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ON RECOGNITION BY PRIME GRAPH OF THE PROJECTIVE SPECIAL LINEAR GROUP OVER GF(3)

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ABSTRACT. Let G be a finite group. The prime graph of G is denoted by $\Gamma(G)$. We prove that the simple group $\mathrm{PSL}_n(3)$, where $n \geqslant 9$, is quasirecognizable by prime graph; i.e., if G is a finite group such that $\Gamma(G) = \Gamma(\mathrm{PSL}_n(3))$, then G has a unique nonabelian composition factor isomorphic to $\mathrm{PSL}_n(3)$. Darafsheh proved in 2010 that if $p > 3$ is a prime number, then the projective special linear group $\mathrm{PSL}_p(3)$ is at most 2-recognizable by spectrum. As a consequence of our result we prove that if $n \geqslant 9$, then $\mathrm{PSL}_n(3)$ is at most 2-recognizable by spectrum.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *spectrum* of a finite group G which is denoted by $\omega(G)$ is the set of its element orders. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, \ldots, s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, we always suppose that $2 \in \pi_1(G)$. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise nonadjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly if $p \in \pi(G)$, then let $\rho(p, G)$ be some independent set with the maximal number of vertices in $\Gamma(G)$ containing p and let $t(p, G) = |\rho(p, G)|$.

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A finite group G is called *recognizable by prime graph* whenever if for a finite group H, we have $\Gamma(H) = \Gamma(G)$, then $H \cong G$. A nonabelian simple group P is called *quasirecognizable by prime graph* if every finite group whose prime graph is $\Gamma(P)$ has a unique nonabelian composition factor which is isomorphic to P (see [**16**]). Obviously recognition (quasirecognition) by prime graph implies recognition (quasirecognition) by spectrum, but the converse is not true in general. Also some methods of recognition by spectrum cannot be used for recognition by prime graph. If Ω is a nonempty subset of the set of natural numbers, we denote by $h(\Omega)$ the number of nonisomorphic groups G with $\omega(G) = \Omega$. If G is a finite group, then $h(\omega(G))$ is denoted by $h(G)$. If $h(G) = \infty$, then G is called *nonrecognizable by spectrum*. If $h(G) = r$, then G is called *r-recognizable by spectrum*.

Hagie in [**12**], determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [**16, 35**]. A group G is called a CIT group if G is of even order and the centralizer in G of any involution is a 2-group. In [**18**], finite groups with the same prime graph as a CIT simple group are determined. Also in [**19**], it is proved that if $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, then $\mathrm{PSL}_2(p)$ is recognizable by prime graph. In [**17, 23**], finite groups with the same prime graph as $\mathrm{PSL}_2(q)$, where q is not prime, are determined. It is proved that the simple group $F_4(q)$, where $q = 2^n > 2$ (see [**15**]) and ${}^2F_4(q)$ (see [**1**]) are quasirecognizable by prime graph. In [**14**], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11, 13, 19$ and $\Gamma(G) = \Gamma(\mathrm{PGL}_2(p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}_2(p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}_2(13)$ or $\mathrm{PSL}_2(27)$. Then it is proved that if p and $k > 1$ are odd and $q = p^k$ is a prime power, then $\mathrm{PGL}_2(q)$ is uniquely determined by its prime graph [**2**]. In [**20, 21, 22, 24, 27, 28**] finite groups with the same prime graph as $\mathrm{PSL}_n(2)$, $U_n(2)$, $D_n(2)$, $B_n(3)$ and ${}^2D_n(2)$ are obtained. In [**3, 4**], it is proved that ${}^2D_{2^m+1}(3)$ is recognizable by prime graph.

The projective special linear groups defined over a finite field of order 3, called the ternary field, are denoted by $\mathrm{PSL}_n(3)$, $\mathrm{PSL}(n, 3)$, $L_n(3)$ or $A_{n-1}(3)$ as a finite group of Lie type. In this paper as the main result we prove that the simple group $\mathrm{PSL}_n(3)$, where $n \geqslant 9$, is quasirecognizable by prime graph; i.e., if G is a finite group such that $\Gamma(G) = \Gamma(\mathrm{PSL}_n(3))$, then G has a unique nonabelian composition factor isomorphic to $\mathrm{PSL}_n(3)$. In [**8**], it is proved that the projective special linear group $\mathrm{PSL}_p(3)$, where $p > 3$ is a prime number, is at most 2-recognizable by spectrum, i.e., if G is a finite group such that $\omega(G) = \omega(\mathrm{PSL}_p(3))$, where $p > 3$ is an odd prime, then G is isomorphic to $\mathrm{PSL}_p(3)$ or $\mathrm{PSL}_p(3) \cdot 2$, the extension of $\mathrm{PSL}_p(3)$ by the graph automorphism. As a consequence of our result we prove that if $n \geqslant 9$, then $\mathrm{PSL}_n(3)$ is at most 2-recognizable by spectrum, i.e., if G is a finite group such that $\omega(G) = \omega(\mathrm{PSL}_n(3))$, then G is isomorphic to $\mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$, the extension of $\mathrm{PSL}_n(3)$ by the graph automorphism.

In this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [**7**]. Throughout the proof we use the classification of finite simple groups. In [**31**, Tables 2–9], independent sets also independent numbers for all simple groups are listed and we use these results in the proof of the main theorem of this paper.

2. Preliminary results

LEMMA 2.1. [**33**, Theorem 1] *Let G be a finite group with $t(G) \geqslant 3$ and $t(2,G) \geqslant 2$. Then the following hold:*

(1) *there exists a finite nonabelian simple group S such that $S \leqslant \bar{G} = G/K \leqslant$ Aut(S) for the maximal normal soluble subgroup K of G;*
(2) *for every independent subset ρ of $\pi(G)$ with $|\rho| \geqslant 3$ at most one prime in ρ divides the product $|K||\bar{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$;*
(3) *one of the following holds:*
 (a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K||\bar{G}/S|$; in particular, $t(2,S) \geqslant t(2,G)$;*
 (b) *there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2,G) = 2$, and $S \cong \mathrm{Alt}_7$ or $\mathrm{PSL}_2(q)$ for some odd q.*

REMARK 2.1. In Lemma 2.1, for every odd prime $p \in \pi(S)$, we have $t(p,S) \geqslant t(p,G) - 1$.

LEMMA 2.2. [**26**] *Let N be a normal subgroup of G. Assume that G/N is a Frobenius group with Frobenius kernel F and cyclic Frobenius complement C. If $(|N|,|F|) = 1$, and F is not contained in $NC_G(N)/N$, then $p|C| \in \omega(G)$, where p is a prime divisor of $|N|$.*

LEMMA 2.3 (Zsigmondy's Theorem). [**36**] *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for p^n-1, that is, $p' \mid (p^n-1)$ but $p' \nmid (p^m-1)$, for every $1 \leqslant m < n$, (usually p' is denoted by r_n)*
(ii) *$p = 2$, $n = 1$ or 6,*
(iii) *p is a Mersenne prime and $n = 2$.*

LEMMA 2.4. [**13**] *Let G be a finite simple group.*

(1) *If $G = C_n(q)$, then G contains a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n-1)/(2,q-1)$.*
(2) *If $G = {}^2D_n(q)$, and there exists a primitive prime divisor r of $q^{2n-2}-1$, then G contains a Frobenius subgroup with kernel of order q^{2n-2} and cyclic complement of order r.*
(3) *If $G = B_n(q)$ or $D_n(q)$, and there exists a primitive prime divisor r_m of q^m-1 where $m = n$ or $n-1$ such that m is odd, then G contains a Frobenius subgroup with kernel of order $q^{m(m-1)/2}$ and cyclic complement of order r_m.*

REMARK 2.2. [**30**] Let p be a prime number and $(q,p) = 1$. Let $k \geqslant 1$ be the smallest positive integer such that $q^k \equiv 1 \pmod p$. Then k is called *the order of q with respect to p* and we denote it by $\mathrm{ord}_p(q)$. Obviously by the Fermat's little theorem it follows that $\mathrm{ord}_p(q)|(p-1)$. Also if $q^n \equiv 1 \pmod p$, then $\mathrm{ord}_p(q)|n$. Similarly if $m > 1$ is an integer and $(q,m) = 1$, we can define $\mathrm{ord}_m(q)$. If a is an odd prime, then $\mathrm{ord}_a(q)$ is denoted by $e(a,q)$, too. If q is odd, then $e(2,q)=1$ for $q \equiv 1 \pmod 4$ and $e(2,q)=2$ for $q \equiv -1 \pmod 4$.

LEMMA 2.5. [**32**, Proposition 2.4] *Let G be a simple group of Lie type, $B_n(q)$ or $C_n(q)$ over a field of characteristic p. Define*

$$\eta(m) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m/2 & \text{otherwise.} \end{cases}$$

Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$, and l/k is not an odd natural number.

LEMMA 2.6. [**31**, Proposition 2.1] *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $2 \leqslant k \leqslant l$. Then r and s are nonadjacent if and only if $k + l > n$, and k does not divide l.*

LEMMA 2.7. [**31**, Proposition 2.2] *Let $G = {}^2A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod 4; \\ m/2 & \text{if } m \equiv 2 \pmod 4; \\ 2m & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $2 \leqslant \nu(k) \leqslant \nu(l)$. Then r and s are nonadjacent if and only if $\nu(k) + \nu(l) > n$, and $\nu(k)$ does not divide $\nu(l)$.

Let q be a prime. We denote by $D_n^+(q)$ the simple group $D_n(q)$, and by $D_n^-(q)$ the simple group ${}^2D_n(q)$.

LEMMA 2.8. [**32**, Proposition 2.5] *Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic p and let function $\eta(m)$ be defined as in Lemma 2.5. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$, l/k is not an odd natural number, and if $\varepsilon = +$, then the equality chain $n = l = 2\eta(l) = 2\eta(k) = 2k$ is not true.*

LEMMA 2.9. [**5**, Lemma 3.1] *Let G be a finite group satisfying the conditions of Lemma 2.1, and let the groups K and S be as in the claim of Lemma 2.1. Let there exist $p \in \pi(K)$ and $p' \in \pi(S)$ such that $p \nsim p'$ in $\Gamma(G)$, and S contains a Frobenius subgroup with kernel F and cyclic complement C such that $(|F|, |K|) = 1$. Then $p|C| \in \omega(G)$.*

LEMMA 2.10. [**34**, Theorem 1] *Let $L = \mathrm{PSL}_n(q)$, where $n \geqslant 5$ and $q = p^{\alpha}$. If L acts on a vector space W over a field of characteristic p, then $\omega(L) \neq \omega(W \rtimes L)$.*

3. Main Results

THEOREM 3.1. *The simple group $\mathrm{PSL}_n(3)$, where $n \geqslant 9$, is quasirecognizable by prime graph; i.e., if G is a finite group such that $\Gamma(G) = \Gamma(\mathrm{PSL}_n(3))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}_n(3)$.*

PROOF. Let $D = \mathrm{PSL}_n(3)$, where $n \geqslant 9$, and G be a finite group such that $\Gamma(G) = \Gamma(D)$. Using [**32**, Tables 4-8], we conclude that $t(D) = [\frac{n+1}{2}] \geqslant 5$ and $t(2, D) = 2$. Therefore $t(G) \geqslant 5$ and $t(2, G) = 2$. Also $\rho(D) = \{r_i \mid [\frac{n}{2}] < i \leqslant n\}$, where r_i is a primitive prime divisor of $3^i - 1$. Also using [**32**, Table 6], it follows that $\rho(2, \mathrm{PSL}_n(3)) = \{2, r_{n-1}\}$ if n is even and $\rho(2, \mathrm{PSL}_n(3)) = \{2, r_n\}$ if n is odd. Using Lemma 2.1, we conclude that there exists a finite nonabelian simple group S such that $S \leqslant \bar{G} = G/N \leqslant \mathrm{Aut(S)}$, where N is the maximal normal soluble subgroup of G. Also $t(S) \geqslant t(G) - 1 \geqslant 4$ and $t(2, S) \geqslant t(2, G) \geqslant 2$, by Lemma 2.1. Now we consider each possibility for S, by the tables in [**32**].

Step 1. Let $S \cong A_m$, where $m \geqslant 5$.

We know that $t(S) \geqslant 4$. Thus $m \geqslant 17$. So $17 \mid |A_m|$. Since $e(17, 3) = 16$, we conclude that $n \geqslant 16$. Therefore $t(G) = [(n + 1)/2] \geqslant 8$, and so $m > 19$. If p is a prime number such that $p \leqslant m - 17$, then $p \sim 17$ in $\Gamma(S)$. Let $A := [m - 17, m] \cap \mathbb{Z}$. Then $12 = [18/2] + [18/3] - [18/6]$ elements of A are divisible by 2 or 3. Therefore $t(17, S) \leqslant 7$.

On the other hand, we know that $e(17, 3) = 16$. Let $k = e(p, 3)$. Then p is not adjacent to 17 in $\Gamma(G)$ if and only if $16 + k > n$ and $16 \nmid k$ if $16 \leqslant k$, and $k \nmid 16$ if $k \leqslant 16$. There are 16 consecutive numbers in $(n-16, n] \cap \mathbb{Z}$. So 16 can divide exactly one of them. Also at most 5 of them divide 16. Hence $t(17, G) \geqslant 16 - 1 - 5 = 10$. Therefore $7 \geqslant t(17, S) \geqslant t(17, G) - 1 \geqslant 10 - 1 = 9$, which is a contradiction.

Step 2. In this step, we prove that the simple group S is not isomorphic to a simple group of Lie type over $\mathrm{GF(p^\alpha)}$, where $p \neq 3$. Using Table 8 in [**32**], we consider the independent set $B = \{r_i \mid n - 4 \leqslant i \leqslant n\}$ in $\Gamma(G)$, since $n \geqslant 9$ and so $[\frac{n}{2}] < n - 4$. By Lemma 2.1, $|B \cap \pi(S)| \geqslant 4$.

Case 1. *Let $S \cong \mathrm{PSL}_m(q)$, where $q = p^\alpha$, $p \neq 3$.*

We know that $t(S) \geqslant t(G) - 1$. Therefore $[\frac{m+1}{2}] \geqslant t(S) \geqslant [\frac{n+1}{2}] - 1 \geqslant 4$, which implies that $m \geqslant 7$ and $m \geqslant n - 3$. Also $t(p, S) \leqslant 3$ by Table 4 in [**32**]. Thus $t(p, G) \leqslant 4$. So we conclude that $p \notin B$. Therefore p is joined to at least two elements of B in $\Gamma(G)$. In the sequel we consider one case and other cases are similar to it. We assume that p is joined to r_{n-4} and r_{n-3} in $\Gamma(G)$. Let $t = e(p, 3)$ and note that $e(r_{n-4}, 3) = n - 4$ and $e(r_{n-3}, 3) = n - 3$. By Lemma 2.6, one of the following subcases occurs:

(1) $t + n - 3 \leqslant n$ and $t + n - 4 \leqslant n$; (3) $t + n - 4 \leqslant n$ and $t \mid (n - 3)$;
(2) $t + n - 3 \leqslant n$ and $t \mid (n - 4)$; (4) $t|(n - 4)$ and $t \mid (n - 3)$.

Therefore in each case we conclude that $t \leqslant 4$. Therefore $p \in \{2, 5, 13\}$.

• If $p = 5$, then $S \cong \mathrm{PSL}_m(5^\alpha)$. We note that $e(71, 5) = 5$ and $e(71, 3) = 35$. We know that $m \geqslant 6$ and $e(71, 5^\alpha)$ divides $e(71, 5) = 5$. Therefore $71 \in \pi(S) \subseteq \pi(G)$. Now by Lemma 2.6, if $e(x, 5^\alpha) \leqslant m - 5$, then x is joined 71 in $\Gamma(S)$. Therefore $t(71, S) \leqslant 6$. On the other hand, by Lemma 2.6, we conclude that if $e(y, 3) \leqslant n-35$, then $y \sim 71$ in $\Gamma(G)$. Therefore $\rho(71, G) \subseteq \{r_i \mid n - 35 < i \leqslant n\}$. Also we know that $r_i \nsim 71$ if and only if $n - 35 < i \leqslant n$ and $i/35$, $35/i$ are not integers. Let $C = \{n - 34, \ldots, n\}$. Thus there is only one $i \in C$ such that $i/35$ is an integer. Also since $35 = 5 \times 7$, there are at most 4 elements $i \in C$ such that $35/i$ is an integer. Thus $t(71, G) \geqslant 35 - 5 = 30$, which is a contradiction since $29 \leqslant t(71, G) - 1 \leqslant t(71, S) \leqslant 6$.

• If $p = 2$, then $S \cong \mathrm{PSL}_m(2^\alpha)$. Since $e(31, 2) = 5$ and $e(31, 3) = 30$, similarly we get a contradiction.
• If $p = 13$, then $S \cong \mathrm{PSL}_m(13^\alpha)$. We know that $e(30941, 13) = 5$ and $e(30941, 3) = 30940$. Now similarly to the above we get a contradiction.
Case 2. Let $S \cong U_m(q)$, where $q = p^\alpha$, $p \neq 3$. Since $t(S) = [(m + 1)/2]$, similarly to Case 1 we have $m \geqslant 7$ and $m \geqslant n - 3$. By [**32**, Table 4], we have $t(p, S) \leqslant 3$, and similarly to the last case, we conclude that $p = 2, 5, 13$.
• If $p = 2$, then $S \cong U_m(2^\alpha)$. If $m = 7$, then $t(S) = 4$, which implies that $t(G) = 5$ and so $n = 9, 10$. Therefore $757 \in \rho(2, G) \subseteq \pi(S)$. On the other hand $e(757, 2) = 756 = 2^2 \times 3^3 \times 7$ and since $m = 7$, the order of $U_7(2^\alpha)$ shows that $9 \times 7 \mid \alpha$. Now $\pi(2^{63} - 1) \nsubseteq \pi(G)$, which is a contradiction. Therefore $m \geqslant 8$, and so $\pi(2^8 - 1) \subseteq \pi(S)$, which implies that $17 \in \pi(S)$ and since $e(17, 3) = 16$, we get that $n \geqslant 16$. This implies that $m \geqslant 13$ and so $31 \in \pi(2^{10\alpha} - 1) \subseteq \pi(S)$. We note that $e(31, 3) = 30$ and $e(31, 2) = 5$. Hence $e(31, 2^\alpha) \mid 5$. We know that if $\nu(e(x, 2^\alpha)) \leqslant m - 10$, then $31 \sim x$ in $\Gamma(S)$, by Lemma 2.7. Therefore $t(31, S) \leqslant 10$. Now we determine $t(31, G)$. As $e(31, 3) = 30$ similarly to the above, we conclude that if $e(x, 3) \leqslant m - 30$, then $31 \sim x$ in $\Gamma(G)$. Let $C = \{n - 29, \dots, n\}$. So 30 divides exactly one element of C. Also since $30 = 2 \times 3 \times 5$, there are at most 8 elements in C such that $30/i$ is an integer. Thus $22 \leqslant t(31, G)$. Therefore $21 \leqslant t(31, G) - 1 \leqslant t(31, S) \leqslant 10$, which is a contradiction.
• If $p = 5$, then $S \cong U_m(5^\alpha)$. Since $m \geqslant 7$, we have $449 \in \pi(S)$. Also $e(449, 3) = 448$ and $e(449, 5) = 14$. Now similarly to the above we get a contradiction.
• If $p = 13$, then similarly to the above by using $e(157, 3) = 78$ and $e(157, 13) = 6$, we get a contradiction.
Case 3. Let $S \cong D_m(q)$, where $q = p^\alpha$, $p \neq 3$. Then $t(S) \geqslant t(G) - 1$ implies that $m \geqslant 5$. Similarly to the last cases, we conclude that $p = 2$, 5 or 13.

If $p = 2$, then we note that $e(31, 2) = 5$. Therefore $e(31, 2^\alpha) \mid 5$. Thus for every $x \in \pi(S)$, such that $2\eta(e(x, 2^\alpha)) \leqslant 2n - 10 - 2$, we have $x \sim 31$ in $\Gamma(S)$. Therefore $t(31, S) \leqslant 12$. On the other hand, as we mentioned above $t(31, G) \geqslant 22$, which is a contradiction. Similarly if $p = 5$, then we use $t(31, S)$ and if $p = 13$, then we use $t(157, S)$ and similarly to the above we get a contradiction.
Case 4. Let $S \cong B_m(q)$ or $S \cong C_m(q)$, where $q = p^\alpha$ and $p \neq 3$.

So $(3m + 5)/4 \geqslant [(3m + 5)/4] = t(S) \geqslant t(G) - 1 = [(n + 1)/2] - 1 \geqslant 4$. Thus similarly $m \geqslant 4$ and $p = 2$, 5 or 13. For $p = 5$, we note that $e(31, 3) = 30$ and $e(31, 5) = 3$. By Lemma 2.5 we know that if $\eta(e(x, 5^\alpha)) \leqslant m - 3$, then $x \sim 31$ in $\Gamma(S)$. Therefore $t(31, S) \leqslant 6$, which is a contradiction since $t(31, G) \geqslant 22$.

If $p = 13$, then using $e(157, 3) = 78$ and $e(157, 13) = 6$, we get a contradiction.

If $p = 2$, then $S \cong B_m(2^\alpha)$. Now we note that $e(17, 3) = 16$, $e(17, 2) = 8$. Then $e(17, 2^\alpha) \mid 8$ and so $\eta(e(17, 2^\alpha)) \leqslant 4$. If $\eta(e(x, 2^\alpha)) \leqslant m - 4$, then $x \sim 17$ in $\Gamma(S)$ by Lemma 2.5. So only 8 elements, where $\eta(e(x, 2^\alpha)) > m - 4$ may not be joined 17 in $\Gamma(S)$. Hence the independent set which contains 17 has at most 9 elements in $\Gamma(S)$. On the other hand, if $r_i \nsim 17$ in $\Gamma(G)$, then $m - 16 < i \leqslant m$; and $i/16$, $16/i$ are not integers. Then 16 divides one of the numbers in $[m - 15, m] \cap \mathbb{Z}$. Also at most 5 numbers in this interval can divide 16. So at least 10 elements are not adjacent to 17 in $\Gamma(G)$. Therefore $\rho(17, G)$ has at least 11 elements and we get a

contradiction since $10 \leqslant t(17, G) - 1 \leqslant t(17, S) \leqslant 9$.
Case 5. Let $S \cong {}^2D_m(q)$, where $q = p^\alpha$, $p \neq 3$. Therefore $(3m + 4)/4 \geqslant t(S) \geqslant 4$ implies that $3m \geqslant 2n - 6$ and $m \geqslant 4$. Now we consider the following cases.
• Let $n \geqslant 11$. Then $B' = \{r'_i \mid n - 5 \leqslant i \leqslant n\}$ is an independent set in $\Gamma(G)$. Since $t(p, S) \leqslant 4$, we conclude that p is joined to at least two elements of B in $\Gamma(G)$. In each case similarly to the previous cases we conclude that $p = 2$, 5, 11 or 13.

If $p = 2$, then since $e(31, 3) = 30$ and $e(31, 2) = 5$, similarly to the last cases we get a contradiction. Also we know that $e(31, 3) = 30$, $e(31, 5) = 3$; $e(7321, 3) = 1830$, $e(7321, 11) = 8$ and $e(157, 3) = 78$, $e(157, 13) = 6$. Hence for $p = 5$, 11 and 13 we get a contradiction.
• Let $n = 9$ and $S \cong {}^2D_m(q)$, where $p \in \pi(\mathrm{PSL}_9(3)) \setminus \{3\}$.

If $p \in \{2, 13, 41, 757, 1093\}$, then $\pi(p^8 - 1) \nsubseteq \pi(G)$. Also for $p \in \{5, 7, 11\}$, we see that $\pi(p^6 - 1) \nsubseteq \pi(G)$.
• If $n = 10$, then similarly we get a contradiction.
Case 6. In this case we prove that S is not isomorphic to an exceptional simple group. Let $S \cong F_4(q)$, $E_6(q)$ or ${}^2E_6(q)$, where $q = p^\alpha$ and $p \in \pi(G)$. Then $t(S) \leqslant 5$ and so $9 \leqslant n \leqslant 12$. Easily we can see that for each $3 \neq p \in \pi(G)$, $\pi((p^8 - 1)(p^{12} - 1)) \nsubseteq \pi(G)$, which is a contradiction since $\pi((p^8 - 1)(p^{12} - 1)) \subseteq \pi(S)$.

If $S \cong E_7(q)$, where $q = p^\alpha$, then $t(S) = 8$ and so $t(G) \leqslant 9$. Therefore $9 \leqslant n \leqslant 18$. Similarly to the last case for each $3 \neq p \in \pi(G)$, we can get a contradiction.

If $S \cong E_8(q)$, where $q = p^\alpha$, then $9 \leqslant n \leqslant 24$ and for each $p \in \pi(\mathrm{PSL}_n(3))$ we have $\pi(p^{10} - p^5 + 1) \nsubseteq \pi(G)$, which is a contradiction.

If $S \cong {}^2F_4(2^{2n+1})$, then $9 \leqslant n \leqslant 12$. If $n = 9$ or $n = 10$, then $757 \in \rho(2, G)$ and so $757 \in \pi({}^2F_4(2^{2n+1}))$. We know that $e(757, 2) = 756$ and so $756 \mid 12(2n+1)$. Therefore $7 \mid (2n + 1)$, and so $\pi(2^7 - 1) \subseteq \pi(S) \subseteq \pi(G)$, which is a contradiction.

If $n = 11$, 12, then $3851 \in \rho(2, G)$ and similarly we get a contradiction, since $e(3851, 2) = 3850$.

If $S \cong {}^2B_2(2^{2n+1})$, then similarly we get a contradiction.
Step 3. Now we consider the simple groups of Lie type over $\mathrm{GF}(3^\alpha)$. In the sequel, we use r'_k for a primitive prime divisor of $(3^\alpha)^k - 1$.
Case 1. Let $S \cong \mathrm{PSL}_m(q)$, where $q = 3^\alpha$.
By [**32**, Table 6], $r_{n-1} \in \pi(S)$ or $r_n \in \pi(S)$. Also $m \geqslant n - 3$.
(**I**) Let n be odd and so $r_n \in \rho(2, S) = \{2, r'_m, r'_{m-1}\}$.
• If $r_n = r'_m$, then $n \mid \alpha m$ and so $n \leqslant \alpha m$. On the other hand, using Zsigmondy's Theorem, we conclude that $\alpha m \leqslant n$, since $\pi(S) \subseteq \pi(G)$. Therefore $\alpha m = n$.

Also we know that $m \geqslant n - 3$. If $\alpha \geqslant 2$, then $n = \alpha m \geqslant 2m \geqslant 2n - 6$, which implies that $6 \geqslant n$ and this is a contradiction. Thus $\alpha = 1$, and so $m = n$. Therefore $S = \mathrm{PSL}_n(3)$.
• If $r_n = r'_{m-1}$, then $n \mid \alpha(m - 1)$ and so $n \leqslant \alpha(m - 1)$. Also by Zsigmondy's Theorem, $\alpha(m - 1) \leqslant n$. Hence $\alpha(m - 1) = n$. On the other hand, $(3^{\alpha m} - 1) \mid |S|$ and since $\pi(S) \subseteq \pi(G)$, we conclude that $\alpha m \leqslant n$, which is a contradiction.
(**II**) Let n be even and so $r_{n-1} \in \pi(S)$. Therefore $r_{n-1} \in \rho(2, S) = \{2, r'_m, r'_{m-1}\}$.
• If $r_{n-1} = r'_{m-1}$, then $(n - 1) \mid \alpha(m - 1)$ and $\alpha(m - 1) \leqslant n$, since $\pi(S) \subseteq \pi(G)$.

Hence $\alpha(m-1)=n-1$. We know that $m \geqslant n-3$. If $\alpha \geqslant 2$, then

$$n-1=\alpha(m-1) \geqslant 2(m-1) \geqslant 2(n-3)-2 \geqslant 2n-8.$$

Hence $n \leqslant 7$, which is a contradiction. Thus $\alpha = 1$ and so $m = n$. Therefore $S \cong \mathrm{PSL}_n(3)$.

• If $r_{n-1} = r'_m$, then $(n-1) \mid \alpha m$. Also $\alpha m \leqslant n$, since $\pi(S) \subseteq \pi(G)$. Therefore $\alpha m = n-1$. If $\alpha \geqslant 2$, we get that $n-1 = \alpha m \geqslant 2m \geqslant 2n-6$. Thus $n \leqslant 5$, which is a contradiction. Thus $\alpha = 1$ and so $m = n-1$. Therefore $S \cong \mathrm{PSL}_{n-1}(3)$. So $r_n \in \pi(K) \cup \pi(\bar{G}/S)$. Also we note that $\pi(\bar{G}/S) \subseteq \pi(\mathrm{Out(S)}) = \{2\}$. So $r_n \in \pi(K)$. We note that there exists a Frobenius subgroup of $\mathrm{PSL}_{n-1}(3)$ of the form $3^{n-2} : (3^{n-2}-1)/d$, where $d = (n-1, 2)$. On the other hand, $r_n \nsim r_{n-1}$ in $\Gamma(G)$. So by Lemma 2.9, we conclude that r_n is joined to r_{n-2} in $\Gamma(G)$, which is a contradiction.

Case 2. Let $S \cong U_m(q)$, where $q = 3^\alpha$. Then $m \geqslant n-3$ and $r_n \in \pi(S)$ or $r_{n-1} \in \pi(S)$.

(**I**) Let n be odd and so $r_n \in \pi(S)$. Using [**32**, Table 4] we must consider four cases, since $r_n \in \{r'_{2m}, r'_{2m-2}, r'_m, r'_{m/2}\}$.

• If $r_n = r'_{2m}$, then m is odd by [**32**, Table 4]. Also similar to the previous cases, $n \mid 2\alpha m$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we conclude that $2\alpha m \leqslant n$. Therefore $2\alpha m = n$, which is a contradiction since n is odd.

• If $r_n = r'_{2m-2}$, then m is even and $n \mid 2\alpha(m-1)$. Also since $\pi(S) \subseteq \pi(G)$, we get that $2\alpha(m-1) \leqslant n$. Thus $n = 2\alpha(m-1)$, which is a contradiction since n is odd.

• If $r_n = r'_m$, then $4 \mid m$, by [**32**, Table 4]. Also $n \mid \alpha m$. Thus $n = \alpha m$, a contradiction since n is odd.

• Let $r_n = r'_{m/2}$. Thus $n \mid \alpha m/2$, which implies that $n \leqslant \alpha m/2 \leqslant n$, since $\pi(S) \subseteq \pi(G)$. Therefore $n = \alpha m/2$. Hence $\alpha m = 2n$ and $r_{\alpha m} \in \pi(S) \subseteq \pi(G)$, which is a contradiction.

(**II**) Let n be even and so $r_{n-1} \in \pi(S)$.

• Let $r_{n-1} = r'_{2m}$. Thus $(n-1) \mid 2\alpha m$. So $n-1 \leqslant 2\alpha m \leqslant n$. Therefore $2\alpha m = n-1$, which is a contradiction, since n is even.

• Let $r_{n-1} = r'_{2m-2}$. So $(n-1) \mid 2\alpha(m-1)$. So similarly to the above, we conclude that $2\alpha(m-1) = n-1$, which is a contradiction since n is even.

• Let $r_{n-1} = r'_m$. So $n-1 = \alpha m$. If $\alpha \geqslant 2$, then $n-1 = \alpha m \geqslant 2m \geqslant 2n-6$. Therefore $5 \geqslant n$, which is a contradiction. If $\alpha = 1$, then $m = n-1$. Therefore $S = U_{n-1}(3)$. Since $n-1$ is odd, we conclude that $(3^{n-1}+1) \mid |S|$. Hence $r_{2(n-1)} \in \pi(G)$, which is a contradiction.

• Let $r_{n-1} = r'_{m/2}$. Thus m is even and $n-1 = \alpha m/2$. We know that $n-1$ is odd and so α is odd. If $\alpha \geqslant 3$, then $n-1 \geqslant 3m/2 \geqslant 3(n-3)/2$. Therefore $7 \geqslant n$, which is a contradiction. If $\alpha = 1$, then $m = 2n-2$. So $r_{2n-2} \in \pi(S) \subseteq \pi(G)$, which is a contradiction.

Case 3. Let $S \cong B_m(q)$, where $q = 3^\alpha$.

Since $t(S) \geqslant t(G) - 1$. We have $3m > 2n - 11$. Also $\rho(2, S) = \{2, r'_m, r'_{2m}\}$.

(**I**) Let n be odd and so $r_n \in \rho(2, G)$. Therefore $r_n = r'_m$ or $r_n = r'_{2m}$.

• Let $r_n = r'_m$. Then m is odd by [**32**, Table 6]. Also $n \mid \alpha m$. Hence $n \leqslant \alpha m \leqslant n$, since $\pi(S) \subseteq \pi(G)$. Thus $n = \alpha m$. Obviously α is odd. If $\alpha \geqslant 5$, then $n = \alpha m \geqslant$

$5m \geqslant \frac{10}{3}n - \frac{55}{3}$. So $55 \geqslant 7n$, which is a contradiction. If $\alpha = 1$, then $n = m$. So $S \cong B_n(3^\alpha)$. Hence $r_{2n} \in \pi(S) \subseteq \pi(G)$, which is a contradiction. If $\alpha = 3$, then $n = 3m$. Hence $S \cong B_{n/3}(27)$. Thus $\pi(27^{2n/3} - 1) = \pi(3^{2n} - 1) \subseteq \pi(S) \subseteq \pi(G)$, which is a contradiction.

• Let $r_n = r'_{2m}$. So m is even. Therefore similarly to the above, we conclude that $n = 2\alpha m$, which is a contradiction since n is odd.

(**II**) Let n be even and so $r_{n-1} \in \rho(2, G)$. Then $r_{n-1} = r'_m$ or $r_{n-1} = r'_{2m}$. Similarly to the above we get a contradiction.

Case 4. Let $S = D_m(q)$, where $q = 3^\alpha$. Similarly, we conclude that if $m \not\equiv 3 \pmod 4$, then $3m \geqslant 2n - 2$ and if $m \equiv 3 \pmod 4$, then $3m > 2n - 4$, since $t(S) \geqslant t(G) - 1$. Therefore in each case we have $3m > 2n - 4$. We know that $\rho(2, S) = \{2, r'_{m-1}, r'_m, r'_{2m-2}\}$. Also if $r'_{m-1} \in \rho(2, S)$, then m is even and if $r'_m \in \rho(2, S)$, then m is odd.

(**I**) If n is odd, then $r_n \in \pi(S)$.

• Let $r_n = r'_m$. So n and m are odd. Similarly to the above, we conclude that $n = \alpha m$. If $\alpha = 1$, then $n = m$. So $S \cong D_n(3)$. Hence $r_{2n-2} \in \pi(S) \subseteq \pi(G)$, which is a contradiction. If $\alpha \geqslant 2$, then $n = \alpha m \geqslant 2m \geqslant 4(n - 2)/3$, which is a contradiction.

• Let $r_n = r'_{m-1}$. So n is odd and m is even. Similarly, we conclude that $n = \alpha(m - 1)$. Now since $\pi((3^\alpha)^{2(m-1)} - 1) \subseteq \pi(S) \subseteq \pi(G)$, we get a contradiction.

• Let $r_n = r'_{2m-2}$. Then $n \mid 2\alpha(m - 1)$, and so $n = 2\alpha(m - 1)$, which is a contradiction since n is odd.

(**II**) If n is even, then $r_{n-1} \in \pi(S)$.

• Let $r_{n-1} = r'_{m-1}$. Hence $(n - 1) \mid \alpha(m - 1)$ and $\alpha(m - 1) \leqslant n$. Hence $n - 1 = \alpha(m - 1)$. Then $r_{2\alpha(m-1)} \in \pi(S) \subseteq \pi(G)$, which is a contradiction.

• Let $r_{n-1} = r'_m$. So n is even and m is odd. Similarly to the above, we conclude that $(n - 1) \mid \alpha m$. Thus $\alpha m = n - 1$ and so α is odd. Also we know that $3m > 2n - 4$. If $\alpha \geqslant 3$, then $n - 1 = \alpha m \geqslant 3m \geqslant 2n - 4$. Therefore $n \leqslant 3$, which is a contradiction.

If $\alpha = 1$, then $n - 1 = m$ and so $S \cong D_{n-1}(3)$. Hence $r_{2n-2} \in \pi(S) \subseteq \pi(G)$, which is a contradiction.

• If $r_{n-1} = r'_{2m-2}$, then $n - 1 = 2(m - 1)\alpha$, which is a contradiction since n is even.

Case 5. Let $S \cong {}^2D_m(q)$, where $q = 3^\alpha$.

Similarly to the above, we conclude that $3m > 2n - 10$. By [**32**, Table 6] it follows that if $r'_{2m-2} \in \rho(2, S)$, then m is odd.

(**I**) If n is odd, then $r_n \in \pi(S)$. Hence $r_n = r'_{2m}$ or $r_n = r'_{2m-2}$. If $r_n = r'_{2m}$ or r'_{2m-2}, then similarly to the above, we conclude that $n = 2\alpha m$ or $2\alpha(m - 1)$, which is a contradiction since n is odd.

(**II**) If n is even, then $r_{n-1} \in \pi(S)$. If $r_{n-1} = r'_{2m}$, or r'_{2m-2}, then $n - 1 = 2\alpha m$ or $2\alpha(m - 1)$, which is a contradiction since $n - 1$ is odd.

Case 6. Let $S \cong F_4(q)$, where $q = 3^\alpha$.

Since $t(S) = 5$, $t(G) \leqslant 6$ and so $9 \leqslant n \leqslant 12$. We know that $\pi(3^{12\alpha} - 1) \subseteq \pi(F_4(q))$. Therefore $\alpha = 1$ and so $q = 3$, $n = 12$. Now $r_{11} \in \rho(2, G) \subseteq \pi(S)$, which is a contradiction.

Similarly, we conclude that S can not be isomorphic to $E_6(q)$ and ${}^2E_6(q)$.

Case 7. Let $S \cong {}^2G_2(3^{2m+1})$, where $m \geqslant 1$. Since $t(S) = 5$, we get that $9 \leqslant n \leqslant 12$. Similarly to the previous case if $n \geqslant 11$, then we get a contradiction since $3 \in \rho(S)$, for each independent set $\rho(S)$. Therefore $t(G) = 5$ and so $n = 9$ or $n = 10$.

Now Zsigmondy's Theorem implies that $6(2m + 1) \leqslant 10$, which is a contradiction.

Step 4. In this step we prove that S is not isomorphic to a sporadic simple group. If $S \cong J_4$, then $43 \mid |S|$ and since $e(43, 3) = 42$ we have $n \geqslant 42$. So $t(G) \geqslant [\frac{42+1}{2}] = 21$, which is a contradiction since $t(J_4) = 7$. For the rest of sporadic simple groups $t(S) \leqslant 5$ and so $9 \leqslant n \leqslant 12$. Hence $\{757, 1093\} \cap \pi(S) \neq \emptyset$, which is a contradiction.

Therefore the quasirecognition of $\mathrm{PSL}_n(3)$, where $n \geqslant 9$, is proved. □

Theorem 3.2. *If $\Gamma(G) = \Gamma(\mathrm{PSL}_n(3))$, where $n \geqslant 9$, then $\mathrm{PSL}_n(3) \leqslant G/N \leqslant \mathrm{Aut}(\mathrm{PSL_n}(3))$, where N is a 3-group for even n and N is a $\{2, 3\}$-group for odd n.*

Proof. By Theorem 3.1, we know that $\mathrm{PSL}_n(3) \leqslant G/N \leqslant \mathrm{Aut}(\mathrm{PSL_n}(3))$. Similarly to [**20**], we can assume that N is an elementary abelian p-group for some $p \in \pi(G)$. Now we prove that $\mathrm{PSL}_n(3)$ acts faithfully on N. For this reason, we prove that $C = C_G(N) \leqslant N$. Since C is a normal subgroup of G, if $C \nleqslant N$, then CN/N is a nontrivial normal subgroup of G/N. As the proof of the main theorem in [**20**] shows that $\mathrm{socle}(G/N) \cong \mathrm{PSL}_n(3)$ and so CN/N has a subgroup isomorphic to $\mathrm{PSL}_n(3)$. Therefore $r_{n-1}, r_n \in \pi(\mathrm{PSL}_n(3))$ implies that r_{n-1}, r_n divide the order of $CN/N \cong C/(C \cap N)$. Hence $p \sim r_n$ and $p \sim r_{n-1}$ in $\Gamma(G)$, which implies that $p = 1$, by Lemma 2.6. Therefore $C \leqslant N$ and $\mathrm{PSL}_n(3)$ acts faithfully on N. Also $\mathrm{PSL}_n(3)$ contains Frobenuis subgroups of the form $3^{n-1} : (3^{n-1} - 1)/(n, 2)$ and $3^{n-2} : (3^{n-2} - 1)/(n - 1, 2)$. Hence if $p \neq 3$, then using Lemma 2.2 it follows that $p \sim r_{n-1}$ and $p \sim r_{n-2}$ in $\Gamma(G)$. Therefore $p = 2$ or $p = 3$, using Lemma 2.6. Now if n is even, then $2 \nsim r_{n-1}$, which is a contradiction. Therefore if n is odd, then N is a $\{2, 3\}$-group and if n is even, then N is a 3-group. □

Corollary 3.1. *Let $\Gamma(G) = \Gamma(\mathrm{PSL}_n(3))$, where $n \geqslant 9$. Then $G/N \cong \mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$, the extension of $\mathrm{PSL}_n(3)$ by the graph automorphism, where N is a 3-group, if n is even and N is a $\{2, 3\}$-group, if n is odd.*

Proof. We know that using the notations of [**7**], $f = 1$, $g = 2$ and $d = (n, 2)$. By the assumption, we know that $\mathrm{PSL}_n(3) \leqslant \bar{G} := G/N \leqslant \mathrm{Aut}(\mathrm{PSL_n}(3))$. Let $S = \mathrm{PSL}_n(3)$. Then $\bar{G}/S \leqslant \mathrm{Aut(S)}$. Now if ϕ is a diagonal automorphism of S, and ψ is a graph automorphism of S, then $S \cdot \phi$ and $S \cdot (\phi\psi)$ have elements of orders $2r_{n-1}$ and $2r_n$, which is a contradiction, since in the prime graph of G we have $2 \nsim r_{n-1}$ if n is even and $2 \nsim r_n$ if n is odd. Therefore $\bar{G} \cong S$ or $S \cdot \psi$, the extension of $\mathrm{PSL}_n(3)$ by the graph automorphism. □

Theorem 3.3. *Let $\omega(G) = \omega(\mathrm{PSL}_n(3))$, where $n \geqslant 9$. Then $G \cong \mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$, the extension of $\mathrm{PSL}_n(3)$ by the graph automorphism.*

PROOF. Using Corollary 3.1 we know that if n is even, then $G/N \cong \mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$, where N is a 3-group. Now using Lemma 2.10 it follows that $N = 1$. Therefore $G \cong \mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$.

Similarly if n is odd, then N can not be a 3-group. Also as we stated in the proof of Theorem 3.2, $\mathrm{PSL}_n(3)$ has a Frobenius subgroup of the form $3^{n-1} : (3^{n-1} - 1)$. Now if $2 \mid |N|$, then using Lemma 2.9 we get that $2(3^{n-1} - 1) \in \omega(\mathrm{PSL}_n(3))$, which is a contradiction by [**6**]. Therefore in each case we have $G \cong \mathrm{PSL}_n(3)$ or $\mathrm{PSL}_n(3) \cdot 2$. $\square$

REMARK 3.1. In [**29**], it is proved that $h(\mathrm{PSL}_3(3)) = \infty$. Also $\mathrm{PSL}_4(3)$ and $\mathrm{PSL}_5(3)$ are recognizable by spectrum (see [**10, 25**]). In [**11**], it is shown that $h(\mathrm{PSL}_6(3)) = 2$. In [**9**], it is proved that $h(\mathrm{PSL}_7(3)) = 2$ and $h(\mathrm{PSL}_8(3)) = 1$. Also in [**8**] for each prime number $p > 3$, the following conjectures arise.

CONJECTURE 1. If $p \equiv 1 \pmod 3$, then $\mathrm{PSL}_p(3)$ is 2-recognizable by spectrum.

CONJECTURE 2. If $p \equiv 2 \pmod 3$, then $\mathrm{PSL}_p(3)$ is recognizable by spectrum.

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References

1. Z. Akhlaghi, M. Khatami, B. Khosravi, *Quasirecognition by prime graph of the simple group* ${}^2F_4(q)$, Acta Math. Hungar. **122**(4) (2009), 387–397.
2. Z. Akhlaghi, B. Khosravi, M. Khatami, *Characterization by prime graph of* $\mathrm{PGL}(2, p^k)$ *where* p *and* $k > 1$ *are odd*, Internat. J. Algebra Comput. **20**(7) (2010), 847–873.
3. A. Babai, B. Khosravi, N. Hasani, *Quasirecognition by prime graph of* ${}^2D_p(3)$ *where* $p = 2^n + 1 \geqslant 5$ *is a prime*, Bull. Malays. Math. Sci. Soc. (2) **32**(3) (2009), 343–350.
4. A. Babai, B. Khosravi, *Recognition by prime graph of* ${}^2D_{2^m+1}(3)$, Sib. Math. J. **52**(5) (2011), 788–795.
5. ______, *On the composition factors of a group with the same prime graph as* $B_n(5)$, Czechoslovak Math. J. **62 (137)**(2) (2012), 469–486.
6. A. A. Buturlakin, *Spectra of finite linear and unitary groups*, Algebra Logic **47**(2) (2008), 91–99.
7. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Oxford 1985.
8. M. R. Darafsheh, *Recognition of the projective special linear group over* GF(3), Internat. J. Algebra Comput. **18**(5) (2008), 925–933.
9. ______, *On the recognition of the simple groups* $L_7(3)$ *and* $L_8(3)$ *by the spectrum*, Acta Math. Sinica (English Series) **26**(3) (2010), 477–488.
10. M. R. Darafsheh, A. R. Moghaddamfar, *A characterization of some finite groups by their element orders*, Algebra Colloq. **7** (2000), 466-Ű476.
11. M. R. Darafsheh, Y. Farjami, A. Sadrudini, *On groups with the same set of order elements*, Int. Math. Forum **1**(27) (2006), 1325-Ű1334.
12. M. Hagie, *The prime graph of a sporadic simple group*, Comm. Algebra **31**(9) (2003), 4405–4424.
13. H. He, W. Shi, *Recognition of some finite simple groups of type* $D_n(q)$ *by spectrum*, Internat. J. Algebra Comput. **19**(5) (2009), 681–698.
14. M. Khatami, B. Khosravi, Z. Akhlaghi, *NCF-distinguishablity by prime graph of* $\mathrm{PGL}(2, p)$, *where* p *is a prime*, Rocky Mountian J. Math. **41**(5) (2011), 1523–1545.
15. B. Khosravi, A. Babai, *Quasirecognition by prime graph of* $F_4(q)$ *where* $q = 2^n > 2$, Monatsh. Math. **162**(3) (2011), 289–296.
16. A. Khosravi, B. Khosravi, *Quasirecognition by prime graph of the simple group* ${}^2G_2(q)$, Sib. Math. J. **48**(3) (2007), 570–577.

17. B. Khosravi, B. Khosravi, B. Khosravi, 2-*Recognizability of* $\mathrm{PSL}(2, p^2)$ *by the prime graph*, Sib. Math. J. **49**(4) (2008), 749–757.
18. ______, *Groups with the same prime graph as a CIT simple group*, Houston J. Math. **33**(4) (2007), 967–977.
19. ______, *On the prime graph of* $\mathrm{PSL}(2, p)$ *where* $p > 3$ *is a prime number*, Acta. Math. Hungar. **116**(4) (2007), 295–307.
20. ______, *A characterization of the finite simple group* $L_{16}(2)$ *by its prime graph*, Manuscripta Math. **126** (2008), 49–58.
21. B. Khosravi, *Quasirecognition by prime graph of* $L_{10}(2)$, Sib. Math. J. **50**(2) (2009), 355–359.
22. ______, *Some characterizations of* $L_9(2)$ *related to its prime graph*, Publ. Math. Debrecen **75**(3-4) (2009), 375–385.
23. ______, *n-Recognition by prime graph of the simple group* $\mathrm{PSL}(2, q)$, J. Algebra Appl. **7**(6) (2008), 735–748.
24. B. Khosravi, H. Moradi, *Quasirecognition by prime graph of finite simple groups* $L_n(2)$ *and* $U_n(2)$, Acta. Math. Hungar. **132**(1-2) (2011), 140–153.
25. S. Lipschutz, W. J. Shi, *Finite groups whose element orders do not exceed twenty*, Progr. Nat. Sci. **10** (2000), 11-Ű21.
26. ______, *Characterizations of finite groups by the set of orders of their elements*, Algebra Logic **36**(1) (1997), 23–32.
27. Z. Momen, B. Khosravi, *On r-recognition by prime graph of* $B_p(3)$ *where p is an odd prime*, Monatsh. Math. **166**(2) (2012), 239-Ű253.
28. ______, *Groups with the same prime graph as the orthogonal group* $B_n(3)$, Sib. Math. J. **54**(3) (2013), 487–500.
29. W. Shi, *On the simple* K_3 *groups*, J. Southwest Teachers Univ. (Natural Sci.) **13**(3) (1988), 1–4. [in Chinese]
30. W. Sierpiński, *Elementary theory of numbers*, Panstwowe Wydawnictwo Naukowe, Warsaw (1964), (Monografie Matematyczne, Vol. 42).
31. A. V. Vasil'ev, E. P. Vdovin, *An adjacency criterion in the prime graph of a finite simple group*, Algebra Logic **44**(6) (2005), 381–405.
32. ______, *Cocliques of maximal size in the prime graph of a finite simple group*, Algebra Logic **50**(4) (2011), 291-Ű322.
33. A. V. Vasil'ev, I. B. Gorshkov, *On the recognition of finite simple groups with a connected prime graph*, Sib. Math. J. **50**(2) (2009), 233–238.
34. A. V. Zavarnitsin, *Properties of element orders in covers for* $\mathrm{PSL}_n(q)$ *and* $U_n(q)$, Sib. Math. J. **49**(2) (2008), 246–256.
35. ______, *On the recognition of finite groups by the prime graph*, Algebra Logic **43**(4) (2006), 220–231.
36. K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), 265–284.

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