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COMPLEX VALUED PROBABILITY LOGICS

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Abstract. We present two complex valued probabilistic logics, LCOMP_B and LCOMP_S, which extend classical propositional logic. In LCOMP_B one can express formulas of the form $B_{z,\rho}\alpha$ meaning that the probability of α is in the complex ball with the center z and the radius ρ, while in LCOMP_S one can make statements of the form $S_{z,\rho}\alpha$ with the intended meaning – the probability of propositional formula α is in the complex square with the center z and the side 2ρ. The corresponding strongly complete axiom systems are provided. Decidability of the logics are proved by reducing the satisfiability problem for LCOMP_B (LCOMP_S) to the problem of solving systems of quadratic (linear) inequalities.

1. Introduction

In measure theory, complex measures generalize the concept of measures by letting them have complex values. It is well known that complex measures are used for characterization of linear functionals on the space of all continuous complex-valued functions that vanish at infinity (Riesz Theorem) [**9**]. By adding the assumption that the measure of the entire space is equal to 1, we obtain complex valued probabilities [**1, 3**]. Complex valued probabilities have proven to be useful in applications. For example, in [**12**] the author considers relativistic quantum mechanics based on complex probability theory. In this approach, a wave function Ψ is not treated as "the state of the system". Ψ represents the best estimate of the complex probability of finding particle at some point in a measure space. Actually, it says what is known about the system. Thus, the collapse of the wave function represents learning a new fact about the system and therefore leads to calculation of new complex probabilities. In this way realistic quantum theory gives simple explanations of several paradoxical problems in quantum mechanics. In [**13**] complex probabilities are applied to the concepts of degradation of systems and estimation of remaining useful life of faulty components in the field of prognostic based on reliability. Complex probabilities also appear in Markov Stochastic Processes, where in [**14**] the authors

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consider the transition probability matrix P of a discrete Markov chain and give a motivating example for calculating P^r where $n < r < n+1$, for some $n \in \mathbf{Z}$. It turns out that, in general, P^r is a complex matrix, i.e., it belongs to the class of generalized stochastic matrices.

In this paper, we develop two logics for reasoning about complex valued probabilities. Since the field of complex numbers is not ordered, we can not use the standard probabilistic operators of the form $P_{\geqslant r}\alpha$, with the intended meaning "The probability of α is greater or equal to r" [**4, 5, 6, 7, 8**]. We had a similar situation formalizing p-adic valued probabilities [**10**], where we introduced probabilistic operators of the form $K_{r,\rho}\alpha$ that have the intuitive meaning "The probability of α belongs to the p-adic ball with the center r and the radius ρ". Similarly, in this paper we use complex balls (in the first logic) and squares (in the second logic) to estimate probabilities of events. The main differences between axiomatizations for the logics from [**10**] and those presented in this paper, as well as in the proofs of the corresponding statements, follow from the fact that the strong triangle inequality holds for p-adic fields, but not for complex numbers.

We consider two logics:

- LCOMP_B, with the probabilistic operators $B_{z,\rho}\alpha$ meaning "The probability of α is in the ball with the center z and the radius ρ",
- LCOMP_S, with the probabilistic operators $S_{z,\rho}\alpha$ meaning "The probability of α is in the square with the center z and the side $2 \cdot \rho$"

For these logics, the corresponding axiom systems with infinitary rules of inference are given and proved to be sound and strongly complete.

Decidability for the logic LCOMP_B is proved by reducing the satisfiability problem to the problem of solving systems of quadratic inequalities, which is known to be in PSPACE [**1**]. For the logic LCOMP_S the same reduction can be done to the linear systems solving problem, which implies NP-completeness of the logic.

The rest of the paper is organized as follows: In Sections 2 and 3 we present the logics LCOMP_B and LCOMP_S, respectively. Section 2 is divided into 4 subsections, within which we present Syntax and Semantics, and then axioms and inference rules, prove the corresponding soundness and completeness and discuss decidability. Finally, the concluding remarks are given in Section 4.

2. The logic LCOMP_B

In this section, we present the probability logic LCOMP_B in which we use probabilistic operators of the form $B_{z,\rho}\alpha$. The intended meaning of these operators is: "The probability of α belongs to the ball $B[z,\rho]$".

Let $|\cdot|$ denote the standard real absolute value, ($|x| = x$ if $x \geqslant 0$, $|x| = -x$ if $x < 0$). If $\mathbf{C}$ is a field of complex numbers and $z = x + iy \in \mathbf{C}$, then we use $\|\cdot\|$ for the complex norm, $\|x+iy\| = \sqrt{x^2+y^2}$. Let $B[z,\rho] = \{z_1 \in \mathbf{C} : \|z - z_1\| \leqslant \rho\}$ be a complex ball with the center z and the radius ρ, $\mathrm{CQ} = \{a + ib \mid a, b \in \mathbf{Q}\}$ the set of complex numbers with rational coordinates, and $\mathbf{Q}^+$ the set of all non-negative rationals.

2.1. Syntax and semantics. Suppose that Var is a countable set of propositional letters. By $\mathrm{For}_{\mathrm{Cl}}$ we will denote the set of all propositional formulas over Var. Propositional formulas will be denoted by α, β and γ. The set $\mathrm{For}_{\mathrm{CP}}$ of all probabilistic formulas is defined as the least set satisfying the following conditions:

- If $\alpha \in \mathrm{For}_{\mathrm{Cl}}$, $z \in \mathrm{CQ}$, $\rho \in \mathbf{Q}^+$, then $B_{z,\rho}\alpha$ is a probabilistic formula;
- If φ, ϕ are probabilistic formulas, then $(\neg\varphi)$, $(\varphi \wedge \phi)$ are probabilistic formulas.

Probabilistic formulas will be denoted by φ, ϕ and θ. The set For of all LCOMP_B-formulas is a union of $\mathrm{For}_{\mathrm{Cl}}$ and $\mathrm{For}_{\mathrm{CP}}$. Formulas will be denoted by A, B and C, indexed if necessary. The other classical connectives $(\vee, \Rightarrow, \Leftrightarrow)$ can be defined as usual. We denote both $\alpha \wedge \neg\alpha$ and $\varphi \wedge \neg\varphi$ by $\bot$, letting the context to determine the meaning. Also, we use $\top$ for $\alpha \vee \neg\alpha$ and $\varphi \vee \neg\varphi$.

Definition 2.1. An LCOMP_B-model is a structure $M = \langle W, H, \mu, v\rangle$ where:

- W is a nonempty set of elements called worlds;
- H is an algebra of subsets of W;
- $\mu : H \longrightarrow \mathbf{C}$ is a measure (additive function) such that $\mu(W) = 1$;
- $v : W \times \mathrm{Var} \to \{\mathrm{true}, \mathrm{false}\}$ is a valuation which associated with every world $w \in W$ a truth assignment $v(w, \cdot)$ on propositional letters; the valuation $v(w, \cdot)$ is extended to classical propositional formulas as usual.

If M is an LCOMP_B-model, we will denote by $[\alpha]$ the set of all worlds w such that $v(w, \alpha) = \mathrm{true}$. An LCOMP_B-model $M = \langle W, H, \mu, v\rangle$ is *measurable* if $[\alpha] \in H$ for every formula $\alpha \in \mathrm{For}_{\mathrm{Cl}}$. In this paper we focus on the class of all measurable LCOMP_B-models. Thus, when we write “LCOMP_B-model" we mean “measurable LCOMP_B-model".

Definition 2.2. Let $M = \langle W, H, \mu, v\rangle$ be an LCOMP_B-model. The satisfiability relation $\models$ is inductively defined as follows:

- If $\alpha \in \mathrm{For}_{\mathrm{Cl}}$,, then $M \models \alpha$ iff $v(w, \alpha) = \mathrm{true}$ for every $w \in W$.
- If $\alpha \in \mathrm{For}_{\mathrm{Cl}}$, $z \in \mathrm{CQ}$, $\rho \in \mathbf{Q}^+$, then $M \models B_{z,\rho}\alpha$ iff $\|\mu([\alpha]) - z\| \leqslant \rho$.
- If $\varphi \in \mathrm{For}_{\mathrm{CP}}$,, then $M \models \neg\varphi$ iff it is not $M \models \varphi$.
- If $\varphi, \psi \in \mathrm{For}_{\mathrm{CP}}$, then $M \models \varphi \wedge \psi$ iff $M \models \varphi$ i $M \models \psi$. □

Note that for all $\rho \in \mathbf{Q}^+$, $z \in \mathrm{CQ}$, $M \models B_{z,\rho}\alpha$ means that $\mu([\alpha])$ belongs to the complex ball with the center z and the radius ρ. If $\rho = 0$,, then we obtain that the probability of α is equal to z.

2.2. Axiomatization. The axiom system AX_{LCOMP_B} of the logic LCOMP_B contains the following axioms and inference rules:

Axioms

A1. Substitutional instances of tautologies;
A2. $B_{z,\rho}\alpha \Rightarrow B_{z,\rho'}\alpha$, whenever $\rho' \geqslant \rho$;
A3. $B_{z_1,\rho_1}\alpha \wedge B_{z_2,\rho_2}\beta \wedge B_{0,0}(\alpha \wedge \beta) \Rightarrow B_{z_1+z_2,\rho_1+\rho_2}(\alpha \vee \beta)$;
A4. $B_{z_1,\rho_1}\alpha \Rightarrow \neg B_{z_2,\rho_2}\alpha$, if $\|z_1 - z_2\| > \rho_1 + \rho_2$;
A5. $B_{z_1,\rho_1}\alpha \Rightarrow B_{z_2,\rho_1+\rho_2}\alpha$, if $\|z_1 - z_2\| \leqslant \rho_2$;

Inference rules

R1. From A and $A \Rightarrow B$ infer B. Here A and B are either both propositional, or both probabilistic formulas;
R2. From α infer $B_{1,0}\alpha$;
R3. If $n \in \mathbf{N}$, from $\varphi \Rightarrow \neg B_{z,\frac{1}{n}}\alpha$ for every $z \in \mathrm{CQ}$, infer $\varphi \Rightarrow \perp$;
R4. From $\alpha \Rightarrow \perp$, infer $B_{0,0}\alpha$;
R5. If $z \in \mathrm{CQ}$, from $\varphi \Rightarrow B_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, infer $\varphi \Rightarrow B_{z,\rho}\alpha$;
R6. From $(\alpha \Leftrightarrow \beta)$, infer $(B_{z,\rho}\alpha \Rightarrow B_{z,\rho}\beta)$;

We will briefly discuss the meaning and the scope of the axioms and inference rules. Axiom A1 provides validity of all classical tautologies. Axiom A2 corresponds to the obvious property of balls: a ball of smaller radius is contained in a ball of larger radius. Axiom A3 corresponds to the additivity of measures. Axiom A4 provides that the measure of a formula cannot belong to two disjoint balls. Axiom A5 allows that the following holds: if the measure of α belongs to the ball B, it belongs to some larger ball B', such that $B \subseteq B'$.

Rule R2 can be considered as the rule of necessitation in modal logic, but it can be applied only to theorems. Rule R3 provides that for every classical formula α and every $n \in \mathbf{N}$, there must be some $z \in \mathrm{CQ}$ such that the measure of α belongs to the ball $B[z, \frac{1}{n}]$. Rule R4 guarantees that contradiction has the measure 0. Rule R5 expresses the next property: if the measure of α is arbitrarily close to some number $z \in \mathrm{CQ}$,, then the measure of α is equal to z. Finally, rule R6 says that equivalent classical formulas have the same measure. Note that the rules R3 and R5 are infinitary.

Definition 2.3. A formula A is *deducible* from the set T of formulas (denoted $T \vdash A$) if there is a sequence (called a proof) of formulas A_0, $A_1 \ldots A_n$, where $A_n = A$ such that every A_i is either an instance of some axiom, or it is a formula from the set T, or it can be derived from the preceding formulas by some inference rule. The length of a proof is a successor ordinal. As it is usual, $T \nvdash \alpha$ means that $T \vdash \alpha$ does not hold. A formula A is a *theorem* ($\vdash A$) iff it is deducible from the empty set. A set of formulas T is *consistent* if there are $\alpha \in \mathrm{For}_{\mathrm{Cl}}$ and $\varphi \in \mathrm{For}_{\mathrm{CP}}$ such that $T \nvdash \alpha$ and $T \nvdash \varphi$. A consistent set T of formulas is said to be *maximal consistent* if it has the following properties:

- For every $\alpha \in \mathrm{For}_{\mathrm{Cl}}$, if $T \vdash \alpha$,, then both α and $B_{1,0}\alpha$ are in T;
- For every $\varphi \in \mathrm{For}_{\mathrm{CP}}$, either $\varphi \in T$ or $\neg\varphi \in T$.

A set of formulas T is *deductively closed* if for every $A \in \mathrm{For}$, if $T \vdash A$, then $A \in T$.

2.3. Soundness and completeness.

Theorem 2.1 (Deduction theorem). *Let T be a set of formulas and A and B both classical or both propositional formulas. Then, $T, A \vdash B$ implies $T \vdash A \Rightarrow B$.*

Proof. We use transfinite induction on the length of the proof of B from $T \cup \{A\}$. For example, we consider the case where $B = (\varphi \Rightarrow \perp)$ obtained from $T \cup \{A\}$ by an application of rule R3 and $A \in \mathrm{For}_P$. Then for some $n \in \mathbf{N}$, $\alpha \in \mathrm{For}_{\mathrm{Cl}}$:

$T, A \vdash \varphi \Rightarrow \neg(B_{z,\frac{1}{n}}\alpha)$ for every $z \in \mathrm{CQ}$

$T \vdash A \Rightarrow (\varphi \Rightarrow \neg(B_{z,\frac{1}{n}}\alpha))$ for every $r \in \mathrm{CQ}$, by induction hypothesis
$T \vdash A \wedge \varphi \Rightarrow \neg(B_{z,\frac{1}{n}}\alpha)$ for every $r \in \mathrm{CQ}$, by classical reasoning
$T \vdash A \wedge \varphi \Rightarrow \perp$, by rule R3
$T \vdash A \Rightarrow (\varphi \Rightarrow \perp)$.
The other cases follow similarly. □

THEOREM 2.2. *Every consistent set can be extended to a maximal consistent set.*

PROOF. Let T be a consistent theory, $\overline{T}$ the set of all classical formulas that are consequences of T, α_0, α_1,... an enumeration of all formulas from $\mathrm{For}_{\mathrm{Cl}}$, and φ_0, φ_1,... an enumeration of all formulas from $\mathrm{For}_{\mathrm{CP}}$. Let $f : \mathbf{N} \to \mathbf{N^2}$ be any bijection (i.e., f is of the form $f(i) = (\pi_1(i), \pi_2(i))$[1]). We define a sequence of theories T_i in the following way:

(1) $T_0 = T \cup \overline{T} \cup \{B_{1,0}\alpha | \alpha \in \overline{T}\}$;
(2) For every $i \geqslant 0$,
  (a) If $T_{2i} \cup \{\varphi_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{\varphi_i\}$;
  (b) Otherwise, if $T_{2i} \cup \{\varphi_i\}$ is inconsistent, then:
    (i) If $\varphi_i = (\psi \Rightarrow B_{z,\rho}\alpha)$, then $T_{2i+1} = T_{2i} \cup \{\neg\varphi_i, \psi \Rightarrow \neg B_{z,\rho+p^{-n}}\alpha\}$ for some $n \in \mathbf{N}$ such that T_{2i+1} is consistent,
    (ii) Otherwise $T_{2i+1} = T_{2i} \cup \{\neg\varphi_i\}$;
(3) For every $i \geqslant 0$, $T_{2i+2} = T_{2i+1} \cup \{B_{z,\frac{1}{\pi_1(i)}}\alpha_{\pi_2(i)}\}$ for some $z \in \mathrm{CQ}$ such that T_{2i+2} is consistent.

We show that for every i, T_i is consistent.

The set T_0 is consistent since it contains consequences of a consistent set. The sets obtained by the steps 2a are obviously consistent. The step 2b (ii) produces consistent sets too. For if $T_{2i}, \varphi_i \vdash \perp$, by Deduction Theorem we have $T_{2i} \vdash \neg\varphi_i$, and since T_{2i} is consistent so is $T_{2i} \cup \{\neg\varphi_i\}$. Let us consider the step 2b(i). Suppose that $\varphi_i = (\psi \Rightarrow B_{z,0}\alpha)$, $T_{2i} \cup \{\varphi_i\}$ is inconsistent and that for every $n \in \mathbf{N}$, $T_{2i} \cup \{\neg(\psi \Rightarrow B_{z,0}\alpha), \psi \Rightarrow \neg B_{z,p^{-n}}\alpha\}$ is inconsistent. Then:

$T_{2i}, \neg(\psi \Rightarrow B_{z,\rho}\alpha), \psi \Rightarrow \neg B_{z,\rho+\frac{1}{n}}\alpha \vdash \perp$, for every $n \in \mathbf{N}$

$T_{2i}, \neg(\psi \Rightarrow B_{z,\rho}\alpha) \vdash \neg(\psi \Rightarrow \neg B_{z,\rho+\frac{1}{n}}\alpha)$, for every $n \in \mathbf{N}$, by Deduction theorem

$T_{2i}, \neg(\psi \Rightarrow B_{z,\rho}\alpha) \vdash \psi \Rightarrow B_{z,\rho+\frac{1}{n}}\alpha$, for every $n \in \mathbf{N}$, by classical tautology $\neg(\alpha \Rightarrow \neg\beta) \Rightarrow (\alpha \Rightarrow \beta)$

$T_{2i}, \neg(\psi \Rightarrow B_{z,\rho}\alpha) \vdash \psi \Rightarrow B_{z,\rho}\alpha$, by rule R5

$T_{2i} \vdash \neg(\psi \Rightarrow B_{z,\rho}\alpha) \Rightarrow (\psi \Rightarrow B_{z,\rho}\alpha)$, by Deduction theorem

$T_{2i} \vdash \psi \Rightarrow B_{z,\rho}\alpha$.

[1]For instance, we can consider the function $f(x) = (\pi_1(x), \pi_2(x))$, where $\pi_1(x) = (x+1)_1$, $\pi_2(x) = \left[\frac{1}{2}\left(\left[\frac{x+1}{2^{(x+1)_1}}\right] \dot{-} 1\right)\right]$. Here, $(x)_i$ is the degree of the i-th prime number in the factorization of x and

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geqslant y \\ 0 & \text{otherwise.} \end{cases}$$

Note that f is the inverse function of the function $F : \mathbf{N^2} \to \mathbf{N}$, $F(m,n) = 2^m(2n+1) - 1$.

Since $T_{2i} \cup \{\psi \Rightarrow B_{z,\rho}\alpha\}$ is not consistent, from $T_{2i} \vdash \psi \Rightarrow B_{z,\rho}\alpha$ it follows that T_{2i} is not consistent, a contradiction.

Next, consider the step 3. Suppose that for every $z \in \mathrm{CQ}$ the set $T_{2i+1} \cup \{B_{z,\frac{1}{\pi_1(i)}}\alpha_{\frac{1}{\pi_2(i)}}\}$ is inconsistent. Let $T_{2i+1} = T_0 \cup T^+_{2i+1}$, where T^+_{2i+1} is set of all formulas from For_P which were added to T_0 in previous steps of the construction. Then:

$T_0, T^+_{2i+1}, B_{z,\frac{1}{\pi_1(i)}}\alpha_{\pi_2(i)} \vdash \perp$, for every $z \in \mathrm{CQ}$

$T_0, T^+_{2i+1} \vdash \neg B_{z,\frac{1}{\pi_1(i)}}\alpha_{\pi_2(i)}$ for every $z \in \mathrm{CQ}$, by Deduction theorem

$T_0 \vdash (\bigwedge_{\varphi \in T^+_{2i+1}} \varphi) \Rightarrow \neg B_{z,\frac{1}{\pi_1(i)}}\alpha_{\pi_2(i)}$, for every $z \in \mathrm{CQ}$, by Deduction theorem

$T_0 \vdash (\bigwedge_{\varphi \in T^+_{2i+1}} \varphi) \Rightarrow \perp$, by rule R3.

Therefore $T_{2i+1} \vdash \perp$, a contradiction.

Let $T^* = \bigcup_{i<\omega} T_i$. It remains to show that T^* is maximal and consistent. The steps 1 and 2 of the above construction guarantee that T^* is maximal. We continue by showing that T^* is a deductively closed set which does not contain all formulas, and, as a consequence, that T^* is consistent.

First we show that T^* does not contain all formulas. Let $\alpha \in \mathrm{For}_{\mathrm{Cl}}$. According to the construction of T_0, α and $\neg\alpha$ cannot be simultaneously in T_0. Suppose that $\varphi \in \mathrm{For}_{\mathrm{CP}}$. Then for some i, j, $\varphi = \varphi_i$ and $\neg\varphi = \varphi_j$. Since $T_{\max(2i,2j)+1}$ is consistent, T^* cannot contain both φ and $\neg\varphi$.

Next we show that T^* is deductively closed. If $\alpha \in \mathrm{For}_{\mathrm{Cl}}$ and $T^* \vdash \alpha$, then by the construction of T_0, $\alpha \in T^*$ and $B_{1,0}\alpha \in T^*$. Let $\varphi \in \mathrm{For}_{\mathrm{CP}}$. Notice that if $\varphi = \varphi_j$ and $T_i \vdash \varphi_j$, it must be $\varphi \in T^*$ because $T_{\max(i,2j)+1}$ is consistent. Suppose that the sequence $\varphi_1, \varphi_2, \dots \varphi$ forms the proof of φ from T^*. If the sequence is finite, there must be a set T_i such that $T_i \vdash \varphi$. Then, similarly as above, $\varphi \in T^*$. Thus suppose that the sequence is countably infinite.We can show that for every i, if φ_i is obtained by an application of an inference rule, and all premises belong to T^*, then there must be $\varphi_i \in T^*$. If the rule is a finitary one, then there must be a set T_j which contains all premises and $T_j \vdash \varphi_i$. Reasoning as above, we conclude that $\varphi_i \in T^*$. So, let us now consider the infinitary rules.

Suppose that $\varphi_i = (\psi \Rightarrow \perp)$ is obtained from the set $\{\varphi_z = (\psi \Rightarrow \neg B_{z,\frac{1}{n}}\alpha) \mid z \in \mathrm{CQ}\}$ of premises, by rule R3 and for some $\alpha \in \mathrm{For}_{\mathrm{Cl}}, n \in \mathbf{N}$. By the induction hypothesis $\varphi_z \in T^*$ for every $z \in \mathrm{CQ}$. By the step 3 of the construction there must be some z' and some l such that $\psi \Rightarrow B_{z',\frac{1}{n}}\alpha$ belongs to T_l. Since all premises belongs to T^*, for some k, $\psi \Rightarrow \neg B_{z',\frac{1}{n}}\alpha \in T_k$. If $m = \max(l,k)$, then $\psi \Rightarrow \neg B_{z',\frac{1}{n}}\alpha, \psi \Rightarrow B_{z',\frac{1}{n}}\alpha \in T_m$. Thus $T_m \vdash \psi \Rightarrow B_{z',\frac{1}{n}}\alpha$ and $T_m \vdash \psi \Rightarrow \neg B_{z',\frac{1}{n}}\alpha$ so $T_m \vdash \psi \Rightarrow \perp$ Then, in the same way as above, we have $\psi \Rightarrow \perp \in T^*$. Finally, the case $\varphi_i = (\psi \Rightarrow B_{z,\rho}\alpha)$ follows similarly. □

Let T^* be a maximal consistent set obtained from a consistent set T by the construction from Theorem 2.2. According to the step (3), T^* has the following property: For every formula $\alpha \in \mathrm{For}_{\mathrm{Cl}}$ and every $n \in \mathbf{N}$ there is at least one $z \in \mathrm{CQ}$ such that $B_{z,\frac{1}{n}}\alpha \in T^*$. Since T^* is deductively closed, using axiom A5, we can obtain countably many numbers $z \in \mathrm{CQ}$ such that $B_{z,\frac{1}{n}}\alpha \in T^*$. Now, for each formula $\alpha \in \mathrm{For}_{\mathrm{Cl}}$ we make a sequence z_n in the following way:

- For every $n \in \mathbf{N}$ we arbitrarily chose any number z such that $B_{z,\frac{1}{n}}\alpha \in T^*$ and this z will be the n-th number of the sequence, i.e., $z_n = z$.

In this way we obtain the sequence $z(\alpha)$=z_0, z_1, $z_2 \dots$ where $B_{z_n,\frac{1}{n}}\alpha \in T^*$.

Notice that it is possible that $z_m = z_k$, for some $m \neq k$.

Lemma 2.1. *Let $z(\alpha)$ be defined as above. Then, $z(\alpha)$ is a Cauchy sequence* (*with respect to the norm* $\|\cdot\|$).

Proof. Let ε be arbitrary. Choose n_0 such that $\frac{2}{n_0} \leqslant \varepsilon$. If $n, m \geqslant n_0$, then according to the definition of $z(\alpha)$, $B_{z_n,\frac{1}{n}}\alpha, B_{z_m,\frac{1}{m}}\alpha \in T^*$. We claim that $\|z_n - z_m\| \leqslant \frac{1}{n} + \frac{1}{m}$. To prove this, suppose that $\|z_n - z_m\| > \frac{1}{n} + \frac{1}{m}$. Then:

$T^* \vdash B_{z_n,\frac{1}{n}}\alpha$

$T^* \vdash B_{z_m,\frac{1}{m}}\alpha$

$T^* \vdash B_{z_n,\frac{1}{n}}\alpha \Rightarrow \neg B_{z_m,\frac{1}{m}}\alpha$ by axiom A4

$T^* \vdash \neg B_{z_m,\frac{1}{m}}\alpha$ by rule R1, which contradicts the consistency of T^*.

Thus, $\|z_n - z_m\| \leqslant \frac{1}{n} + \frac{1}{m} \leqslant \frac{2}{n_0} \leqslant \varepsilon$. □

In the following lemma we will show that this limes does not depend on the choice of z_k's.

Lemma 2.2. *Let $\alpha \in T^*$, $\alpha \in \mathrm{For}_{\mathrm{Cl}}$. Suppose that $(z_n)_{n\in\mathbf{N}}$, and $(z'_n)_{n\in\mathbf{N}}$ are two different sequences obtained by the above given construction* (*i.e., for at least one m, $z_m \neq z'_m$*). *Then* $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z'_n$.

Proof. According to axiom A4, as in the previous lemma we conclude that for every n, $\|z_n - z'_n\| \leqslant \frac{2}{n}$. Suppose that $\lim_{n\to\infty} z_n = a$. Let ε be arbitrary. Then there is n'_0 such that for $n \geqslant n'_0$ $\|z_n - a\| \leqslant \frac{\varepsilon}{2}$. Choose n''_0 such that $n''_0 > \frac{4}{\varepsilon}$ and let $n_0 \geqslant \max\{n'_0, n''_0\}$. Then for $n \geqslant n_0$:

$$\|z'_n - a\| = \|(z'_n - z_n) + (z_n - a)\| \leqslant \|z'_n - z_n\| + \|z_n - a\| \leqslant \frac{2}{n} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{n\to\infty} z'_n = a$. □

Next we define a canonical model. Let $M_{T^*} = \langle W, H, \mu, v\rangle$, where:

- $W = \{w \mid w \models \overline{T}\}$ contains all classical propositional interpretations that satisfy the set $\overline{T}$ of all classical consequences of the set T,
- $H = \{[\alpha] : \alpha \in \mathrm{For}_{\mathrm{Cl}}\}$
- $\mu : H \to C$: Let $z(\alpha) = (z_n)_{n\in\mathbf{N}}$. Then

$$\mu([\alpha]) = \begin{cases} z & \text{if } B_{z,0}\alpha \in T^* \\ \lim_{n\to\infty} z_n & \text{otherwise} \end{cases}$$

- for every world w and every propositional letter $p \in \mathrm{Var}$, $v(w, p) = \text{true}$ iff $w \models p$.

The axioms guarantee that everything is well defined. For example, by the classical reasoning we can show that $\{[\alpha] : \alpha \in \mathrm{For}_{\mathrm{Cl}}\}$ is an algebra of subsets of W. The rule R6 implies that if $[\alpha] = [\beta]$, then $\mu([\alpha]) = \mu([\beta])$. From the axioms A2, A3 and rule R4 it follows that μ is finitely additive probability measure.

THEOREM 2.3 (Strong completeness). *A set of formulas T is consistent iff it has an* LCOMP_B*-model.*

PROOF. ($\Leftarrow$) This direction follows from the soundness of the above axiomatic system.

($\Rightarrow$) In order to prove this direction we construct $M_{T^*} = (W, H, \mu, v)$ as above, and show, by induction on complexity of formulas, that for every formula A, $M_{T^*} \models A$ iff $A \in T^*$. For instance, we will consider the case where $A = B_{z,\rho}\alpha$ for some $z \in \mathrm{CQ}$, $\rho \in Q$ and $\alpha \in \mathrm{For}_{\mathrm{Cl}}$.

Suppose that $B_{z,\rho}\alpha \in T^*$. First we assume that $\rho > 0$. Let $z(\alpha) = (z_n)_{n\in\mathbf{N}}$ and $\mu([\alpha]) = \lim^{p}_{n\to\infty} z_n$. Thus $(\forall\varepsilon)(\exists n_0)(\forall n)(n \geqslant n_0 \to \|\mu([\alpha]) - z_n\| \leqslant \varepsilon)$.

We distinguish the following cases:

- There is $\rho_1 < \rho$ such that $B_{z,\rho_1}\alpha \in T^*$. Then $\rho - \rho_1 > 0 > \frac{1}{m}$, for some $m \in \mathbf{N}$. Let $\varepsilon = \rho - \rho_1 - \frac{1}{m} > 0$. Let n_0 be such that for every $n \geqslant n_0$, $\|\mu([\alpha]) - z_n\| \leqslant \rho - \rho_1 - \frac{1}{m}$. If $n \geqslant \max\{n_0, m\}$, then from $T^* \vdash B_{z,\rho_1}$ and $T^* \vdash B_{z_n,\frac{1}{n}}\alpha$, using axiom A4 we obtain $\|z - z_n\| \leqslant \rho_1 + \frac{1}{n}$. Thus $\|z - \mu([\alpha])\| = \|z - z_n + z_n - \mu([\alpha])\| \leqslant \|z - z_n\| + \|z_n - \mu([\alpha])\| \leqslant \rho_1 + \frac{1}{n} + \rho - \rho_1 - \frac{1}{m} \leqslant \rho$. Therefore $M_{T^*} \models B_{z,\rho}\alpha$.
- There is no $\rho_1 < \rho$ such that $B_{z,\rho_1}\alpha \in T^*$. But from $T^* \vdash B_{z,\rho}\alpha$ and axiom A2, we obtain $T^* \vdash B_{z,\rho+\frac{1}{n}}\alpha$, for every $n \in \mathbf{N}$. Therefore, according to the previous considerations, we obtain $\|z - \mu([\alpha])\| \leqslant \rho + \frac{1}{n}$, for every $n \in \mathbf{N}$, and thus $\|z - \mu([\alpha])\| \leqslant \rho$.

Finally, let $\rho = 0$, i.e., $T^* \vdash B_{z,0}\alpha$. Then, according to the definition of measure μ in the canonical model, $\mu([\alpha]) = z$, i.e., $\|\mu([\alpha]) - z\| = 0$. Thus, $M_{T^*} \models B_{z,0}\alpha$.

For the other direction, suppose that $M_{T^*} \models B_{z,\rho}\alpha$. Let $\rho > 0$. As in the previous direction, we consider the following cases.

- $\|\mu([\alpha]) - z\| < \rho$, that is, there is $\rho_1 < \rho$ such that $M_{T^*} \models B_{z,\rho_1}\alpha$. Choose $k \in \mathbf{N}$ such that $\rho - \rho_1 > \frac{1}{k}$. Let $\mu([\alpha]) = \lim^{p}_{n\to\infty} z_n$. Then, there is n_0 such that $(\forall n)(n \geqslant n_0 \to \|\mu([\alpha]) - z_n\| \leqslant \frac{1}{k})$. Let $n \geqslant n_0$ and $\frac{1}{n} + \frac{1}{k} + \rho_1 \leqslant \rho$. Then from $\|z_n - \mu([\alpha])\| \leqslant \frac{1}{k}$ and $\|z - \mu([\alpha])\| \leqslant \rho_1$, we obtain $\|z - z_n\| \leqslant \rho_1 + \frac{1}{k}$. Therefore, $T^* \vdash B_{z_n,\frac{1}{n}}\alpha$
  $T^* \vdash B_{z_n,\frac{1}{n}}\alpha \Rightarrow B_{z,\frac{1}{n}+\rho_1+\frac{1}{k}}\alpha$, by A5
  $T^* \vdash B_{z,\frac{1}{n}+\rho_1+\frac{1}{k}}\alpha$
  $T^* \vdash B_{z,\rho}\alpha$, using A2
- There is no $\rho_1 < \rho$ such that $\|\mu([\alpha]) - z\| < \rho_1$. Since $\|\mu([\alpha]) - z\| < \rho$, we have $\|\mu([\alpha]) - z\| < \rho + \frac{1}{n}$, for every $n \in \mathbf{N}$. Therefore, using the previous considerations we obtain $T^* \vdash B_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, and according to rule R5, $T^* \vdash B_{z,\rho}\alpha$.

If $\rho = 0$, then $\|\mu([\alpha]) - z\| = 0$, so $\|\mu([\alpha]) - z\| < \frac{1}{n}$ for every $n \in \mathbf{N}$. Thus $T^* \vdash B_{z,0+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, and by rule R5, Thus $T^* \vdash B_{z,0}\alpha$. $\square$

2.4. Decidability. In this section, we analyze decidability of the satisfiability problem for LCOMP_B-formulas. Since there is a procedure for deciding satisfiability of classical propositional formulas, we will consider only $\mathrm{For}_{\mathrm{CP}}$-formulas.

Let $\varphi \in \mathrm{For}_{\mathrm{CP}}$. If $p_1, \dots, p_n$ are all propositional letters appearing in φ, then an atom of a formula φ is a formula of the form $\pm p_1 \wedge \ldots \wedge \pm p_n$, where $\pm p_i$ is either p_i or $\neg p_i$. It can be shown, using classical propositional reasoning, that φ is equivalent to a formula

$$\mathrm{DNF}(\varphi) = \bigvee_{i=1,m} \Big(\bigwedge_{j=1,k_i} \pm B_{z_{i,j}, \frac{1}{n_{i,j}}} \alpha_{i,j} \Big)$$

where $\pm B_{z_{i,j}, \frac{1}{n_{i,j}}} \alpha_{i,j}$ denotes either $B_{z_{i,j}, \frac{1}{n_{i,j}}} \alpha_{i,j}$ or $\neg B_{z_{i,j}, \frac{1}{n_{i,j}}} \alpha_{i,j}$

φ is satisfiable iff at least one disjunct from $\mathrm{DNF}(\varphi)$ is satisfiable.

Fix some i and consider disjunct $D_i = \bigwedge_{j=1,k_i} \pm B_{z_{i,j}, \frac{1}{n_{i,j}}} \alpha_{i,j}$ from $\mathrm{DNF}(\varphi)$. Let $p_1, \dots, p_n$ be all propositional letters appearing in D_i. Every propositional formula α is equivalent to the complete disjunctive normal form, denoted $\mathrm{FDNF}(\alpha)$.

If $\models (\alpha \Leftrightarrow \beta)$, then according to the rule R6, for every model M and every $z \in \mathrm{CQ}$, $\rho \in \mathbf{Q}^+$, $M \models B_{z,\rho}\alpha$ iff $M \models B_{z,\rho}\beta$. Thus, disjunct D_i is satisfiable iff the formula $\bigwedge_{j=1,k_i} \pm B_{z_{i,j}, \frac{1}{n_{i,j}}} \mathrm{FDNF}(\alpha_{i,j})$ is satisfiable. Since for two different atoms a_i and a_j, $[a_i] \cap [a_j] = \emptyset$, for every model M, $\mu[a_i \vee a_j] = \mu[a_i] + \mu[a_j]$. Hence, disjunct D_i is satisfiable iff the following system is satisfiable

$$\sum_{t=1}^{2^n} z_t = 1$$

$$J_1 = \begin{cases} \Big(\sum\limits_{a_t \in \alpha_{i,1}} x_t - a_1\Big)^2 + \Big(\sum\limits_{a_t \in \alpha_{i,1}} y_t - b_1\Big)^2 \leqslant \rho_1^2 & \text{if } \pm B_{z_1,\rho_1}\alpha_{i,1} = B_{z_1,\rho_1}\alpha_{i,1} \\ \Big(\sum\limits_{a_t \in \alpha_{i,1}} x_t - a_1\Big)^2 + \Big(\sum\limits_{a_t \in \alpha_{i,1}} y_t - b_1\Big)^2 > \rho_1^2 & \text{if } \pm B_{z_1,\rho_1}\alpha_{i,1} = \neg B_{z_1,\rho_1}\alpha_{i,1} \end{cases}$$

$$\vdots$$

$$J_{k_i} = \begin{cases} \Big(\sum\limits_{a_t \in \alpha_{i,k_i}} x_t - a_{k_i}\Big)^2 + \Big(\sum\limits_{a_t \in \alpha_{i,k_i}} y_t - b_{k_i}\Big)^2 \leqslant \rho_{k_i}^2 & \text{if } \pm B_{z_{k_i},\rho_{k_i}}\alpha_{i,k_i} = B_{z_{k_i},\rho_{k_i}}\alpha_{i,k_i} \\ \Big(\sum\limits_{a_t \in \alpha_{i,k_i}} x_t - a_{k_i}\Big)^2 + \Big(\sum\limits_{a_t \in \alpha_{i,k_i}} y_t - b_{k_i}\Big)^2 > \rho_{k_i}^2 & \text{if } \pm B_{z_{k_i},\rho_{k_i}}\alpha_{i,k_i} = \neg B_{z_{k_i},\rho_{k_i}}\alpha_{i,k_i} \end{cases}$$

where $a_t \in \alpha_{i,j}$ denote that the atom a_t appears in $\mathrm{FDNF}(\alpha_{i,j})$, $z_t = \mu([a_t]) = x_t + iy_t$ and $z_j = a_j + ib_j$, $1 \leqslant j \leqslant k_i$.

As we can see, our formulas can be coded in the existential fragment of the RCF. It is well known that the SAT problem for this fragment is PSPACE complete [**1**], so PSAT for LCOMP_B is in PSPACE.

Therefore, we have the following result.

THEOREM 2.4. *The satisfiability problem for* LCOMP_B*-formulas is decidable.*

3. The logic LCOMP_S

In this section we present the probability logic LCOMP_S in which we use probabilistic operators $S_{z,\rho}\alpha$ meaning "The probability of α belongs to the square parallel to the x axis, with the center z and the side 2ρ". Let $[a, b] = \{x \in \mathbf{R} \mid a \leqslant x \leqslant b\}$

denote interval on real line. The most of the notions defined in Section 2 are also used for the logic LCOMP_S. The main, but important differences are:

- In the definition of all probabilistic formulas $\mathrm{For}_{\mathrm{CP}}^S$, basic probabilistic formulas are of the form $S_{z,\rho}\alpha$ where $\alpha \in \mathrm{For}_{\mathrm{Cl}}$, $z \in \mathrm{CQ}$ and $\rho \in \mathbf{Q}^+$.
- If $M = \langle W, H, \mu, v\rangle$ is an LCOMP_S model, then for $\alpha, \mathrm{For}_{\mathrm{Cl}}$, $z \in \mathrm{CQ}$, $\rho \in \mathbf{Q}^+$ and $z = a + ib$, $M \models S_{z,\rho}\alpha$ iff $\mathrm{Re}(\mu([\alpha])) \in [a-\rho, a+\rho]$ and $\mathrm{Im}(\mu([\alpha])) \in [b-\rho, b+\rho]$.

  Note that, for arbitrary $\rho \in \mathbf{Q}^+$, $z \in \mathrm{CQ}$, $M \models S_{z,\rho}\alpha$ means that $\mu([\alpha])$, belongs to the complex square parallel to the x axis, with the center z and the side 2ρ. If $\rho = 0$, then we obtain that the probability is equal to z.

The axiom system AX_{LCOMP_S} of the logic LCOMP_S contains the following axioms and inference rules:
Axioms

A1. Substitutional instances of tautologies;
A2. $S_{z,\rho}\alpha \Rightarrow S_{z,\rho'}\alpha$, whenever $\rho' \geqslant \rho$;
A3. $S_{z_1,\rho_1}\alpha \wedge S_{z_2,\rho_2}\beta \wedge S_{0,0}(\alpha \wedge \beta) \Rightarrow S_{z_1+z_2,\rho_1+\rho_2}(\alpha \vee \beta)$;
A4. $S_{z_1,\rho_1}\alpha \Rightarrow \neg S_{z_2,\rho_2}\alpha$, if $[a_1-\rho_1, a_1+\rho_1] \cap [a_2-\rho_2, a_2+\rho_2] = \emptyset$ or $[b_1-\rho_1, b_1+\rho_1] \cap [b_2-\rho_2, b_2+\rho_2] = \emptyset$, where $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$;
A5. $S_{z_1,\rho_1}\alpha \Rightarrow S_{z_2,\rho_2}\alpha$, if $[a_1-\rho_1, a_1+\rho_1] \subseteq [a_2-\rho_2, a_2+\rho_2]$ and $[b_1-\rho_1, b_1+\rho_1] \subseteq [b_2-\rho_2, b_2+\rho_2]$

Inference rules

R1. From A and $A \Rightarrow B$ infer B. Here A and B are either both propositional, or both probabilistic formulas;
R2. From α infer $S_{1,0}\alpha$;
R3. If $n \in \mathbf{N}$, from $\varphi \Rightarrow \neg S_{z,\frac{1}{n}}\alpha$ for every $z \in \mathrm{CQ}$, infer $\varphi \Rightarrow \perp$;
R4. From $\alpha \Rightarrow \perp$, infer $S_{0,0}\alpha$;
R5. If $z \in \mathrm{CQ}$, from $\varphi \Rightarrow S_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, infer $\varphi \Rightarrow S_{z,\rho}\alpha$;
R6. From $(\alpha \Leftrightarrow \beta)$, infer $(S_{z,\rho}\alpha \Rightarrow S_{z,\rho}\beta)$;

Axiom A2 corresponds to the obvious property of squares: a square with smaller side is contained in a square with larger side and the same center. Axiom A3 corresponds to the additivity of measures. Axiom A4 provides that the measure of formula cannot belong to two disjoint squares. Axiom A5 allows that the following holds: if the measure of the formula α belongs to the square S, it belongs to a larger square S', such that $S \subseteq S'$. Rule R3 provides that for every classical formula α and every $n \in \mathbf{N}$, there must be some $z \in \mathrm{CQ}$ such that the measure of α belongs to the square $S[z, \frac{1}{n}]$. Rule R5 expresses the following property: if the measure of α is arbitrary close to some number $z \in \mathrm{CQ}$, then the measure of α is equal to z.

Construction of maximal consistent extensions of consistent sets of formulas can be done similarly as above. If T^* is maximal consistent set of formulas, then for each formula $\alpha \in \mathrm{For}_{\mathrm{Cl}}$ we make a sequence $z(\alpha) = z_0, z_1, \ldots$, where $S_{z_j,\frac{1}{j}}\alpha \in T^*$.

Lemma 3.1. *For every $\alpha \in \mathrm{For}_{\mathrm{Cl}}$, $z(\alpha)$ is a Cauchy sequence* (*with respect to the norm* $\|\cdot\|$).

PROOF. Let ε be arbitrary. Choose n_0 such that $\frac{4}{n_0} \leqslant \varepsilon$. If $n, m \geqslant n_0$, then $S_{z_n,\frac{1}{n}}\alpha, S_{z_m,\frac{1}{m}}\alpha \in T^*$. Let $z_{n/m} = x_{n/m} + iy_{n/m}$. Then, there exist x and y such that $x \in [x_n - \frac{1}{n}, x_n + \frac{1}{n}] \cap [x_m - \frac{1}{m}, x_m + \frac{1}{m}]$ and $y \in [y_n - \frac{1}{n}, y_n + \frac{1}{n}] \cap [y_m - \frac{1}{m}, y_m + \frac{1}{m}]$. To prove that, suppose that $[x_n - \frac{1}{n}, x_n + \frac{1}{n}] \cap [x_m - \frac{1}{m}, x_m + \frac{1}{m}] = \emptyset$ or $[y_n - \frac{1}{n}, y_n + \frac{1}{n}] \cap [y_m - \frac{1}{m}, y_m + \frac{1}{m}] = \emptyset$. Then:

$T^* \vdash S_{z_n,\frac{1}{n}}\alpha$

$T^* \vdash S_{z_m,\frac{1}{m}}\alpha$

$T^* \vdash S_{z_n,\frac{1}{n}}\alpha \Rightarrow \neg S_{z_m,\frac{1}{m}}\alpha$ by Axiom A4

$T^* \vdash \neg S_{z_m,\frac{1}{m}}\alpha$ by rule R1, which contradicts the consistency of T^*.

Thus, we have

$$x_n \in [x - \tfrac{1}{n}, x + \tfrac{1}{n}], \quad x_m \in [x - \tfrac{1}{m}, x + \tfrac{1}{m}],$$
$$y_n \in [y - \tfrac{1}{n}, y + \tfrac{1}{n}], \quad y_m \in [y - \tfrac{1}{m}, y + \tfrac{1}{m}].$$

Therefore $|x_n - x_m| \leqslant \frac{1}{n} + \frac{1}{m}$ and $|y_n - y_m| \leqslant \frac{1}{n} + \frac{1}{m}$ and hence

$$\|z_n - z_m\| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} \leqslant 2(\frac{1}{n} + \frac{1}{m}) \leqslant \frac{4}{n_0} \leqslant \varepsilon. \qquad \square$$

LEMMA 3.2. *Let $\alpha \in T^*$, $\alpha \in \mathrm{For}_{\mathrm{Cl}}$. Suppose that $(z_n)_{n\in\mathbf{N}}$, and $(z'_n)_{n\in\mathbf{N}}$ are two different sequences obtained by the above given construction (i.e., for at least one m, $z_m \neq z'_m$). Then* $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z'_n$.

PROOF. Suppose that $\lim_{n\to\infty} z_n = a$. Let ε be arbitrary. Then there is n'_0 such that for $n \geqslant n'_0$ $\|z_n - a\| \leqslant \frac{\varepsilon}{2}$. Choose n''_0 such that $n''_0 > \frac{8}{\varepsilon}$ and let $n_0 \geqslant \max\{n'_0, n''_0\}$. Then for $n \geqslant n_0$ as in the previous Lemma we conclude that $\|z_n - z'_n\| \leqslant \frac{4}{n_0} \leqslant \frac{\varepsilon}{2}$. Thus

$$\|z'_n - a\| = \|(z'_n - z_n) + (z_n - a)\| \leqslant \|z'_n - z_n\| + \|z_n - a\| \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{n\to\infty} z'_n = a$. $\square$

THEOREM 3.1 (Strong completeness). *A set of formulas T is consistent iff it has an* LCOMP$_S$*-model.*

PROOF. We analyze formulas of the form $A = S_{z,\rho}\alpha$ for some $z \in \mathrm{CQ}$, $\rho \in Q$ and $\alpha \in \mathrm{For}_{\mathrm{Cl}}$.

Suppose that $S_{z,\rho}\alpha \in T^*$ where $z = x + iy$. Let $z(\alpha) = (z_n)_{n\in\mathbf{N}}$ and $\mu([\alpha]) = \lim^p_{n\to\infty} r_n = a + ib$.

First assume that $\rho > 0$. We distinguish the following cases:

- There is $\rho_1 < \rho$ such that $S_{z,\rho_1}\alpha \in T^*$. Then $\rho - \rho_1 > 0 > \frac{1}{m}$, for some $m \in \mathbf{N}$. Let $\varepsilon = \rho - \rho_1 - \frac{1}{m} > 0$. Then, there is n_0 such that for every $n \geqslant n_0, \|\mu([\alpha]) - z_n\| \leqslant \rho - \rho_1 - \frac{1}{m}$. Let $n \geqslant \max\{n_0, m\}$ and $z_n = x_n + iy_n$. Then from $\sqrt{(x_n - a)^2 + (y_n - b)^2} \leqslant \varepsilon$ we obtain $|x_n - a|, |y_n - b| \leqslant \varepsilon$. Since $T^* \vdash S_{z,\rho_1}\alpha$ and $T^* \vdash S_{z_n,\frac{1}{n}\alpha}$, using axiom A4, we conclude that there is $T \in [x - \rho_1, x + \rho_1] \cap [x_n - \frac{1}{n}, x_n - \frac{1}{n}]$. Thus $x - \rho_1 \leqslant T \leqslant x + \rho_1$, $x_n - \frac{1}{n} \leqslant T \leqslant x_n - \frac{1}{n}$ and therefore $|x - x_n| \leqslant \rho_1 + \frac{1}{n}$. In the same way, we obtain that $|y - y_n| \leqslant \rho_1 + \frac{1}{n}$. Then $|a - x| = |(a - x_n) + (x_n - x)| \leqslant |a - x_n| + |x_n - x| \leqslant$

$\varepsilon + \rho_1 + \frac{1}{n} = \rho - \rho_1 - \frac{1}{m} + \rho_1 + \frac{1}{n} \leqslant \rho$. Thus, $a \in [x-\rho, x+\rho]$. Similarly, $b \in [y-\rho, y+\rho]$. Hence, $M_{T^*} \models S_{z,\rho}\alpha$.

- There is no $\rho_1 < \rho$ such that $S_{z,\rho_1}\alpha \in T^*$. But from $T^* \vdash S_{z,\rho}\alpha$ and axiom A2, we obtain $T^* \vdash S_{z,\rho+\frac{1}{n}}\alpha$, for every $n \in \mathbf{N}$. Therefore, we obtain $M_{T^*} \models S_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$. Thus $a \in [x-\rho+\frac{1}{n}, x+\rho+\frac{1}{n}]$ and $b \in [y-\rho+\frac{1}{n}, y+\rho+\frac{1}{n}]$ for every $n \in \mathbf{N}$, and hence $a \in [x-\rho, x+\rho]$, $b \in [y-\rho, y+\rho]$, that is $M_{T^*} \models S_{z,\rho}\alpha$.

Finally, let $\rho = 0$, i.e., $T^* \vdash S_{z,0}\alpha$. Then, according to the definition of measure μ in the canonical model, $\mu([\alpha]) = z$, i.e., $a \in [z-0, z+0]$, $b \in [z-0, z+0]$. Thus, $M_{T^*} \models S_{z,0}\alpha$.

For the other direction, suppose that $M_{T^*} \models S_{z,\rho}\alpha$, $z = x+iy$ and $\mu([\alpha]) = a+ib$. Let $\rho > 0$. As in the previous direction, we consider the following cases.

- There is $\rho_1 < \rho$ such that $M_{T^*} \models S_{z,\rho_1}\alpha$. Choose $m \in \mathbf{N}$ such that $\rho - \rho_1 > \frac{1}{m}$. Let $\mu([\alpha]) = \lim_{n\to\infty} z_n$, where $z_n = x_n + iy_n$. Then, there is n_0 such that $(\forall n)(n \geqslant n_0 \to \|\mu([\alpha]) - z_n\| \leqslant \rho - \rho_1 - \frac{1}{m}$, and therefore $|x_n - a| \leqslant \rho - \rho_1 - \frac{1}{m}$, $|y_n - b| \leqslant \rho - \rho_1 - \frac{1}{m}$. Since $M_{T^*} \models S_{z,\rho_1}\alpha$ it follows that $|x-a| \leqslant \rho_1$, $|y-b| \leqslant \rho_1$. Thus $|x - x_n| = |(x-a) + (a-x_n)| \leqslant |x-a| + |x_n - a| \leqslant \rho - \rho_1 - \frac{1}{m} + \rho_1 = \rho - \frac{1}{m} \leqslant \rho - \frac{1}{n}$. In the same way we obtain $|y - y_n| \leqslant \rho - \frac{1}{n}$ Therefore $[x_n - \frac{1}{n}, x_n + \frac{1}{n}] \subseteq [x-\rho, x+\rho]$, $[y_n - \frac{1}{n}, y_n + \frac{1}{n}] \subseteq [y-\rho, y+\rho]$ and hence form $T^* \vdash S_{z_n,\frac{1}{n}}\alpha$, using axiom A5 we obtain
  $T^* \vdash S_{z_n,\frac{1}{n}}\alpha \Rightarrow S_{z,\rho}\alpha$, by A5 and by rule R1, $T^* \vdash S_{z,\rho}\alpha$.
- There is no $\rho_1 < \rho$ such that $M_{T^*} \models S_{z,\rho_1}\alpha$. Then, from $M_{T^*} \models S_{z,\rho}\alpha$ we obtain $M_{T^*} \models S_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$. Therefore, we obtain $T^* \vdash S_{z,\rho+\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, and according to rule R5, $T^* \vdash S_{z,\rho}\alpha$.

If $\rho = 0$, then from $M_{T^*} \models S_{z,0}\alpha$ follows $M_{T^*} \models S_{z,\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$. Thus $T^* \vdash S_{z,\frac{1}{n}}\alpha$ for every $n \in \mathbf{N}$, and by rule R5, $T^* \vdash S_{z,0}\alpha$. $\square$

Theorem 3.2. *The satisfiability problem for* LCOMP$_S$*-formulas is NP-complete.*

Proof. Let φ be a probabilistic LCOMP$_S$-formula. Similarly as in Section2.4, we consider one disjunct D from the DNF(φ) and we conclude that φ is satisfiable iff the system of the following form is satisfiable:

$$\sum_{t=1}^{2^n} z_t = 1$$

$$J_1 = \begin{cases} a_1 - \rho_1 \leqslant \sum\limits_{a_t \in \alpha_1} x_t \leqslant a_1 + \rho_1 \text{ and} & \\ \qquad b_1 - \rho_1 \leqslant \sum\limits_{a_t \in \alpha_1} y_t \leqslant b_1 + \rho_1 & \text{if } \pm S_{z_1,\rho_1}\alpha_1 = S_{z_1,\rho_1}\alpha_1 \\ \sum\limits_{a_t \in \alpha_1} x_t \leqslant a_1 - \rho_1 \text{ or } \sum\limits_{a_t \in \alpha_1} x_t \geqslant a_1 + \rho_1 \text{ or} & \\ \qquad \sum\limits_{a_t \in \alpha_1} y_t \geqslant b_1 + \rho_1 \text{ or } \sum\limits_{a_t \in \alpha_1} y_t \leqslant b_1 - \rho_1 & \text{if } \pm S_{z_1,\rho_1}\alpha_1 = \neg S_{z_1,\rho_1}\alpha_1 \end{cases}$$

$$\vdots$$

$$J_k = \begin{cases} a_k - \rho_k \leqslant \sum\limits_{a_t \in \alpha_k} x_t \leqslant a_k + \rho_k \text{ and} & \\ \qquad b_k - \rho_k \leqslant \sum\limits_{a_t \in \alpha_k} y_t \leqslant b_k + \rho_k & \text{if } \pm S_{z_k,\rho_k}\alpha_k = S_{z_k,\rho_k}\alpha_k \\ \sum\limits_{a_t \in \alpha_k} x_t \leqslant a_k - \rho_k \text{ or } \sum\limits_{a_t \in \alpha_k} x_t \geqslant a_k + \rho_k \text{ or} & \\ \qquad \sum\limits_{a_t \in \alpha_k} y_t \geqslant b_k + \rho_k \text{ or } \sum\limits_{a_t \in \alpha_k} y_t \leqslant b_k - \rho_k & \text{if } \pm S_{z_k,\rho_k}\alpha_k = \neg S_{z_k,\rho_k}\alpha_k \end{cases}$$

where $a_t \in \alpha_j$ denotes that the atom a_t appears in $\mathrm{FDNF}(\alpha_j)$, $z_t = \mu([a_t]) = x_t + iy_t$ and $z_j = a_j + ib_j$,$1 \leqslant j \leqslant k$.

Thus, the statement is proved since PSAT for LCOMP_S-satisfiability is reduced to the linear systems solving problem, similarly as in [**3**]. □

4. Conclusion

We have presented two similar logics that formalize reasoning about complex valued probabilities. The following example, however, shows that it is not possible simply to replace operators of the forms $B_{z,\rho}$ and $S_{z,\rho}$ and keep the meaning of formulas. Namely, since $[2,4] \subset [1,17]$ and $[3,5] \subset [2,18]$, $S_{3+4i,1}\alpha \Rightarrow S_{9+10i,8}\alpha$ is LCOMP_S-valid, but its counterpart $B_{3+4i,1}\alpha \Rightarrow B_{9+10i,8}\alpha$, is not LCOMP_B-valid, because it is obvious that $\frac{5}{2} + \frac{7}{2}i \in B[3+4i,1]$ but $\frac{5}{2} + \frac{7}{2}i \notin B[9+10i,8]$. Similarly, since $B[4+4i,2] \cap B[9+6i,3] = \emptyset$, $B_{4+4i,2}\alpha \Rightarrow \neg B_{9+6i,3}\alpha$ is LCOMP_B-valid, while $S_{4+4i,2}\alpha \Rightarrow \neg S_{9+6i,3}\alpha$ is not LCOMP_S-valid, because the corresponding squares have a common side.

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