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ON THE FARTHEST POINTS IN CONVEX METRIC SPACES AND LINEAR METRIC SPACES

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Abstract. We prove some results on the farthest points in convex metric spaces and in linear metric spaces. The continuity of the farthest point map and characterization of strictly convex linear metric spaces in terms of farthest points are also discussed.

1. Introduction

Consider the geometric problem: Given a nonempty bounded subset C of a metric space (X, ρ) and a point $x \in X$, does there exist a point $y \in C$ which is farthest from x? Is such a y unique? Precisely speaking, we consider the existence and uniqueness problem of a point $y \in C$ satisfying $\rho(x, y) = \sup\{\rho(x, z) : z \in C\}$. Such a y is called a farthest point in C from x. The set of all such $y \in C$ is denoted by $F_C(x)$ and when there is exactly one y satisfying $\rho(x, y) = \sup\{\rho(x, z) : z \in C\}$ then $F_C(x)$ is denoted by $q_C(x)$. If C is a nonempty closed subset of a metric space (X, ρ), then a nearest point in C is defined analogously. A point $z \in C$ is a nearest point to $x \in X$ if $\rho(x, z) = \inf\{\rho(x, y) : y \in C\}$. The set of all such $z \in C$ is denoted by $P_C(x)$. The geometric nature of the space X is very much involved in discussing farthest and nearest points.

Farthest points have applications in the study of extremal structure of sets, characterization of compact convex sets, finding deviation of sets, and they are important building blocks of convex sets which are extensively applied in programming [**11**]. It is strange; rather unfortunate that very little has been done in the theory of farthest points as compared to the theory of nearest points. Moreover,

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most of the literature available in the theory of farthest points is either in Hilbert spaces or in normed linear spaces (see e.g., [**2**, **3**, **5**, **8**, **10**, **11**, **13**] and references therein). The development of farthest point theory in more general spaces is a challenging one. Some attempts have been made in this direction. Ever since Takahashi [**17**] introduced convex metric spaces, efforts are being made to extend results from the theory of Hilbert spaces and normed linear spaces to convex metric spaces [**12**, **14**]. This paper is also a step in this direction, where we discuss some results on farthest points in convex metric spaces and in linear metric spaces. The continuity of the farthest point map and characterization of strictly convex linear metric spaces in terms of farthest points are also discussed.

2. Notations and definitions

We begin with a few definitions and notations. For a metric space (X, ρ) and the closed interval $I = [0, 1]$, a mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$, $\lambda \in I$,

$$\rho(u, W(x, y, \lambda)) \leqslant \lambda\rho(u, x) + (1 - \lambda)\rho(u, y)$$

holds for all $u \in X$. The metric space (X, ρ) together with a convex structure is called a convex metric space [**17**].

Motivated by the linear case, Machado [**9**] defined multiple convex combinations as follows. Let (X, ρ) be a convex metric space. For $x_1, x_2, \ldots, x_n \in X$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$, set

$$\begin{aligned} &W(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \\ &\qquad = W\Big(W\Big(x_1, x_2, \ldots, x_{n-1}, \frac{\lambda_1}{1-\lambda_n}, \frac{\lambda_2}{1-\lambda_n}, \ldots, \frac{\lambda_{n-1}}{1-\lambda_n}\Big), x_n, 1-\lambda_n\Big) \end{aligned}$$

if $\lambda_n \neq 1$ and $W(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 1) = x_n$.

For a subset T of a convex metric space (X, ρ), define convex hull of T [**6**] as

$$\mathrm{Conv}(T) = \bigcup_{n \in N} W(t_1, t_2, \ldots, t_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \bigcup_{n \in N} W^n(T),$$

$W^n(T) = W(W^{n-1}(T))$, $n \geqslant 2$, $W^1(T) = \{W(t_1, t_2, \lambda) : \lambda \in [0, 1],\ t_1, t_2 \in T\}$. This implies (see [**6**]) that for every $x \in X$,

$$\rho(x, W(t_1, t_2, \ldots, t_n, \lambda_1, \lambda_2, \ldots, \lambda_n)) \leqslant \lambda_1\rho(x, t_1) + \lambda_2\rho(x, t_2) + \ldots + \lambda_n\rho(x, t_n),$$

$\lambda_i \geqslant 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $t_i \in T$.

A convex metric space (X, ρ) is called an *M-space* [**7**] if for every two points $x, y \in X$ with $\rho(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that $B[x, r] \cap B[y, \lambda - r] = \{z_r\}$, where $B[x, r] = \{y \in X : \rho(x, y) \leqslant r\}$.

Example 2.1. Let (X, ρ) be a closed ball of $S_{2,r}$ of radius η with $\pi r/4 < \eta < \pi r/2$. Since X is convex and contains no diametral point pairs of the $S_{2,r}$, (X, ρ) is an M-space. Here $S_{2,r}$ is the 2-dimensional spherical space of radius r. Its elements are all the ordered 3-tuples $x = (x_1, x_2, x_3)$ of real numbers with $\sum_{i=1}^{3} x_i^2 = r^2$, distance '$\rho$' is defined for each pair of elements x, y to be the smallest nonnegative number xy such that $\cos(xy/r) = r^{-2}\sum_{i=1}^{3} x_i y_i$. The space (X,ρ) is an M-space [**7**].

A metric space (X,ρ) is called *externally convex* [**7**] if for all distinct points x, y such that $\rho(x,y) = \lambda$, and $r > \lambda$ there exists a unique z of X such that $\rho(x,y) + \rho(y,z) = \rho(x,z) = r$.

EXAMPLE 2.2. Consider metric space (X,ρ) consisting of points on lines $y = 1$ and $y = 2$ in the cartesian plane with $x \geqslant 0$. Let the distance $\rho(x,y)$ for $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be given by $|x_1 - x_2|$ if $y_1 = y_2$ and $1 + |x_1| + |x_2|$ if $y_1 \neq y_2$. This satisfies the condition of external convexity. It may be noted that the space X is not a normed linear space as it is not a linear space [**7**].

If X is a strictly convex Banach space, then it is an externally convex M-space [**7**].

If (X,ρ) is a convex metric space, then for each two distinct points $x, y \in X$ and for every $\lambda \in (0,1)$ there exists a point z such that $\rho(x,z) = (1-\lambda)\rho(x,y)$ and $\rho(z,y) = \lambda\rho(x,y)$. For M-space such a z is always unique.

A linear metric space (X,ρ) is said to be *strictly convex* [**1**] (see also [**15**]) if for $x, y \in X$, $x \neq y$, $\rho(x,0) \leqslant r$, $\rho(y,0) \leqslant r$ imply $\rho(\frac{x+y}{2}, 0) < r$, $r > 0$.

A point x in a convex subset C of a linear metric space (X,ρ) is called an *extreme point* if $x_1, x_2 \in C$ and $x = \lambda x_1 + (1-\lambda)x_2$ for $0 < \lambda < 1$ imply $x_1 = x_2 = x$. The set of extreme points of C is denoted by $\operatorname{ext} C$.

EXAMPLE 2.3. If $\mathbb{R}^3$ is given by the maximum norm

$$\|(\xi_1, \xi_2, \xi_3)\| = \max\{|\xi_1|, |\xi_2|, |\xi_3|\}$$

then the unit ball is a cube. The extreme points are the eight vertices of the cube.

Let C be a bounded subset of a metric space (X,ρ) and $x \in X$; then we define the sets $F_C(x) = \{z \in C : z \text{ is a farthest point in } C \text{ from } x\}$ and $\operatorname{far}(C) = \{z \in C : z \text{ is a farthest point in } C \text{ from some point in } X\}$.

A nonempty subset T of a convex metric space (X,ρ) is said to be *convex* [**17**] if $W(x,y,\lambda) \in T$ for every $x, y \in T$ and $\lambda \in I$.

We shall denote by $[x,y]$ the line segment joining the points x and y i.e., $[x,y] = \{z \in X : \rho(x,z) + \rho(z,y) = \rho(x,y)\}$. The set $[x,y\rangle = \{z \in X : \rho(x,y) + \rho(y,z) = \rho(x,z)\}$ denotes a half ray starting from x and passing through y i.e., it is the union of the line segments $[x,z]$ where $[x,y] \subseteq [x,z]$.

Throughout the paper $\operatorname{cl}(T)$ will stand for closure of T, ∂T for boundary of T, $X \setminus T$ for complement of T in X.

3. On nearest and farthest points

We have the following interesting relationship between the nearest and the farthest points in convex metric spaces (for Banach spaces, see [**10**, Remark 2.2]). Suppose C is a nonempty bounded closed subset of an externally convex M-space (X,ρ). If $z \in C$ is a farthest point from an $x \in X$, then z is also a nearest point in C. Indeed, z is a nearest point in C from any point which is on the line connecting x and z and lies on the opposite side of z to x. So, if there exists no nearest point in C, then there also exists no farthest point in C. This is shown by the following proposition.

PROPOSITION 3.1. *Let C be a nonempty bounded closed subset of an externally convex M-space (X,ρ). If $z \in F_C(x)$ then $z \in P_C(x')$ for every $x' \in [x,z\rangle \smallsetminus [x,z]$.*

PROOF. Suppose $z \in F_C(x)$ then $\rho(x,z) \geqslant \rho(x,y)$ for all $y \in C$. Let $x' \in [x,z\rangle \smallsetminus [x,z]$ be arbitrary. Consider

$$\begin{aligned}\rho(x',z) = \rho(x',x) - \rho(x,z) &\leqslant \rho(x',x) - \rho(x,y) \quad \text{for all } y \in C\\ &\leqslant \rho(x',y) + \rho(x,y) - \rho(x,y) = \rho(x',y) \quad \text{for all } y \in C,\end{aligned}$$

i.e., $\rho(x',z) \leqslant \rho(x',y)$ for all $y \in C$. Hence $z \in P_C(x')$. □

COROLLARY 3.1. *If there exists no nearest point in C, then there also exists no farthest point in C i.e., if C is antiproximinal (very nonproximinal), then C is antipodal.*

REMARK 3.1. Edelstein and Thomson [**4**] gave an example showing that in c_0 (space of all convergent sequences converging to 0) with the usual norm, there exists a nonempty bounded closed convex symmetric subset S which has no nearest point and hence there exist no farthest points in S.

Concerning farthest points, the following two results were proved in [**14**]:

PROPOSITION 3.2. *Let K be a bounded subset of an M-space (X,ρ) and $k_0 \in F_K(x_0)$ for $x_0 \in X$; then $k_0 \in F_K(x)$ for all $x \in [k_0,x_0\rangle \smallsetminus [k_0,x_0]$.*

PROPOSITION 3.3. *Let K be a bounded subset of a convex metric space (X,ρ) and $x_0 \in X$. Then $k_0 \in F_K(x_0)$ iff k_0 is a farthest point from x_0 in $[k_0,y]$ for each $y \in K$.*

Motivated by Propositions 3.2 and 3.3 the following property was considered in farthest points. A bounded subset T of a convex metric space (X,ρ) is said to have *property* (SF) (see [**5, 14**]) if $x_0 \in X$ and $k_0 \in F_K(x_0)$ imply k_0 is a farthest point from y for all $y \in [x_0,k_0]$ i.e., $k_0 \in F_K(W(x_0,k_0,\lambda))$, $0 \leqslant \lambda \leqslant 1$. Concerning property (SF), the following result was proved in [**14**].

PROPOSITION 3.4. *A bounded subset K of a convex metric space (X,ρ) has property (SF) iff K is a singleton.*

In the light of the above three propositions, we discuss the following property in convex metric spaces (in Banach spaces, this property was introduced by Baronti [**2**]). Let (X,ρ) be a convex metric space and K a nonempty subset of X. We say that $P(x,d)$ is true for some $d \in (0,1)$ (see [**2**]) if $y \in F_K(x)$, $y' \in [x,y]$ such that $\rho(y',y) = (1-t)\rho(x,y)$, $\rho(y',x) = t\rho(x,y)$, for $0 < t \leqslant d$ imply $y \in F_K(y')$). Equivalently, $y \in F_K(x) \Rightarrow y \in F_T(W(x,y,t))$, $0 < t \leqslant d$ i.e., if y is the farthest point from x in K, then y is also the farthest point from $W(x,y,t)$ for $0 < t \leqslant d$.

EXAMPLE 3.1. Let $X = R \setminus \{0\}$ with the usual metric and $K = [-1, 1] \setminus \{0\}$. The property $P(x, d)$ is true for $x = 1, -1$ and $d = 1/2$. $F_K(-1) = 1$, $F_K(x) = 1 \equiv y$ for all $x \in [-1, 0)$, $F_K(ty+(1-t)x) = 1$, $0 < t \leqslant d$. $F_K(1) = -1$, $F_K(x) = -1 \equiv y$ for all $x \in (0, -1]$, $F_K(ty + (1 - t)x) = -1$.

REMARK 3.2. $F_K(-1) = 1$ but $F_K(x) \neq 1$ for all $x \in [-1, 1]$ i.e., for all $x \in [-1, F_K(-1)]$. $F_K(1) = -1$ but $F_K(x) \neq -1$ for all $x \in [F_K(1), 1]$.

EXAMPLE 3.2. Let $K = \{(x, y) : x = -\sqrt{1 - y^2}, -1 \leqslant y \leqslant 1\}$ be a subset of R^2 with the usual metric. Then $P(z, d)$ is true for $z = (1, 0)$ and $d = 1/2$, whereas $P(z, d)$ is not true for $z = (0, 0)$.

The number $r(T) = \inf\{r(T, x) : x \in X\}$ where $r(T, x) = \sup_{y\in T} \rho(x, y)$ is called *Chebyshev radius* of T. A *center* or the *Chebyshev center* of T is a point c, if it exists, such that $r(T, c) = r(T)$.

The following result shows that for a remotal set, property $P(c, d)$ can not be true if c is the center of the set.

PROPOSITION 3.5. *If T is a remotal subset of a convex metric space (X, ρ), then $P(c, d)$, $0 < d < 1$ cannot be true if c is the center of the set T.*

PROOF. Let $P(c, d)$ be true where c is a center of T i.e., $r(T, c) = \inf\{r(T, x) : x \in X\}$, so $r(T, c) \leqslant r(T, x)$ for all $x \in X$. Let $y \in F_T(c)$ and property $P(c, d)$ is true. Then for all $y' \in [c, y]$ such that $\rho(y', y) = (1 - t)\rho(c, y)$, $\rho(y', c) = t\rho(c, y)$, for $0 < t \leqslant d$, we have $y \in F_T(y')$. But this implies $r(T, y') = \rho(y', y) < \rho(c, y) = r(T, c)$, which is not true as $r(T, c) = \inf\{r(T, x) : x \in X\}$. □

4. On remotality and unique remotality of convex hulls

A bounded subset T of a metric space (X, ρ) is said to be *remotal* (*uniquely remotal*) if for each $x \in X$ there exists at least one (exactly one) $t \in T$ such that $\rho(x, t) = \sup\{\rho(x, y) : y \in T\} \equiv \delta(x, T)$. Such a point t is called a *farthest point* from x in T.

For remotal sets in convex metric spaces, we have

PROPOSITION 4.1. *A bounded subset T of a convex metric space (X, ρ) is remotal iff* $\mathrm{Conv}(T)$ *is remotal.*

PROOF. Let $T' \equiv \mathrm{Conv}(T)$, T be remotal, and $x \in X$ be arbitrary. Then there exists at least one $t \in T$ such that $\rho(x, t) = \sup\{\rho(x, t_1) : t_1 \in T\} \equiv \delta(x, T)$. Let $t' \in T'$ be arbitrary. Then $t' = W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)$, for some $t_1, t_2, \dots, t_m \in T$, $\lambda_1, \lambda_2, \dots, \lambda_m \in [0, 1]$ and $\sum_{i=1}^m \lambda_i = 1$. Consider

$$\begin{aligned}
\rho(x, t') &= \rho(x, W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)) \\
&\leqslant \lambda_1\rho(x, t_1) + \lambda_2\rho(x, t_2) + \ldots + \lambda_m\rho(x, t_m) \\
&\leqslant \lambda_1\rho(x, t) + \lambda_2\rho(x, t) + \ldots + \lambda_m\rho(x, t) \\
&= (\lambda_1 + \lambda_2 + \ldots + \lambda_m)\rho(x, t) = \delta(x, T).
\end{aligned}$$

Thus $\delta(x,T) = \rho(x,t) \leqslant \sup\{\rho(x,t') : t' \in T'\} \leqslant \delta(x,T)$. Hence $\delta(x,T) = \sup\{\rho(x,t') : t' \in T'\} = \rho(x,t)$. Also $t \in T$ implies $t \in T'$ and so t is farthest point from x in T'. Hence $T' = \mathrm{Conv}(T)$ is remotal.

Conversely, suppose that T' is remotal. Let $x \in X$. Since $T \subset T'$, $\sup\{\rho(x,t) : t \in T\} \leqslant \sup\{\rho(x,t') : t' \in T'\}$ i.e., $\delta(x,T) \leqslant \delta(x,T')$. Now let $t'' \in T'$ i.e., $t'' = W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m)$, for some $t_1,t_2,\dots,t_m \in T$, $\lambda_1,\lambda_2,\dots,\lambda_m \in [0,1]$ and $\sum_{i=1}^m \lambda_i = 1$, such that $\rho(x,t'') = \delta(x,T') = \sup\{\rho(x,t') : t' \in T'\}$. Consider

$$\begin{aligned}\delta(x,T') = \rho(x,t'') &= \rho(x, W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m))\\ &\leqslant \lambda_1\rho(x,t_1) + \lambda_2\rho(x,t_2) + \cdots + \lambda_m\rho(x,t_m)\\ &\leqslant \lambda_1\delta(x,T) + \lambda_2\delta(x,T) + \cdots + \lambda_m\delta(x,T)\\ &= (\lambda_1+\lambda_2+\cdots+\lambda_m)\delta(x,T) = \delta(x,T).\end{aligned}$$

i.e., $\delta(x,T') \leqslant \delta(x,T)$. So, we have $\delta(x,T') = \delta(x,T)$. Now t'' is the farthest point from x in T'. If $t'' \in T$, then T is remotal. If $t'' \notin T$, then $t'' = W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m)$ for some $t_1,t_2,\dots,t_m \in T$, $t_1,t_2,\dots,t_m \neq t''$, $\lambda_1,\lambda_2,\dots,\lambda_m \in [0,1]$, and $\sum_{i=1}^m \lambda_i = 1$ and $\rho(x,t_i) \leqslant \delta(x,T)$. Suppose $\rho(x,t_i) < \delta(x,T)$ for at least one i. Consider

$$\begin{aligned}\delta(x,T) = \delta(x,T') = \rho(x,t'') &= \rho(x, W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m))\\ &\leqslant \lambda_1\rho(x,t_1) + \lambda_2\rho(x,t_2) + \cdots + \lambda_m\rho(x,t_m)\\ &< \lambda_1\delta(x,T) + \lambda_2\delta(x,T) + \cdots + \lambda_m\delta(x,T)\\ &= (\lambda_1+\lambda_2+\cdots+\lambda_m)\delta(x,T) = \delta(x,T).\end{aligned}$$

i.e., $\delta(x,T) < \delta(x,T)$, which is not possible. Therefore our supposition is wrong i.e., $\rho(x,t_i) = \delta(x,T)$ for every i and so all the t_i's are the farthest points from x in T and hence T is remotal. □

Remark 4.1. It was remarked in [**3**] that a bounded subset T of a normed linear space is remotal iff $\mathrm{Conv}(T)$ is remotal.

A result similar to Proposition 4.1 is also true for uniquely remotal sets i.e., we have:

Proposition 4.2. *A bounded subset T of a convex metric space (X,ρ) is uniquely remotal iff* $\mathrm{Conv}(T)$ *is uniquely remotal.*

Proof. Let $T' \equiv \mathrm{Conv}(T)$. We first assume that T is uniquely remotal. Let $x \in X$ be arbitrary and let $t' \in T'$. Then $t' = W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m)$, for some $t_1,t_2,\dots,t_m \in T$, $\lambda_1,\lambda_2,\dots,\lambda_m \in [0,1]$, and $\sum_{i=1}^m \lambda_i = 1$. Consider

$$\begin{aligned}\rho(x,t') &= \rho(x, W(t_1,t_2,\dots,t_m,\lambda_1,\lambda_2,\dots,\lambda_m))\\ &\leqslant \lambda_1\rho(x,t_1) + \lambda_2\rho(x,t_2) + \cdots + \lambda_m\rho(x,t_m)\\ &\leqslant \lambda_1\rho(x,q_T(x)) + \lambda_2\rho(x,q_T(x)) + \cdots + \lambda_m\rho(x,q_T(x))\\ &= (\lambda_1+\lambda_2+\cdots+\lambda_m)\rho(x,q_T(x)) = \rho(x,q_T(x)).\end{aligned}$$

Thus $\rho(x, q_T(x)) \leqslant \sup\{\rho(x, t') : t' \in T'\} \leqslant \rho(x, q_T(x))$. Hence $\rho(x, q_T(x)) = \sup\{\rho(x, t') : t' \in T'\}$. Also $t \in T$ implies $t \in T'$ and so t is the farthest point from x in T'. Hence $T' = \mathrm{Conv}(T)$ is remotal.

Now to prove that T' is uniquely remotal. Let $t'' \neq q_T(x) \in T'$, $t'' \notin T$ be any other element such that $\rho(x, t'') = \sup\{\rho(x, t') : t' \in T'\} = \rho(x, q_T(x))$. Therefore, $t'' = W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)$, for some $t_1, t_2, \dots, t_m \in T$, $\lambda_1, \lambda_2, \dots, \lambda_m \in [0, 1]$, $\sum_{i=1}^{m} \lambda_i = 1$ and $t'' \neq t_1, t_2, \dots, t_m$. Also $\rho(x, t_i) \leqslant \rho(x, q_T(x))$. Let $\rho(x, t_i) < \rho(x, q_T(x))$ for at least one i. Consider

$$\begin{aligned}
\rho(x, q_T(x)) = \rho(x, t'') &= \rho(x, W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)) \\
&\leqslant \lambda_1 \rho(x, t_1) + \lambda_2 \rho(x, t_2) + \cdots + \lambda_m \rho(x, t_m) \\
&< \lambda_1 \rho(x, q_T(x)) + \lambda_2 \rho(x, q_T(x)) + \cdots + \lambda_m \rho(x, q_T(x)) \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_m) \rho(x, q_T(x)) = \rho(x, q_T(x)).
\end{aligned}$$

i.e., $\rho(x, q_T(x)) < \rho(x, q_T(x))$, which is not possible. Therefore our supposition is wrong and so $\rho(x, t_i) = \rho(x, q_T(x))$ for every i and so all the t_i's are the farthest points from x in T, which is also not true as T is uniquely remotal. Therefore our supposition that $t'' \notin T$, $t'' \neq q_T(x)$ is also wrong and hence T' is uniquely remotal.

Conversely, suppose that T' is uniquely remotal. Let $x \in X$. Since $T \subset T'$, $\sup\{\rho(x, t) : t \in T\} \leqslant \sup\{\rho(x, t') : t' \in T'\}$ i.e., $\rho(x, q_T(x)) \leqslant \rho(x, q_{T'}(x))$. If $q_{T'}(x) \in T$, then $\rho(x, q_{T'}(x)) \leqslant \rho(x, q_T(x))$ and if $q_{T'}(x) \notin T$, then $q_{T'}(x) = W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)$, for some $t_1, t_2, \dots, t_m \in T$, $\lambda_1, \lambda_2, \dots, \lambda_m \in [0, 1]$, and $\sum_{i=1}^{m} \lambda_i = 1$.

$$\begin{aligned}
\rho(x, q_{T'}(x)) &= \rho(x, W(t_1, t_2, \dots, t_m, \lambda_1, \lambda_2, \dots, \lambda_m)) \\
&\leqslant \lambda_1 \rho(x, t_1) + \lambda_2 \rho(x, t_2) + \cdots + \lambda_m \rho(x, t_m) \\
&\leqslant \lambda_1 \rho(x, q_T(x)) + \lambda_2 \rho(x, q_T(x)) + \cdots + \lambda_m \rho(x, q_T(x)) \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_m) \rho(x, q_T(x)) = \rho(x, q_T(x)).
\end{aligned}$$

Therefore $\rho(x, q_{T'}(x)) \leqslant \rho(x, q_T(x))$ and so $\rho(x, q_{T'}(x)) = \rho(x, q_T(x))$. The same argument as above shows that $q_T(x) = \{q_{T'}(x)\}$. Hence T is uniquely remotal. □

REMARK 4.2. It was remarked in [**11**] that a bounded subset T of a normed linear space is uniquely remotal iff $\mathrm{Conv}(T)$ is uniquely remotal.

5. On continuity of farthest point map

The set-valued map $F_T : X \to 2^T$, where 2^T is the collection of all subsets of T, defined by $F_T(x) = \{t \in T : \rho(x, t) = \sup_{y \in T} \rho(x, y)\}$ is called the *farthest point map* (f.p.m.). For uniquely remotal sets T, the f.p.m. F_T is single-valued and is denoted by q_T.

In the next result, we study the continuity of the farthest point map in externally convex M-spaces.

PROPOSITION 5.1. *Let T be a uniquely remotal subset of an externally convex M-space (X, ρ), then the f.p.m. q_T is continuous at x_0 iff the restriction of q_T to E_{x_0} is continuous, where $E_{x_0} = \{x \in X : \rho(x, q_T(x)) \geqslant \rho(x_0, q_T(x_0))\}$.*

PROOF. Suppose that the restriction of q_T to E_{x_0} is continuous. If T is singleton, say $T = \{t\}$, then $q_T(x) = t$ for all $x \in X$. Let $x \in X$ be arbitrary with $\rho(x, x_0) < \delta$. Then $\rho(q_T(x), q_T(x_0)) = \rho(t, t) = 0 < \varepsilon$ for all ε. Hence q_T is continuous at x_0. So we can assume that T is not a singleton. Let V be a neighborhood of $q_T(x_0)$. Then by the continuity of $q_T\big|_{E_{x_0}}$, there is a neighborhood of x_0, say G, such that $q_T(E_{x_0} \cap G) \subseteq V$. Choose $\delta > 0$, such that the open ball $B_{2\delta}(x_0)$ is contained in G, where $B_{2\delta}(x_0) = \{x \in X : d(x, x_0) < 2\delta\}$. Let $x \in B_\delta(x_0) \setminus E_{x_0}$ and $y \in [q_T(x), x, -[$ be such that $\rho(y, q_T(x)) = \rho(x_0, q_T(x_0))$. Consider

$$\rho(y, q_T(x)) = \rho(y, x) + \rho(x, q_T(x)) \geqslant \rho(y, x) + \rho(x, z) \geqslant \rho(y, z) \text{ for all } z \in T.$$

Thus $q_T(x) = q_T(y)$. So, we get $\rho(y, q_T(y)) = \rho(x_0, q_T(x_0))$. This gives $y \in E_{x_0}$. Consider

$$\begin{aligned}
\rho(y, x_0) &\leqslant \rho(y, x) + \rho(x, x_0) = \rho(y, q_T(x)) - \rho(x, q_T(x)) + \rho(x, x_0) \\
&= \rho(x_0, q_T(x_0)) - \rho(x, q_T(x)) + \rho(x, x_0) \\
&\leqslant \rho(x_0, x) + \rho(x, q_T(x_0)) - \rho(x, q_T(x)) + \rho(x, x_0) \\
&\leqslant \rho(x_0, x) + \rho(x, x_0) = 2\rho(x, x_0) < 2\delta
\end{aligned}$$

i.e., $\rho(y, x_0) < 2\delta$ implies $y \in B_{2\delta}(x_0) \subset G$. So $y \in G \cap E_{x_0}$ gives $q_T(y) \in V$. Therefore $q_T(B_\delta(x_0)) \subseteq V$ as $x \in B_\delta(x_0) \setminus E_{x_0}$ is arbitrary and $q_T(x) \in V$. Hence q_T is continuous at x_0.

Conversely, suppose q_T is continuous at x_0. Since restriction of a continuous map is continuous, $q_T\big|_{E_{x_0}}$ is continuous at x_0. ☐

REMARK 5.1. For normed linear spaces this proposition was proved in [**13**].

6. On unique remotality of closure of a set

PROPOSITION 6.1. *Let T be a nonempty uniquely remotal subset of a strongly externally convex metric space (X, ρ). Then $\mathrm{cl}(T) \equiv \overline{T}$ is also uniquely remotal and the farthest point maps $q_T : X \to T$ and $q_{\overline{T}} : X \to \overline{T}$ coincide.*

PROOF. Let $x \in X$ and set $r = \sup\{\rho(x, z) : z \in T\} = \sup\{\rho(x, z) : z \in \mathrm{cl}(T)\}$. If $r = 0$, then $T = \mathrm{cl}(T) = \{x\}$ and hence the assertion of proposition holds. So let $r > 0$ and suppose that $y \in \mathrm{cl}(T)$ satisfy $\rho(x, y) = r$ (i.e., y is the farthest point in $\mathrm{cl}(T)$ from x). To show that $\mathrm{cl}(T)$ is uniquely remotal, we shall show that $y \in T$ and hence $y = q_T(x)$. By external convexity, there exists a unique $x_0 \in X$ with

$$\rho(x, x_0) = r \text{ and } \rho(y, x_0) = \rho(y, x) + \rho(x, x_0) = 2r \tag{6.1}$$

On the other hand, $T \subset B[x, r]$ implies $\rho(x_0, z) \leqslant \rho(x_0, x) + \rho(x, z) \leqslant 2r$ for all $z \in T$. Hence $\rho(x_0, z) \leqslant 2r$ holds for every $z \in \mathrm{cl}(T)$. Therefore (6.1) implies y is a farthest point in $\mathrm{cl}(T)$ from x_0.

Next, Let $z_0 = q_T(x_0) \in T$. Then, $\rho(x_0, z_0) = 2r$ and hence

$$2r = \rho(x_0, z_0) \leqslant \rho(x_0, x) + \rho(x, z_0) \leqslant 2r$$

holds(using (6.1) and $T \subset B[x,r]$). Therefore, $2r = \rho(x_0, z_0) = \rho(x_0, x) + \rho(x, z_0)$. Together with (6.1) and the uniqueness assertion in the definition of external convexity, this implies $y = z_0 \in T$ (as $x_0 \neq x$). $\square$

REMARK 6.1. It was remarked in [**11**] that in strictly convex normed linear spaces, the unique farthest point property is inherited by closure of the set.

7. A characterization of linear metric spaces

The geometric nature of the underlying space plays an important role in discussing nearest and farthest points. The following proposition shows that structure of the set of farthest points can be used to characterize strictly convex linear metric spaces.

PROPOSITION 7.1. *Let (X, ρ) be a linear metric space; then the following are equivalent:*
(1) *X is strictly convex.*
(2) *For a bounded closed convex subset C of X,* $\mathrm{far}(C) \subset \mathrm{ext}(C)$*.*
(3) *For a compact convex subset C of X,* $\mathrm{far}(C) \subset \mathrm{ext}(C)$*.*

PROOF. (1) $\Rightarrow$ (2) Let X be strictly convex and C a bounded closed convex subset of X. Suppose $\mathrm{far}(C) \not\subset \mathrm{ext}(C)$ i.e., there exists $x' \in \mathrm{far}(C)$ such that $x' \notin \mathrm{ext}(C)$. Then there exist $x, y \in C$, $x \neq y$ such that x' is mid-point of x and y i.e., $x' = \frac{x+y}{2}$. Now $x' \in \mathrm{far}(C)$ implies $x' \in F_C(y')$ for some $y' \in X$ i.e., $\rho(x', y') = \sup\{\rho(z, y') : z \in C\} \equiv r$. Since $x, y \in C$, $\rho(x, y') \leqslant r$, $\rho(y, y') \leqslant r$ and so the strict convexity of X implies $\rho(x', y') < r$, which is absurd. Hence $\mathrm{far}(C) \subset \mathrm{ext}(C)$.
(2) $\Rightarrow$ (3) This is obvious.
(3) $\Rightarrow$ (1) Suppose $\mathrm{far}(C) \subset \mathrm{ext}(C)$ for every compact convex subset C of X. Let if possible, X be not strictly convex. Then there is a segment $[a, b]$ on the sphere $S(0, r)$, $a, b \in X$ (see [**16**]). Let $C = [a, b]$; then $C \subset \mathrm{far}(C)$ as every point of $[a, b]$ is farthest from 0 but $y \notin \mathrm{ext}(C)$ for any $y \in (a, b)$, contradicting the fact that $\mathrm{far}(C) \subset \mathrm{ext}(C)$. Hence X is strictly convex. $\square$

The following example shows that if X is not strictly convex, then a farthest point in C need not be an extreme point of C.

EXAMPLE 7.1. Let $X = \mathbb{R}^2$ be the metric space with $\rho(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and let $C = \{(-1, y) : -1 \leqslant y \leqslant 1\}$. Then $z = (-1, 0)$ is the farthest point from the origin $(0, 0)$ since its distance from each element of C is 1. Hence $\mathrm{far}(C)$ is nonempty as $z \in \mathrm{far}(C)$ but z is not an extreme point of C [**18**].

REMARK 7.1. For normed linear spaces such a characterization of strictly convex normed linear spaces was given in [**10**].

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