PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 94 (108) (2013), 219–228 DOI: 10.2298/PIM1308219P

A NOTE ON CURVATURE-LIKE INVARIANTS OF SOME CONNECTIONS ON LOCALLY DECOMPOSABLE SPACES

Nevena Pušić

Abstract. We consider an n-dimensional locally product space with p and q dimensional components $(p+q=n)$ with parallel structure tensor, which means that such a space is locally decomposable. If we introduce a conformal transformation on such a space, it will have an invariant curvature-type tensor, the so-called product conformal curvature tensor (PC-tensor). Here we consider two connections, (F,g)-holomorphically semisymmetric one and F-holomorphically semisymmetric one, both with gradient generators. They both have curvature-like invariants and they are both equal to PC-tensor.

1. Introduction

In [8], we considered conformal transformations on anti-Kähler spaces (also called Kähler spaces with Norden metrics or B-spaces). Also, we have considered two kinds of holomorphically semi-symmetric connections: one of them is a metric and F-connection and the other one is just an F-connection. We have proved that both of these connections have the same curvature-like invariant, which is equal to one of conformal invariants on such spaces. It was a geometrical motivation for such a consideration on a locally product (decomposable) space.

As we know, a locally product space is an n-dimensional manifold M_n with a (positive definite, but not necessarily) Riemannian metric (g_{ij}) and with structure tensor field $F_j^i \neq \delta_j^i$, satisfying the conditions $F_s^i F_j^s = \delta_j^i$, $g_{st} F_i^s F_j^t = g_{ij}$, where ∇ is the Levi-Civita connection from g. If we put $F_j^s g_{is} = F_{ij}$, then it is clear that $F_{ij} = F_{ji}$ (from the previous formula we are getting $g_{ij} = F_{ti} F_j^t = F_{tj} F_i^t$, then $g_{ij} F_l^j = F_{ti} F_j^t F_l^j$ and, consequently, $F_{il} = F_{ti} \delta_l^t = F_{li}$). There also can hold $\nabla_k F_j^i = 0$ and, consequently, $\nabla_k F_{ij} = 0$. We shall explain such a case later.

At any neighborhood of any point of a locally product space, if it is a (pseudo) Riemannian space, there exists a coordinate system, called a *separating coordinate system*; we can express the metric tensor in such a coordinate system in the

2010 *Mathematics Subject Classification:* Primary 53A30; Secondary 53A40, 53B15.

Key words and phrases: locally product space, conformal transformation, $PC-$ curvature tensor, class of holomorphically semisymmetric connections, Kähler-type identities.

following way

$$ds^2 = g_{\alpha\beta}(x^i)dx^\alpha dx^\beta + g_{rs}(x^i)\,dx^r dx^s,$$

where $\alpha, \beta = 1, \dots, p$; $r, s = p+1, \dots, p+q = n$ $(n = \dim M_n)$, $i = 1, \dots, n$ or, equivalently

$$(g_{ij}) = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{rs} \end{bmatrix}$$

and then its tangent space is a product of two tangent subspaces: $M_n = M_p \times M_q$. The structure tensor satisfies $F^2 = I$. In the separating coordinate system, it shall have the form, by definition [10]

$$F^i_j = \begin{bmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^r_s \end{bmatrix}$$

or, for its covariant form

$$F_{ij} = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & -g_{rs} \end{bmatrix}.$$

It is not hard to prove that, $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$ $(\alpha, \beta, \gamma = 1, \dots, p)$ and $g_{rs} = g_{rs}(x^t)$ $(r, s, t = p+1, \dots, n)$ is equivalent to $\nabla_k F^i_j = 0$ or $\nabla_k F_{ij} = 0$. In such a case the space M_n is called a *locally decomposable space*, because it can be divided into two naturally defined subspaces.

The choice of metric tensor on a locally product space in such a coordinate system (separating) gives us the form of the covariant structure tensor automatically.

A product conformal transformation [1, 6, 8, 9] is a transformation of the metric of a locally product space, given by

$$\overline{g}_{ij} = \rho g_{ij} + \sigma F_{ij}, \tag{1.1}$$

where ρ and σ are scalar functions satisfying

$$\rho_i = \sigma_a F^a_i, \quad \rho^2 - \sigma^2 \neq 0, \tag{1.2}$$

for their partial derivatives ρ_i and σ_i. For details, the author recommends to see [1]. The geometric interpretation of a PC-transformation is a pair of conformal transformations, each of them acting on one of the subspaces M_p or M_q. Then it is not difficult to show that Christoffel symbols of the metric (1.1) are

$$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \delta^i_j p_k + \delta^i_k p_j - g_{jk} p^i + F^i_j q_k + F^i_k q_j - F_{kj} q^i,$$

where

$$p_i = \frac{\rho\rho_i - \sigma\sigma_i}{2(\rho^2 - \sigma^2)}, \quad q_i = \frac{\rho\sigma_i - \sigma\rho_i}{2(\rho^2 - \sigma^2)}, \tag{1.3}$$

which is the consequence of (1.1) and (1.2). If (1.2) were not satisfied, both vectors in the upper equality would be zero vectors. It can be obtained by calculating Christoffel symbols defined by metric (1.1). Then, one can show that the tensor

$$\begin{aligned} PC_{ijkl} = K_{ijkl} &+ \alpha_2 s_{ijkl} + \beta_2 \widetilde{s}_{ijkl} \\ &- 2\big[(\alpha_1\alpha_2 + \beta_1\beta_2)K + (\alpha_1\beta_2 + \alpha_2\beta_1)\overline{K}\,\big] r_{ijkl} \\ &- 2\big[(\alpha_1\beta_2 + \alpha_2\beta_1)K + (\alpha_1\alpha_2 + \beta_1\beta_2)\overline{K}\,\big] \widetilde{r}_{ijkl}, \end{aligned}$$

where K_{ijkl} is the Riemann–Christoffel tensor of Levi-Civita connection, K_{jk} is the Ricci tensor of the same connection, K is its scalar curvature, $\overline{K}_{kj} = K_{ks}F^s_j$, $\overline{K} = \overline{K}_{kj}g^{kj}$ and

$$\begin{aligned}
r_{ijkl} &= g_{ik}g_{jl} - g_{il}g_{jk} + F_{ik}F_{jl} - F_{il}F_{jk},\\
\widetilde{r}_{ijkl} &= F^t_i r_{tjkl},\\
s_{ijkl} &= K_{jl}g_{ik} - K_{jk}g_{il} + K_{ik}g_{jl} - K_{il}g_{jk} + \overline{K}_{jl}F_{ik}\\
&\quad - \overline{K}_{jk}F_{il} + \overline{K}_{ik}F_{jl} - \overline{K}_{il}F_{jk},\\
\widetilde{s}_{ijkl} &= F^t_i s_{tjkl},\\
\alpha_1 &= \frac{n-2}{2[(n-2)^2 - \psi^2]}, \quad \beta_1 = -\frac{\psi}{2[(n-2)^2 - \psi^2]},\\
\alpha_2 &= \frac{n-4}{(n-4)^2 - \psi^2}, \quad \beta_2 = -\frac{\psi}{(n-4)^2 - \psi^2},\\
\psi &= p - q,
\end{aligned}$$

do not depend on the choice of the functions σ and ρ. This tensor is common for all PC-transformations and it is called a product conformal curvature tensor or a PC-tensor. For more details about locally product and locally decomposable spaces, the author recommends to consult [10].

In this paper, we shall consider two kinds of so-called holomorphically semi-symmetric connection on locally decomposable spaces. Originally, a semi-symmetric connection was considered on a Riemannian space, as a connection with torsion tensor which is equal to $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = p_j\delta^i_k - p_k\delta^i_j$. The generalization of such a connection on the spaces with symmetric structure will be holomorphically semi-symmetric connection, with the torsion tensor given by

$$T^i_{jk} = p_j\delta^i_k - p_k\delta^i_j + q_jF^i_k - q_kF^i_j,$$

where the vector q_i is the image of the generator by the structure. Both the metric and the structure tensor will be parallel with respect to the connection with coefficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k - p^i g_{jk} + q_jF^i_k - q^iF_{jk} \tag{1.4}$$

and we shall call this connection an (F,g)-holomorphically semi-symmetric connection. The other one will be

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k + p^i g_{jk} + q_jF^i_k + q^iF_{jk}. \tag{1.5}$$

As just the structure tensor is parallel towards the connection (1.5), we shall call it an F-holomorphically semi-symmetric connection. For more details about such kind of connection, it may be useful to consult [5]. Also, similar problems have been discussed in papers [2,3,4,7].

It is not difficult to prove that the Riemann–Christoffel tensor of a locally decomposable space satisfies the condition of Kähler type

$$K_{ijkl} = K_{abkl}F^a_iF^b_j$$

using the Ricci identity for the structure tensor and Levi-Civita connection. There also holds

$$K_{ijkl} = K_{iabl} F_j^a F_k^b, \tag{1.6}$$

which can be proved using the first Bianchi identity for the same connection. These identities are analogous to those which have been used in [6]. The identity analogous to (1.6), but with minus on the right-hand side is valid on anti-Kähler spaces.

In this paper, we shall use mostly the covariant components of vectors. These are, in fact, components of co-vectors, but as we can lower any upper index using contraction by a component of the metric tensor, we shall still consider and call them components of vectors.

The vector p_j is *the generator of both these connections*. The vector q_j is *its image by the structure*.

2. Curvature tensor and curvature-type invariant of (F, g)-holomorphically semi-symmetric connection.

Now we shall calculate the curvature tensor of the (F, g)-holomorphically semi-symmetric connection. The components of such a connection are given by (1.4). If we calculate the component of its curvature tensor, we obtain, after lowering the upper index

$$R_{ijkl} = K_{ijkl} + g_{ik}p_{lj} - g_{il}p_{kj} + g_{jl}p_{ki} - g_{jk}p_{li} + F_{ik}q_{lj} - F_{il}q_{kj} + F_{jl}q_{ki} - F_{jk}q_{li}, \tag{2.1}$$

where we introduce the abbreviations p_{kj} and q_{kj} for tensors

$$p_{kj} = \nabla_k p_j - p_k p_j - q_k q_j + \tfrac{1}{2} p_s p^s g_{kj} + \tfrac{1}{2} p_s q^s F_{kj}, \tag{2.2}$$

$$q_{kj} = \nabla_k q_j - p_k q_j - q_k p_j + \tfrac{1}{2} p_s p^s F_{kj} + \tfrac{1}{2} p_s q^s g_{kj}. \tag{2.3}$$

We have got these expressions in the process of calculation of the components R_{ijkl}. It is obvious that $q_{kj} = p_{ka} F_j^a$.

Now we want tensor (2.1) to satisfy standard algebraic conditions for a curvature tensor. It is obvious that it is skew-symmetric in the last two indices (just this condition is satisfied automatically). It will also be skew-symmetric in the first two indices, which can be easily checked. If we want it to be invariant under changing of places of the first and the second pair of indices, then there must hold

$$\begin{aligned} 0 = {} & g_{ik}(p_{lj} - p_{jl}) - g_{il}(p_{kj} - p_{jk}) + g_{jl}(p_{ki} - p_{ik}) - g_{jk}(p_{li} - p_{il}) \\ & + F_{ik}(q_{lj} - q_{jl}) - F_{il}(q_{kj} - q_{jk}) + F_{jl}(q_{ki} - q_{ik}) - F_{jk}(q_{li} - q_{il}). \end{aligned} \tag{2.4}$$

After contraction of the upper equation by g^{ik}, we obtain

$$(n-3)(p_{lj} - p_{jl}) + \psi(q_{lj} - q_{jl}) - F_j^a F_l^b (p_{ba} - p_{ab}) = 0. \tag{2.5}$$

We can see that, if p_{lj} is a symmetric tensor, then the tensor q_{lj} is also symmetric. So, if we take into account (2.2) and (2.3), it is easy to see that, if the generator p_i is a gradient, then its image by the structure is also a gradient.

It is not difficult to prove that the generator of connection (1.4) must be a gradient if its curvature tensor is invariant under changing places of the first and the second pair of indices. There is an exception just for cases of low-dimensional subspaces. If we transvect (2.4) by F^{ik}, we obtain

$$(n-3)(q_{lj}-q_{jl})+\psi(p_{lj}-p_{jl}) = F^a_j F^b_l (q_{ba}-q_{ab}). \tag{2.6}$$

From (2.5), it is easy to get

$$p_{ba}-p_{ab} = \frac{1}{n-3}F^s_a F^r_b (p_{rs}-p_{sr}) - \frac{\psi}{n-3}(q_{ba}-q_{ab}).$$

If we transvect the last equation by $F^a_j F^b_l$, we can obtain, using (2.5)

$$\begin{aligned} F^a_j F^b_l (p_{ba}-p_{ab}) &= \frac{1}{n-3}(p_{lj}-p_{jl}) - \frac{\psi}{n-3}F^a_j F^b_l (q_{ba}-q_{ab}) \\ &= (n-3)(p_{lj}-p_{jl}) + \psi(q_{lj}-q_{jl}). \end{aligned}$$

The consequence of the last equation will be

$$p_{lj}-p_{jl} = -\frac{\psi(n-3)}{(n-2)(n-4)}(q_{lj}-q_{jl}) - \frac{\psi}{(n-2)(n-4)}F^a_j F^b_l (q_{ba}-q_{ab}). \tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$q_{lj}-q_{jl} = \frac{1}{n-3}\frac{(n-2)(n-4)+\psi^2}{(n-2)(n-4)-\psi^2}F^a_j F^b_l (q_{ba}-q_{ab}).$$

The last equation deals with a recurrent relation. Using it once again on the right-hand side, we obtain

$$q_{lj}-q_{jl} = \frac{1}{(n-3)^2}\left[\frac{(n-2)(n-4)+\psi^2}{(n-2)(n-4)-\psi^2}\right]^2 (q_{lj}-q_{jl}).$$

Here we have three possibilities

$$(1)\ \ \frac{(n-2)(n-4)+\psi^2}{(n-2)(n-4)-\psi^2} = n-3, \quad \text{or} \quad (2)\ \ \frac{(n-2)(n-4)+\psi^2}{(n-2)(n-4)-\psi^2} = -(n-3),$$
$$\text{or} \quad (3)\ \ q_{lj} = q_{jl}.$$

From the first possibility, we obtain $\psi = \pm(n-4)$. From the second possibility, we obtain $\psi = \pm(n-2)$. This means

(1) $p = n-2, \quad q = 2 \quad$ or $\quad p = 2, \quad q = n-2$;
(2) $p = n-1, \quad q = 1 \quad$ or $\quad p = 1, \quad q = n-1$.

In case 3, it is easy to notice that q_{lj} is symmetric if and only if the vector q_i is a gradient. If we use (2.6) and if the vector q_i is a gradient, then the tensor p_{lj} is also symmetric and it is true, according to (2.2), if and only if the generator of considered connection is a gradient. So, we have proved

THEOREM 2.1. *If the curvature tensor of* (F,g)*-holomorphically semi-symmetric connection* (1.4) *on a locally decomposable space is invariant under changing places of the first and the second pair of indices, then the generator of such a connection is a gradient automatically, except if the dimension of one of space components is* 1 *or* 2*. If the generator is a gradient, then its image by the structure is also a gradient.*

In our following considerations, we shall presume that both the generator and its image by the structure are gradients. The best way to prove it is to presume that $p > 2$, $q > 2$, like in [6]. Then the first Bianchi identity for such a connection will be satisfied automatically.

Now we can transvect (2.1) by g^{il} and obtain

$$R_{jk} = K_{jk} - (n-4)p_{kj} - p^s_s g_{kj} - \psi q_{kj} - F_{jk} q^s_s. \tag{2.8}$$

We shall define $\overline{K}_{jk} = K_{ijkl}F^{il}$ and $\overline{R}_{jk} = R_{ijkl}F^{il}$. Then, we can transvect (2.1) by F^{il} and obtain

$$\overline{R}_{jk} = \overline{K}_{jk} - (n-4)q_{kj} - q^s_s g_{kj} - \psi p_{kj} - F_{jk} p^s_s, \tag{2.9}$$

The scalar functions p^s_s and q^s_s are still unknown. We shall find their form by multiplying both (2.8) and (2.9) by g^{jk} and by contracting these expressions. We shall obtain two new expressions of scalar type:

$$\begin{aligned} 2(n-2)p^s_s + 2\psi q^s_s &= K - R, \\ 2\psi p^s_s + 2(n-2)q^s_s &= \overline{K} - \overline{R}, \end{aligned}$$

where $\overline{K} = \overline{K}_{jk}g^{jk}$ and $\overline{R} = \overline{R}_{jk}g^{jk}$.

Now we are going to solve this system of equations. We obtain

$$\begin{aligned} p^s_s &= \frac{n-2}{2[(n-2)^2 - \psi^2]}(K - R) - \frac{\psi}{2[(n-2)^2 - \psi^2]}(\overline{K} - \overline{R}), \\ q^s_s &= \frac{n-2}{2[(n-2)^2 - \psi^2]}(\overline{K} - \overline{R}) - \frac{\psi}{2[(n-2)^2 - \psi^2]}(K - R). \end{aligned}$$

If we use the following abbreviations

$$\alpha_1 = \frac{n-2}{2[(n-2)^2 - \psi^2]}, \quad \beta_1 = -\frac{\psi}{2[(n-2)^2 - \psi^2]},$$

then

$$p^s_s = \alpha_1(K - R) + \beta_1(\overline{K} - \overline{R}), \quad q^s_s = \alpha_1(\overline{K} - \overline{R}) + \beta_1(K - R),$$

If we substitute this into (2.8) and (2.9), we obtain that

$$\begin{aligned} (n-4)p_{kj} + \psi q_{kj} &= K_{jk} - R_{jk} - [\alpha_1(K - R) + \beta_1(\overline{K} - \overline{R})]g_{jk} \\ &\quad - [\alpha_1(\overline{K} - \overline{R}) + \beta_1(K - R)]F_{jk}, \\ \psi p_{kj} + (n-4)q_{kj} &= \overline{K}_{jk} - \overline{R}_{jk} - [\alpha_1(\overline{K} - \overline{R}) + \beta_1(K - R)]g_{jk} \\ &\quad - [\alpha_1(\overline{K} - \overline{R}) + \beta_1(K - R)]g_{jk} \end{aligned}$$

If we multiply the first of the upper two equations by $n-4$ and the second one by $-\psi$, add results and put new abbreviations

$$\alpha_2 = \frac{n-4}{(n-4)^2 - \psi^2}, \quad \beta_2 = \frac{-\psi}{(n-4)^2 - \psi^2},$$

we obtain

$$
\begin{aligned}
p_{kj} = \alpha_2\big[K_{jk} - R_{jk} &- [\alpha_1(K-R) + \beta_1(\overline{K}-\overline{R})]g_{jk} \\
&- [\alpha_1(\overline{K}-\overline{R}) + \beta_1(K-R)]F_{jk}\big] \\
+ \beta_2\big[\overline{K}_{jk} - \overline{R}_{jk} &- [\alpha_1(\overline{K}-\overline{R}) + \beta_1(K-R)]g_{jk} \\
&- [\alpha_1(K-R) + \beta_1(\overline{K}-\overline{R})]g_{jk}\big].
\end{aligned} \tag{2.10}
$$

As we have to calculate $q_{kj} = p_{ka}F^a_j$, we have to notice that

$$\overline{K}_{jk}F^k_t = K_{ijkl}F^{il}F^k_t = K_{ijtk}F^{il}F^k_l = K_{ijtk}g^{ik} = K_{jt}.$$

Using equality (2.1), it is easy to prove $R_{ijkl} = R_{abkl}F^a_iF^b_j$. If we use the fact that this curvature tensor is invariant under changing places of the first and the second pair of indices, then we have $R_{ijkl} = R_{ijab}F^a_kF^b_l$. Then

$$\overline{R}_{jk}F^k_t = R_{ijkl}F^{il}F^k_t = R_{ijtk}F^{il}F^k_l = R_{ijtk}g^{ik} = R_{jt}.$$

Then, it is easy to calculate the tensor q_{kj} using (2.10). We have

$$
\begin{aligned}
q_{kj} = \alpha_2\big[\overline{K}_{jk} - \overline{R}_{jk} &- [\alpha_1(K-R) + \beta_1(\overline{K}-\overline{R})]F_{jk} \\
&- [\alpha_1(\overline{K}-\overline{R}) + \beta_1(K-R)]g_{jk}\big] \\
+ \beta_2\big[K_{jk} - R_{jk} &- [\alpha_1(\overline{K}-\overline{R}) + \beta_1(K-R)]F_{jk} \\
&- [\alpha_1(K-R) + \beta_1(\overline{K}-\overline{R})]g_{jk}\big].
\end{aligned} \tag{2.11}
$$

Substituting (2.10) and (2.11) into (2.1), we see that the tensor

$$
\begin{aligned}
K_{ijkl} &- \big[(\alpha_2K_{jk} + \beta_2\overline{K}_{jk})g_{il} - (\alpha_2K_{jl} + \beta_2\overline{K}_{jl})g_{ik} \\
&\qquad + (\alpha_2K_{il} + \beta_2\overline{K}_{il})g_{jk} - (\alpha_2K_{ik} + \beta_2\overline{K}_{ik})g_{jl}\big] \\
&- \big[(\alpha_2\overline{K}_{jk} + \beta_2K_{jk})F_{il} - (\alpha_2\overline{K}_{jl} + \beta_2K_{jl})F_{ik} \\
&\qquad + (\alpha_2\overline{K}_{il} + \beta_2K_{il})F_{jk} - (\alpha_2\overline{K}_{ik} + \beta_2K_{ik})F_{jl}\big] \\
&+ 2\big[(\alpha_1\alpha_2 + \beta_1\beta_2)K + (\alpha_2\beta_1 + \beta_2\alpha_1)\overline{K}\big](g_{il}g_{jk} - g_{ik}g_{il} + F_{il}F_{jk} - F_{ik}F_{jl}) \\
&+ 2\big[(\alpha_1\alpha_2 + \beta_1\beta_2)\overline{K} + (\alpha_2\beta_1 + \beta_2\alpha_1)K\big](F_{il}g_{jk} - F_{ik}g_{jl} + F_{jk}g_{li} - F_{jl}g_{ki})
\end{aligned}
$$

is identical to the tensor which is constructed in the same way by using the corresponding curvature elements of the (F,g)-holomorphically semi-symmetric connection. Also, it is easy to transform the upper tensor to the form which is identic to (1.3). So, we have proved

THEOREM 2.2. *If the curvature tensor of the (F,g)-holomorphically semi-symmetric connection on a locally decomposable space is invariant under changing places of the first and the second pair of indices, then such a connection has a curvature-type tensor which is equal to the product conformal curvature tensor of such a space and, consequently, it does not depend on the choice of the generator.*

3. Curvature tensor and curvature-type invariant of F-holomorphically semi-symmetric connection on a locally decomposable space

Now we shall calculate components of curvature tensor of the F-holomorphically semi-symmetric connection; its components are given by (1.5). After lowering the upper index, we obtain

$$R_{ijkl} = K_{ijkl} + g_{ik}p_{lj} - g_{il}p_{kj} + g_{jk}\overline{p}_{li} - g_{jl}\overline{p}_{ki} + F_{ik}q_{lj} - F_{il}q_{kj} + F_{jk}\overline{q}_{li} - F_{jl}\overline{q}_{ki}, \tag{3.1}$$

where we induce abbreviations

$$p_{lj} = \nabla_l p_j - S_{lj}, \quad q_{lj} = p_{la}F_j^a, \quad \overline{p}_{li} = \nabla_l p_i + S_{li}, \quad q_{li} = p_{la}F_i^a,$$
$$S_{lj} = p_l p_j + q_l q_j + \tfrac{1}{2}p^s p_s g_{lj} + \tfrac{1}{2}p_s q^s F_{lj}, \quad S_{la}F_j^a = S_{ja}F_l^a.$$

Also the following relations hold and will be necessary for future calculations:

$$S_s^s = \frac{n+4}{2}p_s p^s + \frac{\psi}{2}p_s q^s, \quad S_{lj}F^{lj} = \frac{n+4}{2}p_s q^s + \frac{\psi}{2}p_s p^s, \quad S_{ab}F_l^a F_j^b = S_{lj}.$$

Suppose that curvature tensor (3.1) of such a connection is an algebraic curvature tensor (that means that it satisfies standard algebraic conditions for a curvature tensor) and that the generator p_i of such a connection is a gradient. If the curvature tensor of the connection (3.1) is skew-symmetric in the first two indices and if we take into account symmetry of the tensors S_{lj} and $S_{la}F_j^a$, we obtain

$$0 = g_{ik}\nabla_l p_j - g_{il}\nabla_k p_j + g_{jk}\nabla_l p_i - g_{jl}\nabla_k p_i + F_{ik}\nabla_l q_j - F_{il}\nabla_k q_j + F_{jk}\nabla_l q_i - F_{jl}\nabla_k q_i.$$

If we transvect the relation above by g^{ik}, we obtain

$$(n+1)\nabla_l p = g_{lj}\nabla_s p^s + F_{lj}\nabla_s q^s + F_l^k\nabla_k q_j - \psi\nabla_l q_j, \tag{3.2}$$

and if we transvect it by F^{ik}, we obtain

$$(n+1)\nabla_l q_j + \psi\nabla_j p_l - \nabla_j q_l = g_{lj}\nabla_s q^s + F_{lj}\nabla_s p^s, \tag{3.3}$$

as the generator is a gradient. Now we shall change places of the indices l and j i the upper equation and get

$$(n+1)\nabla_j q_l + \psi\nabla_l p_j - \nabla_l q_j = g_{jl}\nabla_s q^s + F_{jl}\nabla_s p^s.$$

The expression on the right-hand side of both two upper equations is symmetric. If we subtract the second one from the first one, we obtain

$$(n+2)(\nabla_l q_j - \nabla_j q_l) = 0.$$

So, we have proved

Lemma 3.1. *If the generator of F-holomorphically semi-symmetric connection on a locally decomposable space is a gradient and if its curvature tensor is skew-symmetric in the first two indices, then the generator's image by the structure is also a gradient.*

If the generator is a gradient, then the equations (3.2) and (3.3) can be simplified as

$$n\nabla_l p_j + \psi\nabla_l q_j = g_{lj}\nabla_s p^s + F_{lj}\nabla_s q^s,$$
$$\psi\nabla_l p + n\nabla_l q = g_{lj}\nabla_s q^s + F_{lj}\nabla_s p^s.$$

If we multiply the first equation by n, the second one by $-\psi$ and adding them, we obtain

$$\begin{aligned}\nabla_l p_j &= \frac{n\nabla_s p^s - \psi\nabla_s q^s}{n^2-\psi^2} g_{lj} + \frac{n\nabla_s q^s - \psi\nabla_s p^s}{n^2-\psi^2} F_{lj},\\ \nabla_l q_j &= \frac{n\nabla_s q^s - \psi\nabla_s p^s}{n^2-\psi^2} g_{lj} + \frac{n\nabla_s p^s - \psi\nabla_s q^s}{n^2-\psi^2} F_{lj}.\end{aligned}$$

There holds

Lemma 3.2. *If the generator of an F-holomorphically semi-symmetric connection on an almost product space is a gradient and if its curvature tensor is an algebraic curvature tensor, then covariant derivatives of the generator and its image by the structure can be expressed in such a form*

$$\nabla_l p_j = \alpha g_{lj} + \beta F_{lj}, \quad \nabla_l q_j = \beta g_{lj} + \alpha F_{lj}, \tag{3.4}$$

where α and β are scalar functions.

If we want curvature tensor (3.1) to be invariant under changing places of the first and the second pair of indices, we obtain

$$g_{jk}\nabla_l p_i - g_{li}\nabla_k p_j = F_{li}\nabla_k q_j - F_{jk}\nabla_l q_i.$$

It is easy to check that the upper equality will be satisfied authomatically, by the reason of holding of (3.4). It will not be difficult to prove that the curvature tensor of such a connection will also satisfy the first Bianchi identity, because the generator is a gradient and tensors S_{lj} and $S_{la}F^a_j$ are symmetric.

When we substitute the expressions (3.4) into (3.1), we obtain

$$\begin{aligned}R_{ijkl} = K_{ijkl} &- g_{ik}S_{lj} + g_{il}S_{kj} - g_{jl}S_{ki} + g_{jk}S_{li}\\ &- F_{ik}S_{la}F^a_j + F_{il}S_{ka}F^a_j - F_{jl}S_{ka}F^a_i + F_{jk}S_{la}F^a_i.\end{aligned}$$

We know that both tensors $S_{lj}, S_{la}F^a_j$ are symmetric. The only difference between this formula and formula (2.1) is the sign, but it is absolutely irrelevant. In the same manner like in the previous paragraph, we obtain

$$\begin{aligned}S_{kj} = \alpha_2\big[R_{jk} - K_{jk} &- [\alpha_1(R-K) + \beta_1(\overline{R}-\overline{K})]g_{jk}\\ &- [\alpha_1(\overline{R}-\overline{K}) + \beta_1(R-K)]F_{jk}\big]\\ + \beta_2\big[\overline{R}_{jk} - \overline{K}_{jk} &- [\alpha_1(\overline{R}-\overline{K}) + \beta_1(R-K)]g_{jk}\\ &- [\alpha_1(R-K) + \beta_1(\overline{R}-\overline{K})]g_{jk}\big]\end{aligned}$$

and, consequently

$$\begin{aligned}S_{ka}F^a_j = \alpha_2\big[\overline{R}_{jk} - \overline{K}_{jk} &- [\alpha_1(R-K) + \beta_1(\overline{R}-\overline{K})]F_{jk}\\ &- [\alpha_1(\overline{R}-\overline{K}) + \beta_1(R-K)]g_{jk}\big]\\ + \beta_2\big[R_{jk} - K_{jk} &- [\alpha_1(\overline{R}-\overline{K}) + \beta_1(R-K)]F_{jk}\\ &- [\alpha_1(R-K) + \beta_1(\overline{R}-\overline{K})]g_{jk}\big],\end{aligned}$$

as we can easily prove that the curvature tensor of F-holomorphically semi-symmetric connection, if it is an algebraic curvature tensor, satisfies the condition of Kähler type

$$R_{ijkl} = R_{ijab} F_k^a F_l^b$$

and that there, consequently, holds

$$\overline{R}_{jk} F_t^k = R_{ijkl} F^{il} F_t^k = R_{jt}.$$

So, we have proved that there holds

Theorem 3.1. *If the curvature tensor of an F-holomorphically semi-symmetric connection satisfies some standard algebraic conditions for a curvature tensor and if its generator is a gradient, then such a connection has a curvature-type invariant tensor (independent on the choice of the generator) which is equal to the product conformal curvature tensor.*

References

1. V. Petrović, *Product conformal and product concircular curvature tensors*, Rev. Res., Fac. Sci. Univ. Novi Sad **10** (1980), 137–143.
2. M. Prvanović, *Holomorphically projective transformations in a locally product space*, Math. Balkan. **1** (1971), 195–213.
3. ______, *Holomorphically semi-symmetric connections*, Rev. Res., Fac. Sci. Univ. Novi Sad **9** (1979), 91–99.
4. ______, *Product semi-symmetric connections of the locally decomposable Riemannian spaces*, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., N.S. **10** (1979), 17–27.
5. ______, *Some special product semi-symmetric and some special holomorphically semi-symmetric F-connections*, Publ. Inst. Math., Nouv. Sér. **35(49)** (1984), 139–152.
6. ______, *A note on a manifold whose product conformal curvature tensor is semi-symmetric*, Rev. Res. Fac. Sci. Univ. Novi Sad **25**(1) (1995), 179–199.
7. ______, *Locally decomposable Riemannian manifold endowed with some semi-symmetric F-connections*, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Nat. N.S. **22** (1997), 45–56.
8. M. Prvanović, N. Pušić, *Some conformally invariant tensors on anti-Kähler manifolds and their geometrical properties*, Math. Pannonica, to appear.
9. N. Pušić, *On PC-parallel and PC-recurrent product spaces*, Rev. Res. Fac. Sci. Univ. Novi Sad **24**:1 (1994), 209–221.
10. K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, Macmillan Company, New York, 1965.

Department of Mathematics and Computer Science
Faculty of Science
University of Novi Sad
21000 Novi Sad
Serbia
nevena@dmi.uns.ac.rs