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FINITE DIFFERENCE APPROXIMATION OF A PARABOLIC PROBLEM WITH VARIABLE COEFFICIENTS

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Abstract. We study the convergence of a finite difference scheme that approximates the third initial-boundary-value problem for a parabolic equation with variable coefficients on a unit square. We assume that the generalized solution of the problem belongs to the Sobolev space $W_2^{s,s/2}$, $s \leqslant 3$. An almost second-order convergence rate estimate (with additional logarithmic factor) in the discrete $W_2^{1,1/2}$ norm is obtained. The result is based on some nonstandard a priori estimates involving fractional order discrete Sobolev norms.

1. Introduction

For a class of finite difference schemes (FDSs) approximating elliptic boundary-value problems (BVPs) with generalized solutions, convergence rate estimates compatible with the smoothness of the data

$$\|u - v\|_{W_p^k(\omega)} \leqslant Ch^{s-k}\|u\|_{W_p^s(\Omega)}, \qquad s \geqslant k, \tag{1.1}$$

are of great interest (see [**6, 13**]). Here $u = u(x)$ denotes the solution of the BVP, v denotes the solution of the corresponding FDS, h is the discretization parameter, $W_p^k(\omega)$ is the Sobolev space of mesh functions, and C is a positive generic constant, independent of h and u. In the parabolic case, instead of (1.1) it is natural to look for error bounds of the form

$$\|u - v\|_{W_p^{k,k/2}(Q_{h\tau})} \leqslant C\left(h^{s-k} + \tau^{\frac{s-k}{2}}\right)\|u\|_{W_p^{s,s/2}(Q)}, \qquad s \geqslant k, \tag{1.2}$$

where τ is the temporal mesh-size. A standard technique for establishing estimates of such types (see [**6, 13, 14**]) is based on the Bramble–Hilbert lemma [**3, 5**].

For BVPs with an oblique derivative boundary condition a loss of one half of an order in the convergence rate (usually $O(h^{3/2})$) is often observed, caused by

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the approximation of the boundary condition. Nevertheless, improved results are obtained in some cases, mainly for elliptic problems (see [**4, 7**]).

In the present paper, for the FDS approximating initial-boundary value problem (IBVP) for a parabolic equation with variable coefficients and an oblique derivative boundary condition a second order error bound in the discrete $W_2^{1,1/2}$ norm is obtained under minimal smoothness assumptions on the input data. The result is based on some nonstandard a priori estimates involving fractional order discrete Sobolev norms.

2. Formulation of the Problem

As the model problem we consider, in $Q = \Omega \times (0,T) = (0,1)^2 \times (0,T)$, the following initial-boundary value problem for a parabolic equation with variable coefficients:

$$\frac{\partial u}{\partial t} + Lu = f, \qquad (x,t) = (x_1, x_2, t) \in Q, \tag{2.1}$$

$$lu = 0, \qquad (x,t) \in \Gamma \times (0,T) = \partial\Omega \times (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{2.3}$$

where

$$Lu := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big), \quad lu := \sum_{i,j=1}^{2} a_{ij}\,\frac{\partial u}{\partial x_j}\,\cos(\nu, x_i) + \alpha u \tag{2.4}$$

and ν is the unit outward normal to Γ. We assume that the conditions of strong ellipticity are satisfied:

$$a_{ij} = a_{ij}(x) = a_{ji}, \qquad \alpha = \alpha(x),$$

$$c_0 \sum_{i=1}^{2} \xi_i^2 \leqslant \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \leqslant c_1 \sum_{i=1}^{2} \xi_i^2, \quad \forall x \in \bar{\Omega},\ \forall \xi \in \mathbb{R}^2, \quad c_i = \text{const.} > 0, \tag{2.5}$$

$$\alpha_0 \leqslant \alpha(x) \leqslant \alpha_1, \quad \alpha_i = \text{const.} > 0.$$

Let us denote $\Gamma = \bigcup_{i=1}^{2} \bigcup_{k=0}^{1} \Gamma_{ik}$, where $\Gamma_{1k} = \{k\} \times [0,1]$, $\Gamma_{2k} = [0,1] \times \{k\}$, and $\Sigma_{ik} = \Gamma_{ik} \times [0,T]$.

We also assume that the generalized solution of the problem (2.1)–(2.3) belongs to the Sobolev space $W_2^{s,s/2}(Q)$, $2 < s \leqslant 3$, while the data satisfy the following smoothness conditions: $a_{ij} \in W_2^{s-1}(\Omega)$, $\alpha \in W_2^{s-3/2}(\Gamma_{ik})$, $\alpha \in C(\Gamma)$, $f \in W_2^{s-2,s/2-1}(Q)$ and $u_0 \in W_2^{s-1}(\Omega)$.

3. Finite Difference Approximation

Let $n, m \in \mathbb{N}$, $n \geqslant 2$, $m \geqslant 1$, $h = 1/n$ and $\tau = T/m$. We consider the uniform spatial mesh $\bar{\omega}$ with mesh size h on $\bar{\Omega}$ and the uniform temporal mesh $\bar{\omega}_\tau$ with mesh size τ on $[0,T]$. We also denote $\omega = \bar{\omega} \cap \Omega$, $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \bar{\omega}_\tau \cap [0,T)$, $\omega_\tau^+ = \bar{\omega}_\tau \cap (0,T]$, $\gamma = \bar{\omega} \cap \Gamma$, $\bar{\gamma}_{ik} = \bar{\omega} \cap \Gamma_{ik}$, $\gamma_{ik} = \{x \in \bar{\gamma}_{ik} : 0 < x_{3-i} < 1\}$, $\gamma_{ik}^- = \{x \in \bar{\gamma}_{ik} : 0 \leqslant x_{3-i} < 1\}$, $\gamma_{ik}^+ = \{x \in \bar{\gamma}_{ik} : 0 < x_{3-i} \leqslant 1\}$, $\gamma_{ik}^\star = \bar{\gamma}_{ik} \setminus \gamma_{ik}$,

$\gamma^\star = \gamma \setminus \{\bigcup_{i,k}\gamma_{ik}\}$, $\sigma_{ik} = \gamma_{ik} \times \omega_\tau^+$, $\bar\sigma_{ik} = \bar\gamma_{ik} \times \omega_\tau^+$, $i = 1,2$, $k = 0,1$, and $\bar Q_{h\tau} = \bar\omega \times \bar\omega_\tau$.

The finite difference operators are defined in the usual manner [**12**]:

$$v_{x_i} = (v^{+i} - v)/h, \quad v_{\bar x_i} = (v - v^{-i})/h, \quad v_t = (\hat v - v)/\tau, \quad v_{\bar t} = (v - \check v)/\tau,$$

where $v^{\pm i}(x,t) = v(x \pm he_i, t)$, e_i is the unit vector of the axis x_i, $\hat v(x,t) = v(x, t+\tau)$ and $\check v(x,t) = v(x, t-\tau)$.

We also define the Steklov smoothing operators with the step sizes h and τ [**13**]:

$$T_i^+ f(x,t) = \int_0^1 f(x + hx'e_i, t)\,dx' = T_i^- f(x + he_i, t) = T_i f(x + \tfrac{h}{2}e_i, t),$$
$$T_i^{2\pm} f(x,t) = 2\int_0^1 (1 - x') f(x \pm hx'e_i)\,dx', \quad i = 1,2,$$
$$T_t^+ f(x,t) = \int_0^1 f(x, t + \tau t')\,dt' = T_t^- f(x, t+\tau) = T_t f(x, t + \tfrac{\tau}{2}).$$

These operators commute and transform derivatives into differences, for example:

$$T_i^+\left(\frac{\partial u}{\partial x_i}\right) = u_{x_i}, \quad T_i^-\left(\frac{\partial u}{\partial x_i}\right) = u_{\bar x_i}, \quad T_i^2\left(\frac{\partial^2 u}{\partial x_i^2}\right) = u_{\bar x_i x_i}, \quad i = 1,2,$$
$$T_t^+\left(\frac{\partial u}{\partial t}\right) = u_t, \quad T_t^-\left(\frac{\partial u}{\partial t}\right) = u_{\bar t}.$$

We approximate the IBVP (2.1)–(2.3) with the following implicit FDS:

$$\begin{aligned} v_{\bar t} + L_h v &= \tilde f, \qquad x \in \bar\omega, \quad t \in \omega_\tau^+, \\ v(x,0) &= u_0(x), \qquad x \in \bar\omega, \end{aligned} \tag{3.1}$$

where

$$L_h v = \begin{cases} -\frac{1}{2}\sum\limits_{i,j=1}^{2}\Big[\big(a_{ij}\,v_{x_j}\big)_{\bar x_i} + \big(a_{ij}\,v_{\bar x_j}\big)_{x_i}\Big], & x \in \omega \\ \frac{2}{h}\Big(-\frac{a_{11}+a_{11}^{+1}}{2}\,v_{x_1} - a_{12}\,\frac{v_{x_2}+v_{\bar x_2}}{2} + \tilde\alpha v\Big) - \big(a_{12}\,v_{\bar x_2}\big)_{x_1} \\ \qquad -\big(a_{21}\,v_{x_1}\big)_{\bar x_2} - \frac{1}{2}\big(a_{22}\,v_{x_2}\big)_{\bar x_2} - \frac{1}{2}\big(a_{22}\,v_{\bar x_2}\big)_{x_2}, & x \in \gamma_{10} \\ \frac{2}{h}\Big[-\frac{a_{11}+a_{11}^{+1}}{2}\,v_{x_1} - a_{12}v_{x_2} - a_{21}v_{x_1} - \frac{a_{22}+a_{22}^{+2}}{2}\,v_{x_2} \\ \qquad +(\tilde\alpha_1 + \tilde\alpha_2)v\Big], & x = (0,0) \\ \frac{2}{h}\Big[-\frac{a_{11}+a_{11}^{+1}}{2}\,v_{x_1} - a_{12}v_{\bar x_2} + a_{21}v_{x_1} + \frac{a_{22}+a_{22}^{-2}}{2}\,v_{\bar x_2} \\ \qquad +(\tilde\alpha_1 + \tilde\alpha_2)v\Big] - 2\big(a_{12}\,v_{\bar x_2}\big)_{x_1} - 2\big(a_{21}\,v_{x_1}\big)_{\bar x_2}, & x = (0,1) \\ \text{and analogously at the other boundary nodes,} & x \in \gamma \setminus \bar\gamma_{10} \end{cases}$$

$$\tilde{f} = \begin{cases} T_1^2 T_2^2 T_t^- f, & x \in \omega \\ T_i^{2\pm} T_{3-i}^2 T_t^- f, & x \in \gamma_{i,\, 0.5\mp 0.5} \\ T_1^{2\pm} T_2^{2\pm} T_t^- f, & x = (0.5 \mp 0.5,\, 0.5 \mp 0.5) \in \gamma^\star \end{cases}$$

$$\tilde{\alpha} = T_{3-i}^2 \alpha\,, \qquad x \in \gamma_{i0} \cup \gamma_{i1}\,, \qquad i = 1,\, 2$$

and

$$\tilde{\alpha}_i = T_i^{2\pm} \alpha\,, \quad x \in \gamma^\star, \quad x_i = 0.5 \mp 0.5\,, \quad i = 1,\, 2\,.$$

4. Error Analysis

Let u be the solution of the IBVP (2.1)-(2.3), and let v denote the solution of the FDS (3.1). The error $z = u - v$ is defined on $\bar{Q}_{h\tau}$ and satisfies the following conditions

$$\begin{gathered} z_{\bar{t}} + L_h z = \psi, \qquad x \in \bar{\omega}, \quad t \in \omega_\tau^+, \\ z(x,0) = 0, \qquad x \in \bar{\omega}, \end{gathered} \tag{4.1}$$

where

$$\psi = \begin{cases} \xi_{\bar{t}} + \sum_{i,j=1}^2 \eta_{ij,\, \bar{x}_i}\,, & x \in \omega \\ \tilde{\xi}_{\bar{t}} + \dfrac{2}{h}\,\eta_{11} + \dfrac{2}{h}\,\eta_{12} + \tilde{\eta}_{21,\, \bar{x}_2} + \tilde{\eta}_{22,\, \bar{x}_2} + \dfrac{2}{h}\,\zeta, & x \in \gamma_{10} \\ \tilde{\tilde{\xi}}_{\bar{t}} + \dfrac{2}{h}\,\tilde{\eta}_{11} + \dfrac{2}{h}\,\tilde{\eta}_{12} + \dfrac{2}{h}\,\tilde{\eta}_{21} + \dfrac{2}{h}\,\tilde{\eta}_{22} + \dfrac{2}{h}\,(\zeta_1 + \zeta_2)\,, & x = (0,0) \\ \text{and analogously at the other boundary nodes,} & x \in \gamma \setminus \gamma_{10}^- \end{cases}$$

$$\xi = u - T_1^2 T_2^2 u, \qquad x \in \omega$$

$$\tilde{\xi} = u - T_i^2 T_{3-i}^{2\pm} u, \qquad x \in \gamma_{3-i,\, 0.5\mp 0.5}$$

$$\tilde{\tilde{\xi}} = u - T_1^{2\pm} T_2^{2\pm} u, \qquad x = (0.5 \mp 0.5,\, 0.5 \mp 0.5) \in \gamma^\star$$

$$\eta_{ij} = T_i^+ T_{3-i}^2 T_t^- \Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big) - \frac{1}{2}\,\big(a_{ij} u_{x_j} + a_{ij}^{+i} u_{\bar{x}_j}^{+i}\big), \qquad x \in \omega$$

$$\tilde{\eta}_{ii} = T_i^+ T_{3-i}^{2\pm} T_t^- \Big(a_{ii}\,\frac{\partial u}{\partial x_i}\Big) - \frac{a_{ii} + a_{ii}^{+i}}{2}\,u_{x_i}, \qquad x \in \gamma_{3-i,\, 0.5\mp 0.5}^-$$

$$\tilde{\eta}_{i,3-i} = \begin{cases} T_i^+ T_{3-i}^{2+} T_t^- \Big(a_{i,3-i}\,\dfrac{\partial u}{\partial x_{3-i}}\Big) - a_{i,3-i}\,u_{x_{3-i}}, & x \in \gamma_{3-i,0}^- \\ T_i^+ T_{3-i}^{2-} T_t^- \Big(a_{i,3-i}\,\dfrac{\partial u}{\partial x_{3-i}}\Big) - a_{i,3-i}^{+i}\,u_{\bar{x}_{3-i}}^{+i}, & x \in \gamma_{3-i,1}^- \end{cases}$$

$$\zeta = (T_i^2\alpha)u - T_i^2T_t^-(\alpha u), \qquad x \in \gamma_{3-i,0} \cup \gamma_{3-i,1}$$

$$\zeta_i = (T_i^{2\pm}\alpha)u - T_i^{2\pm}T_t^-(\alpha u), \qquad x \in \gamma^\star, \quad x_i = 0.5 \mp 0.5.$$

We define the following discrete inner products and norms:

$$[v,w] = h^2\sum_{x\in\omega} v(x)w(x) + \frac{h^2}{2}\sum_{x\in\gamma\setminus\gamma^\star} v(x)w(x) + \frac{h^2}{4}\sum_{x\in\gamma^\star} v(x)w(x), \qquad |[v]|^2 = [v,v],$$

$$[v,w)_i = h^2\sum_{x\in\omega\cup\gamma_{i0}} v(x)w(x) + \frac{h^2}{2}\sum_{x\in\gamma^-_{3-i,0}\cup\gamma^-_{3-i,1}} v(x)w(x), \qquad |[v\|_i^2 = [v,v)_i,$$

$$(v,w]_i = h^2\sum_{x\in\omega\cup\gamma_{i1}} v(x)w(x) + \frac{h^2}{2}\sum_{x\in\gamma^+_{3-i,0}\cup\gamma^+_{3-i,1}} v(x)w(x), \qquad \|v]|_i^2 = (v,v]_i,$$

$$[v,w) = h^2\sum_{x\in\omega\cup\gamma^-_{10}\cup\gamma^-_{20}} v(x)w(x), \qquad |[v\|^2 = [v,v), \qquad (v,w] = h^2\sum_{x\in\omega\cup\gamma^+_{11}\cup\gamma^+_{21}} v(x)w(x),$$

$$\|v]|^2 = (v,v], \qquad |[v]|^2_{W_2^1(\bar\omega)} = |[v]|^2 + |[v_{x_1}||_1^2 + |[v_{x_2}\|_2^2, \qquad |[v]|_{C(\bar\omega)} = \max_{x\in\bar\omega}|v(x)|,$$

$$[v,w]_{\bar\gamma_{ik}} = h\sum_{x\in\gamma_{ik}} v(x)w(x) + \frac{h}{2}\sum_{x\in\gamma^\star_{ik}} v(x)w(x), \qquad |[v]|^2_{\bar\gamma_{ik}} = [v,v]_{\bar\gamma_{ik}},$$

$$[v,w)_{\gamma^-_{ik}} = h\sum_{x\in\gamma^-_{ik}} v(x)w(x), \qquad |[v||^2_{\gamma^-_{ik}} = [v,v)_{\gamma^-_{ik}}, \qquad \|v\|^2_{\gamma_{ik}} = h\sum_{x\in\gamma_{ik}} v^2(x),$$

$$|v|^2_{W_2^{1/2}(\gamma^-_{ik})} = h^2\sum_{x,\,x'\in\gamma^-_{ik},\ x'\neq x}\left[\frac{v(x)-v(x')}{x_{3-i}-x'_{3-i}}\right]^2, \qquad |[v\|^2_{W_2^{1/2}(\gamma^-_{ik})} = |v|^2_{W_2^{1/2}(\gamma^-_{ik})} + |[v\|^2_{\gamma^-_{ik}},$$

$$|[v\|^2_{\ddot W_2^{1/2}(\gamma^-_{ik})} = |v|^2_{W_2^{1/2}(\gamma^-_{ik})} + h\sum_{x\in\gamma^-_{ik}}\left(\frac{1}{x_{3-i}+h/2} + \frac{1}{1-x_{3-i}-h/2}\right)v^2(x),$$

$$\|v\|^2_\tau = \tau\sum_{t\in\omega^+_\tau} v^2(t), \qquad |[v\|^2_{i,h\tau} = \tau\sum_{t\in\omega^+_\tau}|[v(\cdot,t)\|_i^2,$$

$$\|v\|^2_{\sigma_{ik}} = \tau\sum_{t\in\omega^+_\tau}\|v(\cdot,t)\|^2_{\gamma_{ik}}, \qquad |[v]|^2_{\bar\sigma_{ik}} = \tau\sum_{t\in\omega^+_\tau}|[v(\cdot,t)]|^2_{\bar\gamma_{ik}},$$

$$|[v\|^2_{L_2(\omega^+_\tau,\,\ddot W_2^{1/2}(\gamma^-_{ik}))} = \tau\sum_{t\in\omega^+_\tau}|[v(\cdot,t)\|^2_{\ddot W_2^{1/2}(\gamma^-_{ik})},$$

$$|v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} = \tau^2\sum_{t,\,t'\in\bar\omega_\tau,\ t'\neq t}\frac{|[v(\cdot,t)-v(\cdot,t')]|^2}{(t-t')^2},$$

$$|[v]|^2_{\ddot W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} = |v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} + \tau\sum_{t\in\bar\omega_\tau}\left(\frac{1}{t+\tau/2} + \frac{1}{T-t+\tau/2}\right)|[v(\cdot,t)]|^2,$$

$$|[v]|^2_{L_2(\omega^+_\tau,\,W_2^1(\bar\omega))} = \tau\sum_{t\in\omega^+_\tau}|[v(\cdot,t)]|^2_{W_2^1(\bar\omega)},$$

$$|[v]|^2_{W_2^{1,\,1/2}(Q_{h\tau})} = |[v]|^2_{L_2(\omega^+_\tau,\,W_2^1(\bar\omega))} + |v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))}.$$

We shall prove a suitable a priori estimate for the FDS (4.1) which will be used to estimate its convergence rate.

LEMMA 4.1. *Let a_{ij} and α satisfy the assumptions from Section 2. Then, for a sufficiently small mesh step h, there exist positive constants C_1 and C_2 such that*

$$C_1|[v]|^2_{W_2^1(\bar\omega)} \leqslant [L_h v, v] \leqslant C_2|[v]|^2_{W_2^1(\bar\omega)}.$$

PROOF. The proof immediately follows from

$$\begin{aligned}[L_h v,\, v] &= \frac{1}{2}\sum_{i=1}^{2}\Big\{[a_{ii}v_{x_i},\, v_{x_i})_i + (a_{ii}v_{\bar x_i},\, v_{\bar x_i}]_i + [a_{i,3-i}v_{x_{3-i}},\, v_{x_i}) \\ &\qquad + (a_{i,3-i}v_{\bar x_{3-i}},\, v_{\bar x_i}]\Big\} + h\sum_{x\in\gamma\smallsetminus\gamma^\star}\tilde\alpha\, v^2 + \frac{h}{2}\sum_{x\in\gamma^\star}(\tilde\alpha_1+\tilde\alpha_2)\, v^2 \\ &= \frac{1}{2}\sum_{i,j=1}^{2}\Big\{[a_{ij}v_{x_j},\, v_{x_i}) + (a_{ij}v_{\bar x_j},\, v_{\bar x_i}]\Big\} + \frac{h}{4}\sum_{i=1}^{2}\sum_{k=0}^{1}(-1)^k[a_{ii}^{+i} - a_{ii},\, v_{x_i}^2)_{\gamma^-_{3-i,\,k}} \\ &\qquad + \sum_{x\in\gamma\smallsetminus\gamma^\star}\tilde\alpha\, v^2 + \frac{h}{2}\sum_{x\in\gamma^\star}(\tilde\alpha_1+\tilde\alpha_2)\, v^2\end{aligned}$$

and a discrete imbedding theorem [**12**]. □

LEMMA 4.2. [**2, 7**] *The following inequality holds true:*

$$\big|[v,\, w_{x_{3-i}})_{\gamma^-_{ik}}\big| \leqslant C|[v]\|_{\ddot W_2^{1/2}(\gamma^-_{ik})}\,|[w]|_{W_2^1(\bar\omega)}.$$

LEMMA 4.3. [**7**] *Let v be a mesh function on $\bar\omega$, then*

$$|[v]|_{C(\bar\omega)} \leqslant C\sqrt{\log\tfrac{1}{h}}\,|[v]|_{W_2^1(\bar\omega)}.$$

LEMMA 4.4. [**8**] *The solution of the FDS*

$$v_{\bar t} + L_h v = \varphi, \quad (x,t)\in\bar\omega\times\omega_\tau^+; \qquad v(x,0) = 0, \quad x\in\bar\omega.$$

satisfies the a priori estimate

$$|[v]|_{W_2^{1,\,1/2}(Q_{h\tau})} \leqslant C\bigg(\tau\sum_{t\in\omega_\tau^+}|[\varphi(\cdot,t)]|^2_{-1}\bigg)^{1/2}$$

where

$$|[\varphi(\cdot,t)]|_{-1} := \sup_w \frac{[\varphi(\cdot,t),w]}{|[w]|_{W_2^1(\bar\omega)}}.$$

LEMMA 4.5. [**8**] *The solution of the FDS*

$$v_{\bar t} + L_h v = \phi_{\bar t}, \quad (x,t)\in\bar\omega\times\omega_\tau^+; \qquad v(x,0) = 0, \quad x\in\bar\omega.$$

satisfies the a priori estimate

$$|[v]|_{W_2^{1,\,1/2}(Q_{h\tau})} \leqslant C|[\phi]|_{\ddot W_2^{1/2}(\bar\omega_\tau,\,L_2(\bar\omega))}.$$

LEMMA 4.6. *Let $w \in W_2^r(\Gamma_{ik})$, $0 < r \leqslant 0.5$. Then*

$$|T_{3-i}^+ w|_{W_2^{1/2}(\gamma_{ik}^-)} \leqslant C(r) h^{r-1/2} \, |w|_{W_2^r(\Gamma_{ik})}.$$

PROOF. Without loss of generality let us set $i = 1$ and $k = 0$. Hence

$$|T_2^+ w|^2_{W_2^{1/2}(\gamma_{10}^-)} = h^2 \sum_{i=1}^{n-1} \sum_{j=1,\, j\neq i}^{n-1} \frac{\left[T_2^+ w(0,ih) - T_2^+ w(0,jh)\right]^2}{(ih-jh)^2}$$

$$= 2h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \frac{\left[T_2^+ w(0,ih) - T_2^+ w(0,jh)\right]^2}{(ih-jh)^2}$$

$$= 2h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \left\{ \frac{1}{h^2} \int_{ih}^{ih+h} \int_{jh}^{jh+h} \left[w(0,x) - w(0,x')\right] dx' dx \right\}^2 (ih-jh)^{-2}$$

$$= 2h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \frac{1}{h^4} \left\{ \int_{ih}^{ih+h} \int_{jh}^{jh+h} \frac{\left[w(0,x) - w(0,x')\right]^2}{(x-x')^{1+2r}} \, dx' dx \right\}$$

$$\times \left\{ \int_{ih}^{ih+h} \int_{jh}^{jh+h} \frac{(x-x')^{1+2r}}{(ih-jh)^2} \, dx' dx \right\}$$

$$\leqslant 2h^{-2} 2^{1+2r} h^2 h^{2r-1} \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \left\{ \int_{ih}^{ih+h} \int_{jh}^{jh+h} \frac{\left[w(0,x) - w(0,x')\right]^2}{(x-x')^{1+2r}} \, dx' dx \right\}$$

$$= 2^{1+2r} h^{2r-1} 2 \int_0^1 \int_0^x \frac{\left[w(0,x) - w(0,x')\right]^2}{(x-x')^{1+2r}} \, dx' dx = 2^{1+2r} h^{2r-1} \, |w|^2_{W_2^r(\Gamma_{10})}.$$

□

Let us rearrange the summands in truncation error ψ in the following manner:

$$\tilde{\eta}_{ij} = \eta_{ij} + \eta'_{ij}, \qquad \tilde{\xi} = \xi + \xi', \qquad \tilde{\tilde{\xi}} = \xi + \xi^\star,$$

where

$$\eta'_{ii} = \pm \frac{h}{3} T_i^+ T_t^- \left(\frac{\partial}{\partial x_{3-i}} \Big(a_{ii} \, \frac{\partial u}{\partial x_i} \Big) \right), \qquad x \in \gamma_{3-i,\, 0.5 \mp 0.5},$$

$$\eta'_{i,3-i} = \pm \frac{h}{3} T_i^+ T_t^- \left(\frac{\partial}{\partial x_{3-i}} \Big(a_{i,3-i} \, \frac{\partial u}{\partial x_{3-i}} \Big) \right) \mp \frac{h}{2} T_i^+ T_t^- \left(a_{i,3-i} \, \frac{\partial^2 u}{\partial x_{3-i}^2} \right)$$

$$+ \frac{h}{2} T_i^+ T_t^- \left(\frac{\partial}{\partial x_i} \Big(a_{i,3-i} \, \frac{\partial u}{\partial x_{3-i}} \Big) \right), \qquad x \in \gamma_{3-i,\, 0.5 \mp 0.5},$$

$$\xi' = \mp \frac{h}{3} T_{3-i}^2 \Big(\frac{\partial u}{\partial x_i} \Big), \qquad x \in \gamma_{i,\, 0.5 \mp 0.5},$$

$$\xi^\star = \mp \frac{h}{3} T_2^{2\pm} \Big(\frac{\partial u}{\partial x_1} \Big) \mp \frac{h}{3} T_1^{2\pm} \Big(\frac{\partial u}{\partial x_2} \Big), \qquad x = (0.5 \mp 0.5,\, 0.5 \mp 0.5) \in \gamma^\star.$$

Using the boundary condition (2.2) we further obtain

$$\xi'_{\bar{t}} = \lambda_{i,\, \bar{x}_{3-i}} + \mu_i + \nu_i, \qquad x \in \gamma_{i,\, 0.5 \mp 0.5},$$

where

$$\lambda_i = \pm\frac{h}{3}\,T_{3-i}^{+}T_t^{-}\Big(\frac{a_{i,3-i}}{a_{ii}}\frac{\partial u}{\partial t}\Big),$$
$$\mu_i = \mp\frac{h}{3}\,T_{3-i}^{2}T_t^{-}\bigg(\frac{\partial}{\partial x_{3-i}}\Big(\frac{a_{i,3-i}}{a_{ii}}\Big)\frac{\partial u}{\partial t}\bigg),$$
$$\nu_i = -\frac{h}{3}\,T_{3-i}^{2}T_t^{-}\Big(\frac{\alpha}{a_{ii}}\frac{\partial u}{\partial t}\Big).$$

Similarly, for $x=(0,0)$, we obtain

$$\xi_t^{\star} = \frac{2}{h}\,\lambda_1 - \frac{2}{h}\,\lambda_1^{\star} + \mu_1 + \nu_1 + \frac{2}{h}\,\lambda_2 - \frac{2}{h}\,\lambda_2^{\star} + \mu_2 + \nu_2,$$

where λ_1 and λ_2 are the same as before and

$$\lambda_i^{\star} = \pm\frac{h}{3}\,T_t^{-}\Big(\frac{a_{i,3-i}}{a_{ii}}\frac{\partial u}{\partial t}\Big),$$
$$\mu_i = -\frac{h}{3}\,T_{3-i}^{2+}T_t^{-}\bigg(\frac{\partial}{\partial x_{3-i}}\Big(\frac{a_{i,3-i}}{a_{ii}}\Big)\frac{\partial u}{\partial t}\bigg),$$
$$\nu_i = -\frac{h}{3}\,T_{3-i}^{2+}T_t^{-}\Big(\frac{\alpha}{a_{ii}}\frac{\partial u}{\partial t}\Big),$$

with an analogous representation at other nodes from $\gamma^{\star}$.

Using Lemmas 4.1–4.5, we obtain the following assertion.

THEOREM 4.1. *The finite difference scheme* (4.1) *is stable in the sense of the a priori estimate*

$$\begin{aligned}|[z]|_{W_2^{1,\,1/2}(Q_{h\tau})} \leqslant C\Bigg\{ &|[\xi]|_{\ddot{W}_2^{1/2}(\bar{\omega}_\tau,\,L_2(\bar{\omega}))} + \sum_{i,j=1}^{2}|[\eta_{ij}\|_{i,h\tau} + \sum_{k=0}^{1}\sum_{i=1}^{2}\|\zeta\|_{\sigma_{ik}}\\ &+ h\sum_{k=0}^{1}\sum_{i,j=1}^{2}|[\eta'_{ij}\|_{L_2(\omega_\tau^{+},\,\ddot{W}_2^{1/2}(\gamma^{-}_{3-i,\,k}))} + h\sum_{k=0}^{1}\sum_{i=1}^{2}|[\lambda_i\|_{L_2(\omega_\tau^{+},\,\ddot{W}_2^{1/2}(\gamma_{ik}^{-}))}\\ &+h\sum_{k=0}^{1}\sum_{i=1}^{2}\Big(|[\mu_i]|_{\bar{\sigma}_{ik}}+|[\nu_i]|_{\bar{\sigma}_{ik}}\Big) + h\sqrt{\log\tfrac{1}{h}}\sum_{i=1}^{2}\sum_{x\in\gamma^{\star}}\Big(\|\zeta_i(x,\cdot)\|_\tau+\|\lambda_i^{\star}(x,\cdot)\|_\tau\Big)\Bigg\}.\end{aligned}\tag{4.2}$$

In accordance with Theorem 4.1, the problem of deriving the convergence rate estimate for the FDS (3.1) is reduced to estimating the right-hand side terms in the inequality (4.2).

Let us assume that $\tau \asymp h^2$, i.e., $c_2h^2 \leqslant \tau \leqslant c_3h^2$ for some positive constants c_2 and c_3.

The term η_{ij} at the internal mesh nodes can be estimated in the same manner as in the case of the Dirichlet IBVP (see [**6**]):

$$\tau\sum_{t\in\omega_\tau^{+}} h^2 \sum_{x\in\omega\cup\gamma_{i0}} \eta_{ij}^2 \leqslant Ch^{2s-2}\,\|a_{ij}\|^2_{W_2^{s-1}(\Omega)}\|u\|^2_{W_2^{s,\,s/2}(Q)}, \quad 2<s\leqslant 3.\tag{4.3}$$

In boundary nodes η_{ii} can be decomposed in the following manner

$$\eta_{ii} = \eta_{ii,1} + \eta_{ii,2} + \eta_{ii,3} + \eta_{ii,4}, \qquad x\in\gamma^{-}_{3-i,\,0.5\mp 0.5},$$

where

$$\eta_{ii,1} = T_i^+ T_t^- \Big(a_{ii}\,\frac{\partial u}{\partial x_i}\Big) - \big(T_i^+ a_{ii}\big)\,\Big(T_i^+ T_t^-\,\frac{\partial u}{\partial x_i}\Big),$$

$$\eta_{ii,2} = \Big[\,\big(T_i^+ a_{ii}\big) - \frac{a_{ii} + a_{ii}^{+i}}{2}\Big]\Big(T_i^+ T_t^-\,\frac{\partial u}{\partial x_i}\Big),$$

$$\eta_{ii,3} = \frac{a_{ii} + a_{ii}^{+i}}{2}\Big[\Big(T_i^+ T_t^-\,\frac{\partial u}{\partial x_i}\Big) - \Big(T_i^+\,\frac{\partial u}{\partial x_i}\Big)\Big],$$

$$\eta_{ii,4} = T_i^+ T_{3-i}^{2\pm} T_t^- \Big(a_{ii}\,\frac{\partial u}{\partial x_i}\Big) - T_i^+ T_t^- \Big(a_{ii}\,\frac{\partial u}{\partial x_i}\Big) \mp \frac{h}{3}\,T_i^+ T_t^- \Big(\frac{\partial}{\partial x_{3-i}}\Big(a_{ii}\,\frac{\partial u}{\partial x_i}\Big)\Big).$$

The terms $\eta_{ii,l}$ for $l = 1, 2, 3$ satisfy the same conditions as analogous terms in [**6**] whereby it follows that

$$\tau \sum_{t\in\omega_\tau^+} h^2 \sum_{x\in\gamma_{3-i,0}^-\cup\gamma_{3-i,1}^-} \eta_{ii,l}^2 \leqslant Ch^{2s-2}\,\|a_{ii}\|^2_{W_2^{s-1}(\Omega)}\|u\|^2_{W_2^{s,s/2}(Q)}, \quad 2 < s \leqslant 3. \tag{4.4}$$

For $s > 2.5$ the term $\eta_{ii,4}$ is a bounded linear functional of $w = a_{ii}\,\frac{\partial u}{\partial x_i} \in W_2^{s-1,(s-1)/2}$ which vanishes when $w = 1, x_1, x_2, t$. Using the Bramble–Hilbert lemma [**3, 5**] and properties of multipliers in Sobolev spaces [**10**] we obtain the following result

$$\begin{aligned}\tau \sum_{t\in\omega_\tau^+} h^2 \sum_{x\in\gamma_{3-i,0}^-\cup\gamma_{3-i,1}^-} \eta_{ii,4}^2 &\leqslant Ch^{2s-2}\,\Big\|a_{ii}\,\frac{\partial u}{\partial x_i}\Big\|^2_{W_2^{s-1,(s-1)/2}(Q)} \\ &\leqslant Ch^{2s-2}\,\|a_{ii}\|^2_{W_2^{s-1}(\Omega)}\|u\|^2_{W_2^{s,s/2}(Q)}, \quad 2.5 < s \leqslant 3.\end{aligned} \tag{4.5}$$

Similarly, at the boundary nodes $\eta_{i,3-i}$ can be decomposed in the following manner

$$\eta_{i,3-i} = \eta_{i,3-i,1} + \eta_{i,3-i,2} + \eta_{i,3-i,3} + \eta_{i,3-i,4}, \qquad x \in \gamma_{3-i,\,0.5\mp0.5}^-,$$

where

$$\begin{aligned}\eta_{i,3-i,1} &= T_i^+ T_{3-i}^{2\pm} T_t^- \Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big) - T_i^+ T_t^- \Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big) \\ &\qquad \mp \frac{h}{3}\,T_i^+ T_t^- \Big(\frac{\partial}{\partial x_{3-i}}\Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big)\Big), \\ \eta_{i,3-i,2} &= T_i^+ T_t^- \Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big) - T_t^- \Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big) \\ &\qquad - \frac{h}{2}\,T_i^+ T_t^- \Big(\frac{\partial}{\partial x_i}\Big(a_{i,3-i}\,\frac{\partial u}{\partial x_{3-i}}\Big)\Big), \\ \eta_{i,3-i,3} &= a_{i,3-i}\Big[T_t^-\Big(\frac{\partial u}{\partial x_{3-i}}\Big) - T_{3-i}^{\pm}\Big(\frac{\partial u}{\partial x_{3-i}}\Big) \pm \frac{h}{2}\,T_i^+ T_t^-\Big(\frac{\partial^2 u}{\partial x_{3-i}^2}\Big)\Big], \\ \eta_{i,3-i,4} &= \pm\frac{h}{2}\Big[T_i^+ T_t^-\Big(a_{i,3-i}\,\frac{\partial^2 u}{\partial x_{3-i}^2}\Big) - a_{i,3-i}\,T_i^+ T_t^-\Big(\frac{\partial^2 u}{\partial x_{3-i}^2}\Big)\Big].\end{aligned}$$

The terms $\eta_{i,3-i,1}$ and $\eta_{i,3-i,2}$ satisfy estimates analogous to (4.5). For $s > 2.5$ and $a_{i,3-i} \in C(\bar\Omega)$ the term $\eta_{i,3-i,3}$ is a bounded linear functional of $u \in W_2^{s,s/2}$ which

vanishes when $w = 1, x_1, x_2, t, x_1^2, x_1x_2, x_2^2$. Using the Bramble–Hilbert lemma and the Sobolev imbedding theorem [**1**] we obtain the following result

$$\tau \sum_{t\in\omega_\tau^+} h^2 \sum_{x\in\gamma_{3-i,0}^-\cup\gamma_{3-i,1}^-} \eta_{i,3-i,3}^2 \leqslant Ch^{2s-2}\,\|a_{i,3-i}\|_{C(\bar\Omega)}^2 \|u\|_{W_2^{s,s/2}(Q)}^2 \leqslant Ch^{2s-2}\,\|a_{i,3-i}\|_{W_2^{s-1}(\Omega)}^2 \|u\|_{W_2^{s,s/2}(Q)}^2, \quad 2.5 < s \leqslant 3. \tag{4.6}$$

Term $\eta_{i,3-i,4}$ can be estimated directly. Let us set $i = 2$ and $x = (0, x_2) \in \gamma_{10}^-$. Then

$$\eta_{2,1,4}(0,x_2,t) = \frac{h}{2}\,\frac{2}{h\tau}\int_{x_2}^{x_2+h}\Big(1-\frac{x_2'}{h}\Big)\int_{t-\tau}^{t}\int_{x_2}^{x_2'}\frac{\partial a_{21}}{\partial x_2}(0,x_2'')\frac{\partial^2 u}{\partial x_1^2}(0,x_2',t')\,dx_2''dt'dx_2',$$

while for other boundary nodes, and also for $i = 1$, we have analogous integral representations. Hence

$$\tau \sum_{t\in\omega_\tau^+} h^2 \sum_{x\in\gamma_{3-i,k}^-} \eta_{i,3-i,4}^2 \leqslant Ch^4 \left\|\frac{\partial a_{i,3-i}}{\partial x_i}\right\|_{L_2(\Gamma_{ik})}^2 \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L_2(\Sigma_{ik})}^2 \leqslant Ch^4\,\|a_{i,3-i}\|_{W_2^{s-1}(\Omega)}^2 \|u\|_{W_2^{s,s/2}(Q)}^2, \qquad s > 2.5\,. \tag{4.7}$$

In such a way, from (4.3)-(4.7) one obtains

$$|[\eta_{ij}\|_{i,h\tau} \leqslant Ch^{s-1}\,\|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^{s,s/2}(Q)}, \quad 2.5 < s \leqslant 3. \tag{4.8}$$

The term ξ at the internal mesh nodes is estimated in [**6**]. At the boundary nodes ξ admits an analogous integral representation as for $x \in \omega$. Hence, one immediately obtains

$$|[\xi]|_{\dot W_2^{1/2}(\bar\omega_\tau,\, L_2(\bar\omega))} \leqslant Ch^{s-1}\sqrt{\log\tfrac{1}{h}}\,\|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^{s,s/2}(Q)}, \quad 2 < s \leqslant 3. \tag{4.9}$$

The term ζ on $\sigma_{3-i,k}$ can be represented in the following manner:

$$\zeta = \big(T_i^2\alpha\big)\big(u - T_i^2T_t^-u\big) + \big[\big(T_i^2\alpha\big)\big(T_i^2T_t^-u\big) - T_i^2\big(\alpha\,T_t^-u\big)\big] = \zeta_{01} + \zeta_{02}.$$

For $s > 2.5$ and $\alpha \in C(\Gamma_{3-i,k})$ the term ζ_{01} is a bounded linear functional of $u \in W_2^{s-1,(s-1)/2}(\Sigma_{3-i,k})$ which vanishes when $u = 1$ and $u = x_{3-i}$. Using the Bramble–Hilbert lemma and the Sobolev imbedding theorem we obtain the following result

$$\|\zeta_{01}\|_{\sigma_{3-i,k}} \leqslant Ch^{s-1}\|\alpha\|_{C(\Gamma_{3-i,k})}\|u\|_{W_2^{s-1,(s-1)/2}(\Sigma_{3-i,k})} \leqslant Ch^{s-1}\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2.5 < s \leqslant 3. \tag{4.10}$$

The term ζ_{02} is a bounded linear functional of $(\alpha, T_t^-u) \in W_q^r(\Gamma_{3-i,k}) \times W_{\frac{2q}{q-2}}^p(\Gamma_{3-i,k})$, $q > 2$, which vanishes when $u = 1$ or $T_t^-u = 1$. Using the bilinear version of the Bramble–Hilbert lemma, the Sobolev imbedding theorem and the Hölder inequality,

after summation over the mesh $\gamma_{3-i,k}$ we obtain the following result

$$\|\zeta_{02}\|_{\gamma_{3-i,k}} \leqslant Ch^{r+p}\|\alpha\|_{W_q^r(\Gamma_{3-i,k})}\|T_t^- u\|_{W^p_{2q/(q-2)}(\Gamma_{3-i,k})}$$
$$\leqslant Ch^{r+p}\|\alpha\|_{W_2^{r+\frac12-\frac1q}(\Gamma_{3-i,k})}\|T_t^- u\|_{W_2^{p+\frac1q}(\Gamma_{3-i,k})}, \quad 0\leqslant r\leqslant 1, \quad 1-\tfrac1q<p\leqslant 1.$$

Summing over the mesh ω_τ^+ and using imbedding and trace theorems we obtain

$$\begin{aligned}\|\zeta_{02}\|_{\sigma_{3-i,k}} &\leqslant Ch^{r+p}\,\|\alpha\|_{W_2^{r+\frac12-\frac1q}(\Gamma_{3-i,k})}\|u\|_{W_2^{p+\frac1q,\frac12(p+\frac1q)}(\Sigma_{3-i,k})}\\ &\leqslant Ch^{r+p}\,\|\alpha\|_{W_2^{r+\frac12-\frac1q}(\Gamma_{3-i,k})}\|u\|_{W_2^{p+\frac12+\frac1q,\frac12(p+\frac12+\frac1q)}(Q)}\\ &\leqslant Ch^{r+p}\,\|\alpha\|_{W_2^{r+p-\frac12}(\Gamma_{3-i,k})}\|u\|_{W_2^{r+p+1,\frac12(r+p+1)}(Q)}.\end{aligned}$$

Finally, setting $r+p=s-1$, we get

$$\|\zeta_{02}\|_{\sigma_{3-i,k}} \leqslant Ch^{s-1}\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2<s\leqslant 3. \tag{4.11}$$

From (4.10) and (4.11) it follows that

$$\|\zeta\|_{\sigma_{3-i,k}} \leqslant Ch^{s-1}\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2.5<s\leqslant 3. \tag{4.12}$$

For $x\in\gamma^\star_{3-i,k}$ we set

$$\zeta_i = \left(T_i^{2\pm}\alpha\right)\left(u-T_t^- u\right)+\left[\left(T_i^{2\pm}\alpha\right)\left(T_t^- u\right)-T_i^{2\pm}\left(\alpha\,T_t^- u\right)\right]=\zeta_{i1}+\zeta_{i2}.$$

For $r>0.5$ and $\alpha\in C(\Gamma_{3-i,k})$ the term ζ_{i1} is a bounded linear functional of $u\in W_2^r(0,T)$ which vanishes when $u=1$. Using the Bramble–Hilbert lemma and the Sobolev imbedding theorem, after summation over the mesh ω_τ^+ we obtain

$$\begin{aligned}\|\zeta_{i1}\|_\tau &\leqslant C\tau^r\|\alpha\|_{C(\Gamma_{3-i,k})}\|u(x,\cdot)\|_{W_2^r(0,T)}\\ &\leqslant Ch^{2r}\|\alpha\|_{W_2^{2r-1/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{2r+1,r+1/2}(Q)}, \qquad 0.5<r\leqslant 1,\end{aligned}$$

whereby, setting $2r+1=s$,

$$\|\zeta_{i1}\|_\tau \leqslant Ch^{s-1}\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2<s\leqslant 3. \tag{4.13}$$

For $i=1$ and $x=(0,0)$ we have

$$\begin{aligned}\zeta_{12}(0,0,t) &= \frac{2}{h}\int_0^h\Big(1-\frac{x_1'}{h}\Big)\alpha(x_1',0)\left[T_t^- u(0,0,t)-T_t^- u(x_1',0,t)\right]dx_1'\\ &= \frac{2}{h}\int_0^h\int_0^{x_1'}\Big(1-\frac{x_1'}{h}\Big)\alpha(x_1',0)\,\frac{\partial(T_t^- u)}{\partial x_1}(x_1'',0,t)\,dx_1''dx_1'.\end{aligned}$$

Analogous representations hold for other nodes $x\in\gamma^\star$ and also for $i=2$. Hence

$$|\zeta_{i2}|\leqslant Ch\,\|\alpha\|_{C(\Gamma_{3-i,k})}\Big\|\frac{\partial(T_t^- u)}{\partial x_i}\Big\|_{C(\bar\Omega)}\leqslant Ch\,\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\|T_t^- u\|_{W_2^s(\Omega)}, \quad s>2,$$

whereby, after summation over the mesh ω_τ^+, one obtains

$$\|\zeta_{i2}\|_\tau \leqslant Ch\,\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad s>2. \tag{4.14}$$

In such a way, for $x\in\gamma^\star_{3-i,k}$, from (4.13) and (4.14) we get

$$\|\zeta_i\|_\tau \leqslant Ch\,\|\alpha\|_{W_2^{s-3/2}(\Gamma_{3-i,k})}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad s>2. \tag{4.15}$$

Let us now estimate $|[\eta'_{ij}\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{3-i,k}^-))}$. Using Lemma 4.6 we immediately obtain

$$|\eta'_{ii}|_{W_2^{1/2}(\gamma_{3-i,k}^-)} \leqslant Ch^{r+1/2}\left|T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right)\right|_{W_2^r(\Gamma_{3-i,k})}$$
$$\leqslant Ch^{r+1/2}\left\|T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right)\right\|_{W_2^{r+1/2}(\Omega)}, \qquad 0<r\leqslant 0.5.$$

Using the inequality [**11**]

$$\|F\|_{L_2(0,\varepsilon)} \leqslant C\begin{cases} \varepsilon^r\,\|F\|_{W_2^r(0,1)}, & 0<r<0.5,\\ \varepsilon^{1/2}\,\log\frac{1}{\varepsilon}\,\|F\|_{W_2^{1/2}(0,1)}, & r=0.5,\\ \varepsilon^{1/2}\,\|F\|_{W_2^{1/2}(0,1)}, & r>0.5, \end{cases}$$

where $0<\varepsilon<1$, we obtain

$$h\sum_{x\in\gamma_{3-i,k}^-}\left(\frac{1}{x_i+h/2}+\frac{1}{1-x_i-h/2}\right)(\eta'_{ii})^2$$
$$\leqslant Ch^{2r+1}\log\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right)\right\|^2_{W_2^r(\Gamma_{3-i,k}^-)}$$
$$\leqslant Ch^{2r+1}\log\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right)\right\|^2_{W_2^{r+1/2}(\Omega)}, \quad 0<r<0.5$$

and

$$h\sum_{x\in\gamma_{3-i,k}^-}\left(\frac{1}{x_i+h/2}+\frac{1}{1-x_i-h/2}\right)(\eta'_{ii})^2$$
$$\leqslant Ch^2\log^3\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right)\right\|^2_{W_2^1(\Omega)}.$$

From the obtained inequalities, summing over the mesh ω_τ^+, denoting $r+2.5=s$ and using properties of multipliers in Sobolev spaces, we immediately obtain

$$\|\eta'_{ii}\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{3-i,k}^-))} \leqslant Ch^{s-2}\sqrt{\log\tfrac{1}{h}}\,\|a_{ii}\|_{W_2^{s-1}(\Omega)}\,\|u\|_{W_2^{s,s/2}(Q)}, \tag{4.16}$$

for $2.5<s<3$ and

$$\|\eta'_{ii}\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{3-i,k}^-))} \leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\,\|a_{ii}\|_{W_2^2(\Omega)}\,\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.17}$$

Analogous estimates hold for $\eta'_{i,3-i}$ substituting a_{ii} with $a_{i,3-i}$.

The term λ_i has a similar structure as η'_{ij}. Handling it in the same manner we obtain for $2.5<s<3$

$$\|\lambda_i\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{ik}^-))} \leqslant Ch^{s-2}\sqrt{\log\tfrac{1}{h}}\,\left\|\frac{a_{i,3-i}}{a_{ii}}\right\|_{M(W_2^{s-2}(\Omega))}\,\|u\|_{W_2^{s,s/2}(Q)},$$

where $\|\cdot\|_{M(W_2^{s-2}(\Omega))}$ is the norm in the space of multipliers in $W_2^{s-2}(\Omega)$ [**10**]. Further, for $2.5 < s < 3$ and $q > 2$, using the properties of multipliers in Sobolev spaces and the assumptions (2.4), we obtain

$$\begin{aligned}\left\|\frac{a_{i,3-i}}{a_{ii}}\right\|_{M(W_2^{s-2}(\Omega))} &\leqslant C\left\|\frac{a_{i,3-i}}{a_{ii}}\right\|_{W_q^1(\Omega)} \leqslant C\Big(\|a_{i,3-i}\|_{W_q^1(\Omega)} + \|a_{ii}\|_{W_q^1(\Omega)}\Big)\\ &\leqslant C\Big(\|a_{i,3-i}\|_{W_2^{s-1}(\Omega)} + \|a_{ii}\|_{W_2^{s-1}(\Omega)}\Big).\end{aligned}$$

In such a way, we finally obtain

$$\begin{aligned}(4.18)\quad \|\lambda_i\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{ik}^-))} &\leqslant Ch^{s-2}\sqrt{\log\tfrac{1}{h}}\\ &\times\Big(\|a_{ii}\|_{W_2^{s-1}(\Omega)} + \|a_{i,3-i}\|_{W_2^{s-1}(\Omega)}\Big)\|u\|_{W_2^{s,s/2}(Q)},\quad 2.5 < s < 3\end{aligned}$$

and analogously

$$\begin{aligned}(4.19)\quad \|\lambda_i\|_{L_2(\omega_\tau^+,\ddot{W}_2^{1/2}(\gamma_{ik}^-))} &\leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\\ &\times\Big(\|a_{ii}\|_{W_2^2(\Omega)} + \|a_{i,3-i}\|_{W_2^2(\Omega)}\Big)\|u\|_{W_2^{3,3/2}(Q)},\quad s = 3.\end{aligned}$$

For $r > 0.5$ and a_{ii}, $a_{i,3-i} \in C(\bar\Omega)$ the term $\lambda_i^\star$ is a bounded linear functional of $u \in W_2^r(0,T)$ which vanishes when $u = 1$. Using the Bramble–Hilbert lemma and handling it in the same manner as in the case of ζ_{i1} we obtain

$$(4.20)\qquad \|\lambda_i^\star\|_\tau \leqslant Ch^{s-2}\|a_{i,3-i}\|_{W_2^{s-1}(\Omega)}\|u\|_{W_2^{s,s/2}(Q)},\qquad 2 < s \leqslant 3.$$

The terms ν_i and μ_i can be estimated directly:

$$\begin{aligned}(4.21)\quad |[\nu_i]|_{\bar\sigma_{ik}} &\leqslant Ch\left\|\frac{\alpha}{a_{ii}}\,\frac{\partial u}{\partial t}\right\|_{L_2(\Sigma_{ik})} \leqslant Ch\left\|\frac{\alpha}{a_{ii}}\right\|_{C(\Gamma_{ik})}\|u\|_{W_2^{s,s/2}(Q)}\\ &\leqslant Ch\,\|u\|_{W_2^{s,s/2}(Q)},\qquad s > 2.5\end{aligned}$$

and

$$(4.22)\quad |[\mu_i]|_{\bar\sigma_{ik}} \leqslant Ch\Big(\|a_{ii}\|_{W_2^{s-1}(\Omega)} + \|a_{i,3-i}\|_{W_2^{s-1}(\Omega)}\Big)\|u\|_{W_2^{s,s/2}(Q)},\qquad s > 2.5$$

From (4.2), (4.8), (4.9), (4.12) and (4.15)–(4.22) we obtain the following result.

THEOREM 4.2. *Let the assumptions from Section 2 hold and let $\tau \asymp h^2$. Then the FDS* (3.1) *converges in the norm $W_2^{1,1/2}(Q_{h\tau})$ and the convergence rate estimates*

$$\begin{aligned}&|[z]|_{W_2^{1,1/2}(Q_{h\tau})} \leqslant Ch^{s-1}\sqrt{\log\tfrac{1}{h}}\\ &\times\Big(1 + \max_{i,j}\|a_{ij}\|_{W_2^{s-1}(\Omega)} + \max_{i,k}\|\alpha\|_{W_2^{s-3/2}(\Gamma_{ik})}\Big)\|u\|_{W_2^{s,s/2}(Q)},\qquad 2.5 < s < 3\end{aligned}$$

and

$$|[z]|_{W_2^{1,1/2}(Q_{h\tau})} \leqslant Ch^2\left(\log\tfrac{1}{h}\right)^{\frac{3}{2}}\Big(1+\max_{i,j}\|a_{ij}\|_{W_2^2(\Omega)}+\max_{i,k}\|\alpha\|_{W_2^{3/2}(\Gamma_{ik})}\Big)\|u\|_{W_2^{3,3/2}(Q)}$$

hold.

REMARK 4.1. Note that the obtained error bounds are 'almost' compatible with the smoothness of the data (up to additional logarithmic factors).

REMARK 4.2. The obtained error bounds have been derived under the assumption $\tau \asymp h^2$, which links the temporal mesh-size τ to the spatial mesh-size h, despite the fact that the implicit FDS is unconditionally stable, and therefore from the point of view of stability there should be no limitation on the choice of mesh-sizes. This limitation can be avoided in certain cases by careful study of truncation error functionals (see e.g. [**9**]).

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