=0}^{1} M(l,-1)4^{1-l}\big((1-l)(n-l)c_{n-1-l} + 2c_{n-l}\big)\Bigg) = 0, \ \ n \geqslant 2.$$

Suppose that $f_n \neq 0$ for $n \geqslant 2$, then from the last equation it follows

$$-3c_n + \sum_{l=0}^{1} M(l,-1)4^{1-l}\big((1-l)(n-l)c_{n-1-l} + 2c_{n-l}\big) = 0$$

and it becomes

$$c_{n-1}(n^2 - c_1 n - 2c_1 - 4) = 0. \tag{4.12}$$

Since $c_{n-1} \neq 0$, condition (4.12) is satisfied for $n = -2$ or $n = c_1 + 2$. Hence, we conclude that $f_n = 0$ for $n \geqslant 2$. In such a case the 00 component of the EOM becomes

$$\begin{aligned} &\frac{1}{16\pi G}(-3\alpha^2 + 3\alpha + 2)(t - t_0)^{-2} \\ &\quad + \frac{1}{2} f_0\Big(\frac{3}{2}\alpha^2 - \frac{9}{2}\alpha + 1\Big) + f_1 c_1(3\alpha^2 - 3\alpha + 2)(t - t_0)^{-2} \\ &\quad + 8f_1 M(0,-1)(1 - c_1)(t - t_0)^{-2} + 3\alpha(3 - \alpha)M(0,-1)f_0 \\ &\qquad\qquad - 3\alpha(\alpha - 1)M(1,-1)f_1(t - t_0)^{-2} = 0. \end{aligned} \tag{4.13}$$

In order to satisfy equation (4.13) for all values of time t it has to be

$$f_0 = 0, \quad f_1\Big(\frac{4}{3}c_1^3 + \frac{10}{3}c_1^2 + 2c_1 + 1\Big) = \frac{1}{16\pi G}\Big(\frac{2}{3}c_1^2 + c_1 - 1\Big).$$

The necessary and sufficient condition for the EOM to have a solution is

$$c_1(8c_1^2 + 18c_1 + 3) = 0.$$

Since $c_1 = \frac{1}{8}\big(-9 \pm \sqrt{57}\big)$, the last condition is satisfied.

5. Cubic ansatz: $\Box R = qR^3$

Recall that we are looking for solutions in the form $a(t) = a_0|t - t_0|^\alpha$. In the explicit form the ansatz reads

$$\begin{aligned} &\alpha(\alpha - 1)\Big(3(2\alpha - 1)(t - t_0)^{-4} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha-2}\Big) \\ &\qquad = 18q\Big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\Big)^3. \end{aligned}$$

It yields the following seven possibilities:

(1) $k = 0,\ \alpha = 0,\ q \in \mathbb{R}$, (4) $k = 0,\ \alpha = 1,\ q = 0$,

(2) $k = 0,\ \alpha = \frac{1}{2},\ q \in \mathbb{R}$, (5) $k \neq 0,\ \alpha = 0,\ q = 0$,

(3) $k = -1,\ \alpha = 1,\ q \neq 0,\ a_0 = 1$, (6) $k \neq 0,\ \alpha = 1,\ q = 0$,

(7) $k \neq 0,\ \alpha = \frac{1}{2},\ q = -a_0^4/72$.

Cases (1), (2) and (3) contain scalar curvature $R = 0$, and therefore we will not discuss them. Cases (4), (5) and (6) are also obtained from the quadratic ansatz and have been discussed earlier. The last case contains:

$$a(t) = a_0\sqrt{|t-t_0|}, \quad H(t) = \frac{1}{2(t-t_0)},$$
$$R(t) = \frac{6k}{a_0^2}|t-t_0|^{-1}, \quad R_{00} = \frac{3}{4(t-t_0)^2}.$$

One can derive the following expressions:

$$\begin{aligned}
\Box^n R &= N(n,1)|t-t_0|^{-2n-1}, \quad \Box^n R^{-1} = N(n,-1)|t-t_0|^{1-2n},\\
N(0,1) &= \frac{6k}{a_0^2}, \quad N(0,-1) = N(0,1)^{-1},\\
N(n,1) &= N(0,1)(-1)^n \prod_{l=0}^{n-1}(2l+1)\Big(2l+\frac{1}{2}\Big), \quad n \geqslant 1,\\
N(n,-1) &= N(0,1)^{-1}(-1)^n \prod_{l=0}^{n-1}(2l-1)\Big(2l-\frac{3}{2}\Big), \quad n \geqslant 1,\\
\mathcal{F}(\Box)R &= \sum_{n=0}^{\infty} f_n N(n,1)|t-t_0|^{-2n-1},\\
\mathcal{F}(\Box)R^{-1} &= \sum_{n=0}^{\infty} f_n N(n,-1)|t-t_0|^{1-2n}.
\end{aligned} \tag{5.1}$$

Substituting (5.1) in the trace equation we obtain

$$\begin{aligned}
&-2N(0,1)^{-1}\sum_{n=0}^{\infty} f_n N(n,1)|t-t_0|^{-2n}\\
&+N(0,1)\sum_{n=0}^{\infty} f_n(N(n,-1)-N(0,1)^{-2}N(n,1))|t-t_0|^{-2n}\\
&+3\sum_{n=0}^{\infty} f_n(N(n,-1)-N(0,1)^{-2}N(n,1))|t-t_0|^{-1-2n}\\
&+\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} N(l,-1)\big((1-2l)(-2n+2l+1)N(n-l-1,1)\\
&\qquad\qquad +2N(n-l,1)\big)|t-t_0|^{-2n} = \frac{N(0,1)}{16\pi G}|t-t_0|^{-1}.
\end{aligned}$$

This equation implies the following conditions on the coefficient f_0:

$$f_0 = 0, \quad \frac{N(0,1)}{16\pi G} = 0.$$

Since $N(0,1) \neq 0$, the last equation never holds and therefore there is no solution in this case.

6. Concluding remarks

Using a few new ansätze we have shown that equations of motion for nonlocal gravity model given by action (1.2) yield some bounce cosmological solutions of the form $a(t) = a_0|t - t_0|^\alpha$. When $t \to \infty$ then $R \to 0$ and these solutions lead to $f_0 = 0$ and hence $\Lambda = 0$. In particular, the quadratic ansatz $\Box R = qR^2$ is very promising. Note that ansatz $\Box^n R = c_n R^{n+1}$, $n \geqslant 1$, can be viewed as a special case of ansatz $\Box R = qR^2$.

It is worth noting that equations of motion (2.2) and (2.3) have the de Sitter solutions $a(t) = a_0 \cosh(\lambda t)$, $k = +1$ and $a(t) = a_0 e^{\lambda t}$, $k = 0$, when $f_0 = \frac{-3\lambda^2}{8\pi G} = \frac{-\Lambda}{8\pi G}$, $f_n \in \mathbb{R}$, $n \geqslant 1$.

This investigation can be generalized to some cases with $R^{-p}\mathcal{F}(\Box)R^q$ nonlocal term, where p and q are some natural numbers satisfying $q - p \geqslant 0$. It will be presented elsewhere with discussion of various properties.

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