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DUALITY PRINCIPLE AND SPECIAL OSSERMAN MANIFOLDS

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Abstract. We investigate the connection between the duality principle and the Osserman condition in a pseudo-Riemannian setting. We prove that a connected pointwise two-leaves Osserman manifold of dimension $n \geqslant 5$ is globally Osserman and investigate the relation between the special Osserman condition and the two-leaves Osserman one.

1. Introduction

Let us start with the basic notation and terminology. Let (M, g) be a pseudo-Riemannian manifold of signature $(\nu, n-\nu)$ and ∇ the Levi-Civita connection of (M, g). We define the curvature operator $\mathcal{R}$ of (M, g) by $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and the curvature tensor by $R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)$. We define the Jacobi operator with respect to X by $\mathcal{J}_X(Y) = \mathcal{R}(Y, X)X$.

Very often we work locally and use a restriction to some point $p \in M$. We keep the notation and instead of vector fields we observe tangent vectors from vector space $\mathcal{V} = T_pM$. The notation $\varepsilon_X = g(X, X)$ denotes the norm of $X \in \mathcal{V}$, and it determines various types of vectors. We say that $X \in \mathcal{V}$ is timelike (if $\varepsilon_X < 0$), spacelike ($\varepsilon_X > 0$), null ($\varepsilon_X = 0$), nonnull ($\varepsilon_X \neq 0$), or unit ($\varepsilon_X \in \{-1, 1\}$). In the case of nonnull $X \in \mathcal{V}$, $\mathcal{J}_X$ preserves the nondegenerate hyperspace $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$, and we have the reduced Jacobi operator $\tilde{\mathcal{J}}_X : \{X\}^\perp \to \{X\}^\perp$, given by the restriction $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$.

Let ω_X^p be the characteristic polynomial of restriction of $\mathcal{J}_X$ to a point $p \in M$ of a pseudo-Riemannian manifold (M, g). We say that (M, g) is spacelike (timelike) Osserman at the point $p \in M$ if ω_X^p is constant for unit spacelike (timelike) X. We say that (M, g) is Osserman at the point $p \in M$ if it is both spacelike and timelike Osserman at p. If it holds at each $p \in M$ we say that (M, g) is pointwise Osserman. If the characteristic polynomial ω_X of $\mathcal{J}_X$ is constant for unit spacelike X and unit timelike X (on the tangent bundle) we say that (M, g) is globally Osserman.

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In a pseudo-Riemannian setting, the Jordan normal form plays a crucial role, since the characteristic polynomial does not determine the eigen-structure of a symmetric linear operator. We say that (M, g) (or its curvature tensor R) is a Jordan–Osserman if the Jordan normal form of $\mathcal{J}_X$ is constant on both pseudo-spheres. Diagonalizable Osserman is a special case of Jordan–Osserman such that all Jordan blocks have size 1 (Jacobi operator $\mathcal{J}_X$ is diagonalizable for all non-null X). Diagonalizability is a natural Riemannian-like condition, moreover, it is known that every Jordan–Osserman curvature tensor of non-neutral signature ($n \neq 2\nu$) is necessarily diagonalizable [**15**].

It is worth to emphasize that the spacelike and timelike Osserman conditions at any point are equivalent [**12**]. However, an analogous result is no longer true for the spacelike and timelike Jordan–Osserman conditions, which are equivalent in dimension four [**11**].

In the Riemannian setting ($\nu = 0$), it is known that a local two-point homogeneous space (flat or locally rank one symmetric space) has a constant characteristic polynomial on the unit sphere bundle. Osserman wondered if the converse held [**20**], and this question has been called the Osserman conjecture by subsequent authors. During the solving of some particular cases of the conjecture, the implication

$$\mathcal{J}_X(Y) = \lambda Y \Rightarrow \mathcal{J}_Y(X) = \lambda X \tag{1.1}$$

appeared naturally, and if it holds, it can significantly simplify some calculations. The first results in this topic were given by Chi [**10**], who proved the conjecture in the cases of dimensions $n \neq 4k$, $k > 1$. In his work he used the statement that (1.1) holds, if λ is an extremal (minimal or maximal) eigenvalue of the Jacobi operator.

Rakić used implication (1.1) to formulate the duality principle for Osserman manifolds and proved it in the Riemannian setting [**21**]. After that, the duality principle is reproved by Gilkey [**14**], and it becomes a beneficial tool for the conjecture solution. Moreover, Nikolayevsky used the duality principle [**17**] to prove the Osserman conjecture in all dimensions, except some possibilities in dimension $n = 16$ [**16**–**18**].

It is interesting to investigate the converse of Rakić's theorem. Can we prove that (M, g) is (pointwise) Osserman if and only if the duality principle holds? We proved [**1**, **3**] that this is true in dimension $n = 3$ and for some cases related with Fiedler's tensors. Recently Brozos-Vázquez and Merino [**9**] proved that this is true for the Riemannian manifolds of dimension 4. Finally, Nikolayevsky and Rakić [**19**] gave the affirmative answer to our question in the Riemannian setting.

The generalization of the Osserman conjecture has appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting ($\nu = 1$), an Osserman manifold necessarily has a constant sectional curvature [**6**]. Investigation of Osserman manifolds of signature $(2, 2)$ becomes very popular, and it is worth to mention results from [**7**], which are based on the discussion of possible Jordan normal forms of the Jacobi operator.

The above results provide us with a good motivation and this is why we started to examine the duality principle for Osserman manifolds in a pseudo-Riemannian setting. Implication (1.1) in a pseudo-Riemannian setting looks inaccurate, and therefore we corrected it in the following way [**1**, **5**].

Definition 1.1 (Duality principle). We say that the duality principle holds for a curvature tensor R if for all mutually orthogonal units $X, Y \in \mathcal{V}$, and for all $\lambda \in \mathbb{R}$ there holds

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{1.2}$$

Sometimes it is desirable to use a larger domain for our condition of (1.2), especially if we deal with the converse (of Rakić theorem) problem. We say that the strong duality principle holds for a curvature tensor R if for all $X, Y \in \mathcal{V}$ with $\varepsilon_X \neq 0$, and for all $\lambda \in \mathbb{R}$ equation (1.2) holds [**2**]. This domain (for X and Y) cannot be extended to $\varepsilon_X = 0$, because there exists a 4-dimensional globally Jordan–Osserman manifold such that formula (1.2) does not work at every point [**2**]. However, it is known that the duality principle and the strong duality principle are equivalent in the diagonalizable case [**2**, **5**].

The (strong) duality principle for Osserman manifolds works nicely for every known example, however we failed to prove it in general. In our previous work [**1**,**2**,**5**] we gave the affirmative answer only for the conditions of small index ($\nu \leqslant 1$) or low dimension ($n \leqslant 4$). In this paper, we restrict our attention to small number of eigenvalues of the reduced Jacobi operator.

2. Two-leaves Osserman manifolds

Let us look at small number of eigenvalues of the reduced Jacobi operator. The simplest case is a diagonalizable Osserman curvature tensor whose reduced Jacobi operator has a single eigenvalue, and it has to be a real space form (a manifold of constant sectional curvature) [**13**]. This is the reason why we look at the first nontrivial case, diagonalizable Osserman curvature tensor whose reduced Jacobi operator has two distinct eigenvalues, and we call it two-leaves Osserman.

Definition 2.1 (Two-leaves Osserman). Let R be an Osserman curvature tensor on a vector space $\mathcal{V}$ of the signature $(\nu, n-\nu)$, such that reduced Jacobi operator $\tilde{\mathcal{J}}_X$ is diagonalizable with exactly two distinct eigenvalues $\varepsilon_X \lambda$ and $\varepsilon_X \mu$ for every nonnull $X \in \mathcal{V}$. Then we say that R is two-leaves Osserman.

It is well known [**14**] that R is Osserman if and only if it is k-stein for every $k \in \mathbb{N}$. This is why for pointwise Osserman manifold and each $k \in \mathbb{N}$ there exist smooth functions $C_j : M \to \mathbb{R}$ with

$$\operatorname{Tr}({\mathcal{J}_X}^j) = (\varepsilon_X)^j C_j(p)$$

at each $p \in M$, for every $X \in T_pM$ and $1 \leqslant j \leqslant k$. The question if functions C_j are constant is known as the Schur-like problem. We recall results for a connected pseudo-Riemannian manifold (M, g) of dimension n: if (M, g) is Einstein (1-stein) and $n \neq 2$ then C_1 is constant; if (M, g) is zweistein (2-stein) and $n \notin \{2, 4\}$ then C_2 is constant [**14**]. We are ready now for the proof of the following theorem.

THEOREM 2.1. *If (M, g) is a connected pointwise two-leaves Osserman manifold of dimension $n \geqslant 5$, then (M, g) is a globally Osserman.*

PROOF. Let (M, g) be a connected two-leaves Osserman manifold such that the reduced Jacobi operator $\tilde{\mathcal{J}}_X$ has exactly two distinct eigenvalues $\varepsilon_X\lambda(p)$ and $\varepsilon_X\mu(p)$ with multiplicities $\sigma(p)$ and $\tau(p)$ respectively. The condition $n \geqslant 5$ excludes undesirable cases and we have constant C_1 and C_2. However, for every j we have

$$(\varepsilon_X)^j C_j(p) = \mathrm{Tr}({\mathcal{J}_X}^j) = (\varepsilon_X)^j(\sigma(p)\lambda(p)^j + \tau(p)\mu(p)^j).$$

Let us set $C_0 = n-1$ and look at the system of equations for $j \in \{0, 1, 2\}$:

$$\begin{aligned} \sigma(p) + \tau(p) &= C_0, \\ \sigma(p)\lambda(p) + \tau(p)\mu(p) &= C_1, \\ \sigma(p)\lambda(p)^2 + \tau(p)\mu(p)^2 &= C_2. \end{aligned}$$

From the first two equations we have

$$\sigma = \frac{C_1 - C_0\mu}{\lambda - \mu}, \ \tau = \frac{C_1 - C_0\lambda}{\mu - \lambda},$$

and after the substitution into the third equation

$$C_1(\lambda + \mu) - C_0\lambda\mu = C_2.$$

It is clear that $\tau \geqslant 1$, thus $C_1 - C_0\lambda \neq 0$, and the previous equation gives

$$\mu = \frac{C_2 - C_1\lambda}{C_1 - C_0\lambda}.$$

The other functions can be expressed in terms of λ in the following way

$$\sigma = \frac{C_0C_2 - C_1^2}{C_0\lambda^2 - 2C_1\lambda + C_2}, \ \tau = \frac{(C_1 - C_0\lambda)^2}{C_0\lambda^2 - 2C_1\lambda + C_2}.$$

The variant of this theorem in [**13**] requires that the multiplicities $\sigma(p)$ and $\tau(p)$ are constant, however we shall see that this condition is not necessary. The basic argument of this original result is that σ is an integer by definition.

From the previously derived relation between σ and λ, we see that a concrete σ gives at most two values for λ, which are solutions of the related quadratic equation. Values for σ are integers $1 \leqslant \sigma \leqslant n-2$, and therefore λ can get at most $2(n-2)$ concrete values.

From $C_j(p) = \sigma(p)\lambda(p)^j + \tau(p)\mu(p)^j$, we can see that $C_j(p)$ for $j > 2$ can get at most $2(n-2)$ concrete values. The function $C_j(p)$ is smooth, so it is continuous, and therefore it can get exactly one value, hence C_j are necessarily constant. Thus, all functions (σ, τ, λ, and μ) are constant, which proves that the manifold is globally Osserman. □

It is worth noting that our theorem holds without diagonalizability for the reduced Jacobi operator from the definition of the two-leaves Osserman (Definition 2.1). It is clear that our proof does not depend on the possible Jordan blocks, so we should check only cases with a complex eigenvalue. The existence of complex zero

$\alpha(p)+i\beta(p)$ of the characteristic polynomial of the reduced Jacobi operator implies that its conjugate $\alpha(p)-i\beta(p)$ is also a zero with the same multiplicity $\frac{n-1}{2}$.

At the first sight, we can notice that n is odd, so the signature is not neutral and in a case of Jordan–Osserman manifold, by [**15**] we have a diagonalizable case anyway. Of course, we can do straightforward calculations to get constant $C_1=(n-1)\alpha(p)$ and $C_2=(n-1)(\alpha(p)^2-\beta(p)^2)$, and therefore $\alpha(p)$ and $\beta(p)$ are constant, too. Constant α and β give constant eigenvalues and hence we have a globally Osserman manifold.

Since we proved that the strong duality principle holds for pointwise Osserman manifold of dimension $n \leqslant 4$ [**2**], Theorem 2.1 said that a globally Osserman condition can not help us in the case of the two-leaves Osserman manifold. This is why we can put the problem into a pure algebraic concept with an algebraic Osserman curvature tensor instead of working with an Osserman manifold and the associated tangent bundles.

3. Special Osserman manifolds

The diagonalizability and the fact that $\mathcal{J}_X$ has only two eigenvalues ($\varepsilon_X\lambda$ and $\varepsilon_X\mu$) gives an orthogonal decomposition of the vector space $\mathcal{V}$ for every nonnull X:

$$\mathcal{V}=\mathrm{Span}\{X\}\oplus\mathrm{Ker}(\tilde{\mathcal{J}}_X-\varepsilon_X\lambda\,\mathrm{Id})\oplus\mathrm{Ker}(\tilde{\mathcal{J}}_X-\varepsilon_X\mu\,\mathrm{Id})$$

Let us introduce the following short notation for some important subspaces and their dimensions.

$$\begin{aligned}\mathcal{L}(X)&=\mathrm{Ker}(\tilde{\mathcal{J}}_X-\varepsilon_X\lambda\,\mathrm{Id}), &\dim\mathcal{L}(X)&=\tau\\ \mathcal{M}(X)&=\mathrm{Ker}(\tilde{\mathcal{J}}_X-\varepsilon_X\mu\,\mathrm{Id}), &\dim\mathcal{M}(X)&=\sigma\\ \mathcal{U}(X)&=\mathrm{Span}\{X\}\oplus\mathcal{L}(X), &\dim\mathcal{U}(X)&=\tau+1=n-\sigma\end{aligned}$$

Since each eigenspace of a self-adjoint diagonalizable linear operator is nondegenerate, all mentioned subspaces are nondegenerate. The previous decomposition can be written as

$$\mathcal{V}=\mathrm{Span}\{X\}\oplus\mathcal{L}(X)\oplus\mathcal{M}(X)=\mathcal{U}(X)\oplus\mathcal{M}(X),$$

and an arbitrary $Y\in\mathcal{V}$ can be decomposed as

$$Y=\xi X+Y_L+Y_M,$$

where $Y_L\in\mathcal{L}(X)$ and $Y_M\in\mathcal{M}(X)$. The vectors X, Y_L, and Y_M are all mutually orthogonal, and we denote relevant projections by

$$\Pi_{\mathcal{L}}(X,Y)=Y_L,\ \Pi_{\mathcal{M}}(X,Y)=Y_M,\ \Pi_{\mathcal{U}}(X,Y)=\xi X+Y_L.$$

Many known examples of the two-leaves Osserman curvature tensors have something in common, which motivates us to introduce some additional conditions.

Definition 3.1 (Quasi-special Osserman). We say that R is quasi-special Osserman if it is two-leaves Osserman and for all nonnull $X,Y\in\mathcal{V}$ there holds

$$Y\in\mathcal{U}(X)\Rightarrow\mathcal{U}(X)=\mathcal{U}(Y). \tag{3.1}$$

Definition 3.2 (Special Osserman). We say that R is special Osserman if it is quasi-special Osserman and for all nonnull $X, Z \in \mathcal{V}$ there holds

$$Z \in \mathcal{M}(X) \Rightarrow X \in \mathcal{M}(Z). \tag{3.2}$$

García-Río and Vázquez-Lorenzo introduced the concept of special Osserman manifolds [**8**, **13**]. In our terminology their original definition can be viewed as Definition 3.2, which means that a manifold is two-leaves Osserman with the additional conditions (3.1) and (3.2). They managed to find the complete classification of special Osserman manifolds [**8**, **13**]. Let us recall this results.

Theorem 3.1 (García-Río, Vázquez-Lorenzo). *Complete and simply connected special Osserman manifold is isometric to one of the following:*
1) *an indefinite complex space form of signature* $(2k, 2r)$, $k, r \geqslant 0$,
2) *an indefinite quaternionic space form of signature* $(4k, 4r)$, $k, r \geqslant 0$,
3) *paracomplex space form of signature* (k, k)
4) *a paraquaternionic space form of signature* $(2k, 2k)$, *or*
5) *a Cayley plane over the octaves with definite or indefinite metric tensor, or a Cayley plane over the anti-octaves with indefinite metric tensor of signature* $(8, 8)$.

Let us remark that special Osserman curvature tensors are not the only examples of the two-leaves Osserman. For example, let $\mathcal{V}$ be a vector space furnished with a metric g and a quaternionic structure, where $\{J_1, J_2, J_3 = J_1 J_2\}$ is a canonical basis of that quaternionic structure. Let us define a curvature operator by $\mathcal{R} = \mathcal{R}^{J_1} + \mathcal{R}^{J_2}$, where

$$\mathcal{R}^J(X, Y)Z = g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ$$

presents one of the first examples of an Osserman curvature operator. The associated curvature tensor of so defined $\mathcal{R}$ is a diagonalizable Osserman, whose reduced Jacobi operator has exactly two distinct eigenvalues: $\lambda = -3$ (with multiplicity $\tau = 2$) and $\mu = 0$. Therefore it is the two-leaves Osserman curvature tensor. However, since for every unit X

$$\mathcal{U}(X) = \mathrm{Span}\{X, J_1X, J_2X\} \neq \mathrm{Span}\{X, J_1X, J_3X\} = \mathcal{U}(J_1X),$$

it is not quasi-special Osserman.

At the first sight it seems that there are too many conditions in the definition of a special Osserman curvature tensor. However, additional condition (3.2) which made quasi-special Osserman to be special Osserman, is the duality principle for the value μ. Let us note that eigenspaces $\mathcal{L}(X)$ and $\mathcal{M}(X)$ play different roles now, and that the duality principle for the value λ is already included in condition (3.1).

The main goal in our previous work [**1**,**4**] was to examine if quasi-special Osserman curvature tensors are necessarily special Osserman. Can we prove that strong quasi-special Osserman condition implies the duality principle? Unfortunately, we have not completely answered this question.

We managed to prove that if there is a counterexample, then we have a nonnull A and nonnull $B, C \in \mathcal{M}(A)$ such that the subspaces $\mathcal{H} = \{\Pi_{\mathcal{M}}(B, X) : X \in \mathcal{U}(C)\}$, and $\mathcal{Z} = \mathcal{U}(B) \cap \mathcal{U}(C)$ are totally isotropic spaces (they consist just of null vectors) of the same dimension $\dim \mathcal{Z} = \dim \mathcal{H} = \frac{\tau+1}{2}$. For final result from [**1**, **4**] we introduce the following definition.

Definition 3.3 (Almost-special Osserman). We say that R is almost-special Osserman if it is two-leaves Osserman and for all nonnull $X, Y \in \mathcal{V}$ holds

$$\mathcal{U}(X) \cap \mathcal{U}(Y) \neq 0 \Rightarrow \mathcal{U}(X) = \mathcal{U}(Y).$$

Any special Osserman curvature tensor is an almost-special Osserman, and any almost-special Osserman curvature tensor is a quasi-special Osserman. An almost-special Osserman curvature tensor allows only trivial intersections of $\mathcal{U}$ spaces, and consequently $\dim \mathcal{Z} = 0$, so the following theorem holds.

Theorem 3.2. *Almost-special Osserman curvature tensor is special Osserman.*

At the end, let us give our final remark. The duality principle for an Osserman curvature tensor is a very hard problem. We do not know any counterexample, although we failed to prove the duality principle, even under very strong additional conditions. It seems that a solution of our quasi-special Osserman problem is not far away, because of the strange looking relation $\dim \mathcal{Z} = \dim \mathcal{H} = \frac{\tau+1}{2}$. For example, it immediately solves the problem in the case of even τ. This is why we hope for an affirmative answer of the problem in our future investigations.

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