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ON PARA-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We study a Para-Sasakian manifold admitting a semi-symmetric metric connection whose projective curvature tensor satisfies certain curvature conditions.

1. Introduction

In [**19**], Takahashi introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. In respect of contact geometry, the notion of ϕ-symmetric was introduced and studied by Boeckx, Buecken and Vanhecke [**4**] with several examples. In [**5**], De studied the notion of ϕ-symmetry with several examples for Kenmotsu manifolds. In 1977, Adati and Matsumoto defined para-Sasakian and special para-Sasakian manifolds [**2**], which are special classes of an almost paracontact manifold introduced by Sato [**17**]. Para-Sasakian manifolds have been studied by Tarafdar and De [**20**], De and Pathak [**11**], Matsumoto, Ianus and Mihai [**15**], Matsumoto [**14**] and many others.

Hayden [**13**] introduced semi-symmetric linear connections on a Riemannian manifold. Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^n, g).

A linear connection $\bar{\nabla}$ defined on (M^n, g) is said to be semi-symmetric [**12**] if its torsion tensor T is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$, where η is a 1-form and ξ is a vector field defined by $\eta(X) = g(X,\xi)$, for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n. A semi-symmetric connection $\bar{\nabla}$ is called a semi-symmetric metric connection [**13**] if it further satisfies $\nabla g = 0$. A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M^n, g) has been obtained by Yano [**21**] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{1.1}$$

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We also have $(\bar{\nabla}_X\eta)(Y) = (\nabla_X\eta)Y - \eta(X)\eta(Y) + \eta(\xi)g(X,Y)$. Further, a relation between the curvature tensor $\bar{R}$ of the semi-symmetric metric connection $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection ∇ is given by

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + g(X,Z)QY - g(Y,Z)QX, \tag{1.2}$$

where α is a tensor field of type (0,2) and Q is a tensor field of type (1,1) which is given by

$$\alpha(Y,Z) = g(QY,Z) = (D_Y\eta)(Z) - \eta(Y)\eta(Z) + \tfrac{1}{2}\eta(\xi)g(Y,Z). \tag{1.3}$$

From (1.2) and (1.3), we obtain

$$\begin{aligned}\tilde{\bar{R}}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &- \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W)\\ &- g(Y,Z)\alpha(X,W) + g(X,Z)\alpha(Y,W),\end{aligned}$$

where $\tilde{\bar{R}}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$, $\quad \tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

The semi-symmetric metric connections have been studied by several authors such as Yano [**21**], Amur and Pujar [**1**], Prvanović [**16**], De and Biswas [**10**], Sharfuddin and Hussain [**18**], Binh [**3**], De [**6, 7**], De and De [**8, 9**] and many others.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geqslant 1$, M is locally projectively flat if and only if the projective curvature tensor P vanishes. Here the projective curvature tensor P with respect to the semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \tfrac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \tag{1.4}$$

From (1.4), it follows that

$$\begin{aligned}&\tilde{\bar{P}}(X,Y,Z,W) = \tilde{\bar{R}}(X,Y,Z,W) - \tfrac{1}{2n}[\bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W)],\\ &\tilde{\bar{P}}(X,Y,Z,W) = g(\bar{P}(X,Y)Z,W),\end{aligned}$$

for $X,Y,Z,W \in \chi(M)$, where $\bar{S}$ is the Ricci tensor with respect to the semi-symmetric metric connection. In fact M is projectively flat if and only if it is of constant curvature [**22**]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

The paper is organized as follows: Section 2 is equipped with some prerequisites about P-Sasakian manifolds. In section 3, we establish the relation of the curvature tensor between the Levi-Civita connection and the semi-symmetric metric connection of a P-Sasakian manifold. A P-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and manifold if recurrent with respect to the Levi-Civita connection is studied in Section 4. Section 5 is devoted to study ξ-projectively flat P-Sasakian manifolds with respect to the semi-symmetric metric connection. Finally, we investigate locally ϕ-projectively symmetric P-Sasakian manifolds with respect to the semi-symmetric metric connection.

2. P-Sasakian manifolds

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; \eta(\xi) = 1, \;\; g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.4}$$

for any vector fields X, Y on M. In addition, if (ϕ, ξ, η, g), satisfy the equations

$$d\eta = 0, \;\; \nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

It is known [**2, 17**] that in a P-Sasakian manifold the following relations hold:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3. Curvature tensor of a P-Sasakian manifold with respect to the semi-symmetric metric connection

THEOREM 3.1. *For a P-Sasakian manifold M with respect to the semi-symmetric metric connection $\bar{\nabla}$*

(i) *The curvature tensor $\bar{R}$ is given by* (3.3),
(ii) *The Ricci tensor $\bar{S}$ is given by* (3.5),
(iii) *The scalar curvature $\bar{r}$ is given by* (3.6),
(iv) $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,
(v) $\eta(\bar{R}(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z) + \eta(Y)g(X, \phi Z) - \eta(X)g(Y, \phi Z)$,
(vi) *The Ricci tensor $\bar{S}$ is symmetric,*
(vii) $\bar{S}(Y, \xi) = -(n-1+\gamma)\eta(Y)$,
(viii) $(\bar{\nabla}_W \phi)(X) = -g(X, W)\xi - \eta(X)W + 2\eta(X)\eta(W)\xi - g(X, \phi W)\xi - \eta(X)\phi W$,
(ix) $(\bar{\nabla}_W \eta)(X) = g(X, \phi W) - \eta(X)\eta(W) + g(X, W)$,

(x) $\bar{\nabla}_W\xi = \phi W + W - \eta(W)\xi$.

Proof. Using (2.4) and (2.1) in (1.3), we get

$$\alpha(X,Y) = g(QX,Y) = g(X,\phi Y) - \eta(X)\eta(Y) + \tfrac{1}{2}g(X,Y). \tag{3.1}$$

From (3.1) implies that

$$QX = \phi X - \eta(X)\xi + \tfrac{1}{2}X. \tag{3.2}$$

Again using (3.1) and (3.2) in (1.2), we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ g(X,\phi Z)Y - \eta(X)\eta(Z)Y - g(Y,\phi Z)X\\ &+ \eta(Y)\eta(Z)X + g(X,Z)Y - g(Y,Z)X + g(X,Z)\phi Y\\ &- g(Y,Z)\phi X - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi.\end{aligned} \tag{3.3}$$

From (3.3), we obtain that the curvature tensor $\bar{R}$ satisfies $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$. Using (2.7) and (2.1) in (3.3), implies that

$$\eta(\bar{R}(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) + \eta(Y)g(X,\phi Z) - \eta(X)g(Y,\phi Z).$$

Taking the inner product of (3.3) with W, it follows that

$$\begin{aligned}\bar{\tilde{R}}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &+ g(X,\phi Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W)\\ &- g(Y,\phi Z)g(X,W) + \eta(Y)\eta(Z)g(X,W) + g(X,Z)g(Y,W)\\ &- g(Y,Z)g(X,W) + g(X,Z)g(\phi Y,W) - g(Y,Z)g(\phi X,W)\\ &- g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W).\end{aligned} \tag{3.4}$$

Let $\{e_1,\dots,e_n\}$ be a local orthonormal basis of vector fields in M. Then by putting $X = W = e_i$ in (3.4), summing over i, $1 \leqslant i \leqslant n$, and using (2.1), we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)g(Y,\phi Z) + (n-2)\eta(Y)\eta(Z) - (n-2+\gamma)g(Y,Z), \tag{3.5}$$

where trace of $\phi = \gamma$. Again by putting $Y = Z = e_i$ in (3.5), summing over i, $1 \leqslant i \leqslant n$ and using (2.1), we get

$$\bar{r} = r - 2(n-1)\gamma - (n-1)(n-2) \tag{3.6}$$

where $\bar{r}$ and r are the scalar curvatures with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively. Again putting $Z = \xi$ in (3.5) and using (2.1) and (2.11), we get $\bar{S}(Y,\xi) = -(n-1+\gamma)\eta(Y)$. Using (1.1), (2.1) and (2.6), implies that

$$(\bar{\nabla}_W\phi)(X) = -g(X,W)\xi - \eta(X)W + 2\eta(X)\eta(W)\xi - g(X,\phi W)\xi - \eta(X)\phi W. \tag{3.7}$$

Using (1.1), (2.1) and (2.4), it follows that

$$(\bar{\nabla}_W\eta)(X) = g(X,\phi W) - \eta(X)\eta(W) + g(X,W). \tag{3.8}$$

Again using (1.1), (2.1) and (2.5), we get

$$\bar{\nabla}_W\xi = \phi W + W - \eta(W)\xi. \tag{3.9}$$

□

4. A P-Sasakian manifold (M^n, g) whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection

THEOREM 4.1. *If an n-dimensional P-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection and the associated 1-form A is equal to the associated 1-form η, then the manifold is an η-Einstein manifold.*

DEFINITION 4.1. A P-Sasakian manifold M with respect to the Levi-Civita connection is called recurrent if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z, \tag{4.1}$$

where A is the 1-form.

DEFINITION 4.2. A P-Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(Z,W) = ag(Z,W) + b\eta(Z)\eta(W),$$

where a and b are smooth functions on the manifold.

PROOF. Using (1.1), (2.7), (2.8) and (2.10), we obtain

$$\begin{aligned}(\bar{\nabla}_W R)(X,Y)Z &= \bar{\nabla}_W R(X,Y)Z - R(\bar{\nabla}_W X,Y)Z - R(X,\bar{\nabla}_W Y)Z \\ &\quad - R(X,Y)\bar{\nabla}_W Z = (\nabla_W R)(X,Y)Z - \tilde{R}(X,Y,Z,W)\xi \\ &\quad - \eta(X)R(W,Y)Z - \eta(Y)R(X,W)Z - \eta(Z)R(X,Y)W \\ &\quad + \eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W + \eta(Z)g(X,W)Y \\ &\quad - g(X,W)g(Y,Z)\xi - \eta(Z)g(Y,W)X + g(X,Z)g(Y,W)\xi \\ &\quad + \eta(X)g(Z,W)Y - \eta(Y)g(Z,W)X.\end{aligned} \tag{4.2}$$

Suppose $(\bar{\nabla}_W R)(X,Y)Z = 0$, then from (4.2), we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z &- \tilde{R}(X,Y,Z,W)\xi - \eta(X)R(W,Y)Z - \eta(Y)R(X,W)Z \\ &- \eta(Z)R(X,Y)W + \eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W \\ &+ \eta(Z)g(X,W)Y - g(X,W)g(Y,Z)\xi - \eta(Z)g(Y,W)X \\ &+ g(X,Z)g(Y,W)\xi + \eta(X)g(Z,W)Y - \eta(Y)g(Z,W)X = 0.\end{aligned} \tag{4.3}$$

Using (4.1) in (4.3), we have

$$\begin{aligned}A(W)R(X,Y)Z &- \tilde{R}(X,Y,Z,W)\xi - \eta(X)R(W,Y)Z - \eta(Y)R(X,W)Z \\ &- \eta(Z)R(X,Y)W + \eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W \\ &+ \eta(Z)g(X,W)Y - g(X,W)g(Y,Z)\xi - \eta(Z)g(Y,W)X \\ &+ g(X,Z)g(Y,W)\xi + \eta(X)g(Z,W)Y - \eta(Y)g(Z,W)X = 0.\end{aligned} \tag{4.4}$$

Now contracting X in (4.4) and using (2.1) and (2.7), it follows that

$$\begin{aligned} A(W)S(Y,Z) - \eta(Y)S(Z,W) - \eta(Z)S(Y,W) \\ - (n-1)\eta(Z)g(Y,W) - (n-1)\eta(Y)g(Z,W) = 0. \end{aligned} \tag{4.5}$$

Putting $Y = \xi$ in (4.5) and using (2.1) and (2.11), we obtain

$$S(Z,W) = (1-n)g(Z,W) + (1-n)A(W)\eta(Z). \tag{4.6}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (4.6), we get $S(Z,W) = (1-n)g(Z,W) + (1-n)\eta(W)\eta(Z)$. Therefore, $S(Z,W) = ag(Z,W) + b\eta(Z)\eta(W)$, where $a = (1-n)$ and $b = (1-n)$. □

5. ξ-projectively flat P-Sasakian manifolds with respect to the semi-symmetric metric connection

THEOREM 5.1. *An n-dimensional P-Sasakian manifold is ξ-projectively flat with respect to the semi-symmetric metric connection if and only if the manifold is also ξ-projectively flat with respect to the Levi-Civita connection provided the vector fields X and Y are horizontal vector fields.*

PROOF. Using (3.3) in (1.4), we have

$$\begin{aligned} \bar{P}(X,Y)Z = R(X,Y)Z + g(X,\phi Z)Y - \eta(X)\eta(Z)Y - g(Y,\phi Z)X + \eta(Y)\eta(Z)X \\ + g(X,Z)Y - g(Y,Z)X + g(X,Z)\phi Y - g(Y,Z)\phi X - g(X,Z)\eta(Y)\xi \\ + g(Y,Z)\eta(X)\xi - \tfrac{1}{n-1}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \end{aligned} \tag{5.1}$$

Using (3.5) in (5.1), it follows that

$$\begin{aligned} \bar{P}(X,Y)Z = P(X,Y)Z + \tfrac{1}{n-1}\big[g(X,\phi Z)Y - g(Y,\phi Z)X - \eta(X)\eta(Z)Y \\ + \eta(Y)\eta(Z)X + (1-\gamma)g(X,Z)Y - (1-\gamma)g(Y,Z)X\big] \\ + g(X,Z)\phi Y - g(Y,Z)\phi X - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \end{aligned} \tag{5.2}$$

where

$$P(X,Y)Z = R(X,Y)Z - \tfrac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{5.3}$$

is the projective curvature tensor with respect to the Levi-Civita connection.

Putting $Z = \xi$ in (5.2) and using (2.1), we obtain

$$\bar{P}(X,Y)\xi = P(X,Y)\xi + \tfrac{1}{n-1}[\gamma\eta(Y)X - \gamma\eta(X)Y] + \eta(X)\phi Y - \eta(Y)\phi X. \tag{5.4}$$

Suppose X and Y are orthogonal to ξ; then from (5.4), we get

$$\bar{P}(X,Y)\xi = P(X,Y)\xi,$$

concluding the proof. □

6. Locally ϕ-projectively symmetric P-Sasakian manifolds with respect to the semi-symmetric metric connection

Theorem 6.1. *An n-dimensional P-Sasakian manifold is locally ϕ-projectively symmetric with respect to the semi-symmetric metric connection if and only if the manifold is also locally ϕ-projectively symmetric with respect to the Levi-Civita connection.*

Definition 6.1. A P-Sasakian manifold M with respect to the semi-symmetric metric connection is said to be locally ϕ-projectively symmetric if

$$\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) = 0,$$

for all vector fields X, Y, Z, W are orthogonal to ξ.

Proof. Using (1.1), we get

$$\begin{aligned}(\bar{\nabla}_W P)(X,Y)Z &= \bar{\nabla}_W P(X,Y)Z - P(\bar{\nabla}_W X,Y)Z - P(X,\bar{\nabla}_W Y)Z - P(X,Y)\bar{\nabla}_W Z \\ &= (\nabla_W P)(X,Y)Z + \eta(P(X,Y)Z)W - \eta(X)P(W,Y)Z - \eta(Y)P(X,W)Z \\ &\quad - \eta(Z)P(X,Y)W - \tilde{P}(X,Y,Z,W)\xi + g(X,W)P(\xi,Y)Z \\ &\quad + g(Y,W)P(X,\xi)Z + g(Z,W)P(X,Y)\xi. \end{aligned} \tag{6.1}$$

Putting $X = \xi$ in (5.3) and using (2.8) and (2.11), we have

$$P(\xi,Y)Z = -g(Y,Z)\xi - \tfrac{1}{n-1}S(Y,Z)\xi. \tag{6.2}$$

Putting $Y = \xi$ in (5.3) and using (2.8) and (2.11), it follows that

$$P(X,\xi)Z = g(X,Z)\xi + \tfrac{1}{n-1}S(X,Z)\xi. \tag{6.3}$$

Again putting $Z = \xi$ in (5.3) and using (2.10) and (2.11),

$$P(X,Y)\xi = 0. \tag{6.4}$$

Using (2.7), (5.3), (6.2), (6.3), (6.4) in (6.1), we obtain

$$\begin{aligned}(\bar{\nabla}_W P)(X,Y)Z &= (\nabla_W P)(X,Y)Z - \eta(X)P(W,Y)Z - \eta(Y)P(X,W)Z \\ &\quad - \eta(Z)P(X,Y)W + \eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W \\ &\quad - \tfrac{1}{n-1}\big[\eta(X)S(Y,Z)W - \eta(Y)S(X,Z)W\big] - \tilde{P}(X,Y,Z,W)\xi \\ &\quad - g(X,W)\big[g(Y,Z)\xi + \tfrac{1}{n-1}S(Y,Z)\xi\big] \\ &\quad + g(Y,W)\big[g(X,Z)\xi + \tfrac{1}{n-1}S(X,Z)\xi\big]. \end{aligned} \tag{6.5}$$

Taking covariant differentiation of (5.2) with respect to W and using (3.7), (3.8), (3.9) and (6.5), we get

$$
\begin{aligned}
(\bar{\nabla}_W \bar{P})(X,Y)Z = (\nabla_W P)(X,Y)Z - \eta(X)P(W,Y)Z - \eta(Y)P(X,W)Z \\
- \eta(Z)P(X,Y)W - \tilde{P}(X,Y,Z,W)\xi \\
+ \tfrac{1}{n-1}\big[\eta(Y)S(X,Z)W - \eta(X)S(Y,Z)W - g(X,W)S(Y,Z)\xi + g(Y,W)S(X,Z)\xi \\
- \eta(Z)g(X,\phi W)Y + \eta(Z)g(Y,\phi W)X + 2\eta(X)\eta(Z)\eta(W)Y \\
- 2\eta(Y)\eta(Z)\eta(W)X + (n-2)\eta(Z)g(X,W)Y - (n-2)\eta(Z)g(Y,W)X \\
- \eta(X)g(Z,\phi W)Y + \eta(Y)g(Z,\phi W)X - \eta(X)g(Z,W)Y + \eta(Y)g(Z,W)X\big] \\
- \eta(Z)g(X,W)Y + \eta(Z)g(Y,W)X - g(X,Z)g(Y,W)\xi + g(X,W)g(Y,Z)\xi \\
- \eta(Y)g(X,Z)W + \eta(X)g(Y,Z)W + 4\eta(Y)\eta(W)g(X,Z)\xi - 4\eta(X)\eta(W)g(Y,Z)\xi \\
- 2g(X,Z)g(Y,\phi W)\xi + 2g(X,\phi W)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\phi W + 2\eta(X)g(Y,Z)\phi W.
\end{aligned} \tag{6.6}
$$

Now applying ϕ^2 on both sides of (6.6) and using (2.1) and (2.2), it follows that

$$
\begin{aligned}
\phi^2\big((\bar{\nabla}_W \bar{P})(X,Y)Z\big) = \phi^2\big((\nabla_W P)(X,Y)Z\big) - \eta(X)P(W,Y)Z + \eta(X)\eta(P(W,Y)Z)\xi \\
- \eta(Y)P(X,W)Z + \eta(Y)\eta(P(X,W)Z)\xi - \eta(Z)P(X,Y)W + \eta(Z)\eta(P(X,Y)W)\xi \\
+ \tfrac{1}{n-1}\big[\eta(Y)S(X,Z)W - \eta(Y)\eta(W)S(X,Z)\xi - \eta(X)S(Y,Z)W + \eta(X)\eta(W)S(Y,Z)\xi \\
- \eta(Z)g(X,\phi W)Y + \eta(Z)\eta(Y)g(X,\phi W)\xi + \eta(Z)g(Y,\phi W)X - \eta(Z)\eta(X)g(Y,\phi W)\xi \\
+ 2\eta(X)\eta(Z)\eta(W)Y - 2\eta(Y)\eta(Z)\eta(W)X + (n-2)\eta(Z)g(X,W)Y \\
+ \eta(Z)\eta(Y)g(X,W)\xi - (n-2)\eta(Z)g(Y,W)X - \eta(Z)\eta(X)g(Y,W)\xi \\
- \eta(X)g(Z,\phi W)Y + \eta(Y)g(Z,\phi W)X - \eta(X)g(Z,W)Y + \eta(Y)g(Z,W)X\big] \\
- \eta(Z)g(X,W)Y + \eta(Z)g(Y,W)X - \eta(Y)g(X,Z)W + \eta(Y)\eta(W)g(X,Z)\xi \\
+ \eta(X)g(Y,Z)W - \eta(X)\eta(W)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\phi W + 2\eta(X)g(Y,Z)\phi W.
\end{aligned} \tag{6.7}
$$

Taking X, Y, Z and W are orthogonal to ξ, then from (6.7), we have

$$\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) = \phi^2((\nabla_W P)(X,Y)Z).$$

This completes the proof. □

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