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THE COBURN–SIMONENKO THEOREM FOR SOME CLASSES OF WIENER–HOPF PLUS HANKEL OPERATORS

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Abstract. Wiener–Hopf plus Hankel operators $W(a) + H(b) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ with generating functions a and b from a subalgebra of $L^\infty(\mathbb{R})$ containing almost periodic functions and Fourier images of $L^1(\mathbb{R})$-functions are studied. For a and b satisfying the so-called matching condition

$$a(t)a(-t) = b(t)b(-t), \quad t \in \mathbb{R},$$

we single out some classes of operators $W(a) + H(b)$ which are subject to the Coburn–Simonenko theorem.

1. Introduction

The classical Coburn–Simonenko Theorem states that for a Toeplitz or Wiener–Hopf operator A with a scalar nonzero generating function, at least one of the numbers $\dim\ker A$ or $\dim\operatorname{coker} A$ is equal to zero. Thus if it is known that the corresponding operator is Fredholm with index zero, the Coburn–Simonenko Theorem implies that this operator is invertible. Note that Fredholmness of such operators with generating functions from various classes is well understood. On the other hand, for Toeplitz plus Hankel operators $T(a) + H(b)$ with piecewise continuous generating functions a and b their Fredholm properties can be derived by a direct application of results [**4**, Sections 4.95–4.102], [**12**, Sections 4.5 and 5.7], [**13**]. The case of quasi piecewise continuous generating functions has been studied in [**15**], whereas formulas for the index of the operators $T(a) + H(b)$ considered on various Banach and Hilbert spaces and with various assumptions about the generating functions a and b have been established in [**8**, **14**]. It is also worth mentioning that lately a lot of effort has been spent to obtain information concerning the kernel and cokernel dimensions of Toeplitz plus Hankel or Wiener–Hopf plus Hankel operators. Here we are not going to discuss the history of these investigations in much

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detail, but just mention a few important developments. For example, in the works of Ehrhardt [**9**, **10**] and Ehrhardt and Basor [**1**–**3**], Toeplitz plus Hankel operators have been studied in H^p-spaces on the unit circle $\mathbb{T}$ mainly under the assumption that the generating functions of these operators are piecewise continuous, satisfy an algebraic relation, and that the operators are Fredholm. Wiener–Hopf plus Hankel operators have received less attention in the literature and results are scarce (see, for example, [**5**] and references there). In addition, in most cases the conditions imposed on the generating functions are very restrictive and ensure that the problem can be handled in a more or less straightforward way.

Let us now describe the problem studied in the present paper. Consider the set G of all functions of the form

$$a(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j t} + \int_{-\infty}^{\infty} k(s) e^{its}\, ds, \quad -\infty < t < \infty, \tag{1.1}$$

where $\delta_j \in \mathbb{R}$ are pairwise distinct and

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \quad \int_{-\infty}^{\infty} |k(s)|\, ds < \infty.$$

The set G actually forms a commutative unital Banach algebra under pointwise operations and the norm

$$\|a\| := \sum_{j=-\infty}^{\infty} |a_j| + \int_{-\infty}^{\infty} |k(s)|\, ds.$$

This algebra G contains both the algebra AP_w of all almost periodic functions with absolutely convergent Fourier series and the algebra $\mathcal{L}_0$ of all Fourier transforms of functions from $L^1(\mathbb{R})$. Moreover, the algebra G is the direct sum of AP_w and $\mathcal{L}_0$, and $\mathcal{L}_0$ is an ideal in G. A function $a \in G$ is invertible in G if and only if it satisfies the condition $\inf_{t\in\mathbb{R}} |a(t)| > 0$. Moreover, if $b \in AP_w$, $k \in \mathcal{L}_0$, and $b + k$ is invertible in G, then b is also invertible in AP_w (see [**11**, Chapter VII]). Further, let us introduce the subalgebra G^+ (G^-) of the algebra G, which consists of all functions (1.1) such that all numbers δ_j are nonnegative (nonpositive) and function k vanishes on the negative (positive) semi-axis. It is clear that the functions from G^+ and G^- admit holomorphic extensions to the upper and to the lower half-plane, respectively, and the intersection of the sets G^+ and G^- contains constant functions only.

If $b \in AP_w$, $k \in \mathcal{L}_0$, and the element $a = b + k$ is invertible in G, then the numbers

$$\nu(a) := \lim_{l\to\infty} \frac{1}{2l}[\arg b(t)]_{-l}^{l}, \quad \text{and} \quad n(a) := \frac{1}{2\pi}[\arg(1 + b^{-1}(t)k(t)]_{t=-\infty}^{\infty},$$

are well defined. In particular, the first limit exists because b is an almost periodic function.

Let $\mathbb{R}^+ := (0, \infty)$ and let P be the projection operator from $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ onto $L^p(\mathbb{R}^+)$, that is $P : f \mapsto f|_{\mathbb{R}^+}$. Analogously, Q is the projection operator from $L^p(\mathbb{R})$ onto $L^p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$. In what follows we will identify the space

$L^p(\mathbb{R}^+)$ ($L^p(\mathbb{R}^-)$) with the subspace of $L^p(\mathbb{R})$ consisting of all functions vanishing on $\mathbb{R}^-$ ($\mathbb{R}^+$). Note that $P^2 = P$ and $Q^2 = Q$.

Each function $a \in G$,

$$a(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j t} + \int_{-\infty}^{\infty} k(s)e^{its}\, ds,$$

generates two operators $W^0(a) : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and $W(a) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$\begin{aligned} (W^0(a)f)(t) &:= \sum_{j=-\infty}^{\infty} a_j f(t-\delta_j) + \int_{-\infty}^{\infty} k(t-s)f(s)\, ds, \\ W(a)f &:= PW^0(a)f. \end{aligned} \tag{1.2}$$

These operators belong to the spaces $\mathcal{L}(L^p(\mathbb{R}))$ and $\mathcal{L}(L^p(\mathbb{R}^+))$, respectively, i.e., they are linear bounded operators. Moreover, the mappings $G \to \mathcal{L}(L^p(\mathbb{R}))$ and $G \to \mathcal{L}(L^p(\mathbb{R}^+))$ defined, respectively, by

$$a \mapsto W^0(a) \text{ and } a \mapsto W(a)\,,$$

are injective linear bounded mappings. The function a is referred to as the generating function, or the symbol, for both operators $W^0(a)$ and $W(a)$. The Fredholm theory for the operators $W^0(a)$, $a \in G$ is relatively simple. An operator $W^0(a)$ is semi-Fredholm if and only if a is invertible in G. A proof of this result is implicitly contained in the proof of Theorem 2.4, §2, Chapter VII in [**11**].

Note that the convolution operator (1.2) is shift invariant that is $W^0(a)\tau_v = \tau_v W^0(a)$ for any $v \in \mathbb{R}$, where τ_v is the operator defined by $(\tau_v f)(t) := f(t-v)$. The operator $W(a)$ is called integro-difference operator [**11**, Chapter VII]. It is shown in [**4**, Sections 9.4 and 9.21] that integro-difference operators are indeed Wiener–Hopf integral operators. If a does not vanish identically, then $W(a)$ has a trivial kernel or a dense range in $L^p(\mathbb{R}^+)$ at least for $1 < p < \infty$ and this is the Coburn–Simonenko Theorem for such class of operators (see [**4**, Section 9.5 (d)]).

Now we can formulate the following result.

THEOREM 1.1 (Gohberg/Feldman [**11**]). *If $a \in G$, then the operator $W(a)$ is one-sided invertible in $L^p(\mathbb{R}^+)$ for $1 \leqslant p \leqslant \infty$ if and only if a is invertible in G. Further, if $a \in G$ is invertible in G, then the following assertions are true.*

(i) *If $\nu(a) > 0$, then the operator $W(a)$ is invertible from the left and* $\dim \operatorname{coker} W(a) = \infty$.

(ii) *If $\nu(a) < 0$, then the operator $W(a)$ is invertible from the right and* $\dim \ker W(a) = \infty$.

(iii) *If $\nu(a) = 0$, then the operator $W(a)$ is invertible from the left (right) if $n(a) \geqslant 0$ ($n(a) \leqslant 0$) and*

$$\dim \operatorname{coker} W(a) = n(a) \quad (\dim \ker W(a) = -n(a)).$$

(iv) *If $a \in G$ is not invertible in G, then $W(a)$ is not a semi-Fredholm operator.*

Remark 1.1. Using the Coburn–Simonenko Theorem, one can show that if $W(a)$ is normally solvable and $a \neq 0$, then a is invertible in G, at least in the case where the operator $W(a)$ acts on the space $L^p(\mathbb{R}^+)$, $p \in (1, \infty)$.

Let us introduce Hankel operators. For, consider the operator $J : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defined by $J\varphi := \widetilde{\varphi}$, where $\widetilde{\varphi}(t) := \varphi(-t)$. If $a \in G$ and $1 \leqslant p \leqslant \infty$, then on the space $L^p(\mathbb{R}^+)$ the Hankel operators $H(a)$ and $H(\widetilde{a})$ are defined as follows

$$\begin{aligned} H(a) &: \varphi \mapsto PW^0(a)QJ\varphi, \\ H(\widetilde{a}) &: \varphi \mapsto JQW^0(a)P\varphi. \end{aligned}$$

Note that $JQW^0(a)P = PW^0(\widetilde{a})QJ$, and the last identity is the consequence of the following relations

$$J^2 = I, \quad JQ = PJ, \quad JP = QJ, \quad JW^0(a)J = W^0(\widetilde{a}). \tag{1.3}$$

On the space $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ we also consider the operators $\mathcal{U}$ and $\mathcal{U}^{-1}$ defined by

$$\begin{aligned} (\mathcal{U}\varphi)(t) &:= \varphi(t) - 2\int_{-\infty}^{t} e^{s-t}\varphi(s)\,ds, \quad -\infty < t < \infty, \\ (\mathcal{U}^{-1}\varphi)(t) &:= \varphi(t) - 2\int_{t}^{\infty} e^{t-s}\varphi(s)\,ds, \quad -\infty < t < \infty. \end{aligned}$$

It is well known [**11**] that

$$\mathcal{U} = W^0(\chi), \quad \mathcal{U}^{-1} = W^0(\chi^{-1}),$$

where $\chi(t) := (t - i)/(t + i)$, $\chi^{-1}(t) := (t + i)/(t - i)$, $t \in \mathbb{R}$. Moreover, since $W^0(\chi)W^0(\chi^{-1}) = W^0(\chi\chi^{-1})$, we get $\mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = I$.

One of the aims of this work is to establish a Coburn–Simonenko Theorem for the operators $W(a)+H(a\chi)$ and $W(a)-H(a\chi^{-1})$, where $a \in G$ is invertible. Recall that the semi-Fredholmness of the operators $W(b) + H(c) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $b, c \in G$ implies that the element b is invertible in G at least in the case where $1 < p < \infty$. Indeed, the proof of Theorem 2.30 in [**4**] with the shifts $U^{\pm n}$ and the Toeplitz operators $T(a)$ replaced, respectively, by the translations $\tau_{\pm\nu}$, $\nu \in \mathbb{R}^+$ and the operators $W(b) + H(c)$ implies that $\|W^0(b)f\| \geqslant c\|f\|$ for all $f \in L^p(\mathbb{R})$. But then $W^0(b)$ is semi-Fredholm and, therefore, b is invertible in G. For $p = 1$ this proof does not work. Nevertheless, we conjecture that for $p = 1$ the result is also true. Therefore, the above requirement of the invertibility of the element a is not too restrictive.

Finally, let us also mention that if $a, b \in G$, then $W^0(ab) = W^0(a)W^0(b)$, and if $a \in G^-, c \in G^+$, and $b \in G$, then $W(abc) = W(a)W(b)W(c)$. Moreover, in the following we will make use of the identities

$$\begin{aligned} W(ab) &= W(a)W(b) + H(a)H(\widetilde{b}), \\ H(ab) &= W(a)H(b) + H(a)W(\widetilde{b}). \end{aligned} \tag{1.4}$$

2. Kernels of Wiener–Hopf plus Hankel operators. General properties

In this section we establish certain relations between the kernels of Wiener–Hopf plus Hankel operators and matrix Wiener-Hopf operators in the case where the generating functions $a, b \in G$. The corresponding results for Toeplitz plus Hankel operators $T(a) + H(b)$, $a, b \in L^\infty$ have been obtained recently [**7**]. Taking into account Theorem 1.1 we can always assume that a is invertible in G. Along with the operator $W(a) + H(b)$ let us also consider the Wiener–Hopf minus Hankel operator $W(a) - H(b)$ and the Wiener–Hopf operator $W(V(a, b)))$ defined by the matrix

$$V(a,b) := \begin{pmatrix} a - b\widetilde{b}\widetilde{a}^{-1} & d \\ -c & \widetilde{a}^{-1} \end{pmatrix},$$

where $c := \widetilde{b}\widetilde{a}^{-1}$, $d := b\widetilde{a}^{-1}$.

The following lemma describes connections between the solutions of homogeneous equations with Wiener–Hopf plus/minus Hankel operators and the solutions of the associated homogeneous equation with the matrix Wiener–Hopf operator $W(V(a, b))$.

Lemma 2.1. *Assume that $a, b \in G$, a is invertible in G, and the operators $W(a) \pm H(b)$ are considered on the space $L^p(\mathbb{R}^+)$, $1 \leqslant p \leqslant \infty$.*

- *If $(\varphi, \psi)^T \in \ker W(V(a, b))$, then*

(2.1)
$$(\Phi, \Psi)^T = \frac{1}{2}\big(\varphi - JQW^0(c)\varphi + JQW^0(\widetilde{a}^{-1})\psi, \varphi + JQW^0(c)\varphi - JQW^0(\widetilde{a}^{-1})\psi\big)^T \\ \in \ker \operatorname{diag}\big(W(a) + H(b), W(a) - H(b)\big)$$

- *If $(\Phi, \Psi)^T \in \ker \operatorname{diag}\big(W(a) + H(b), W(a) - H(b)\big)$, then*

$$\big(\Phi + \Psi, P(W^0(\widetilde{b})(\Phi + \Psi) + W^0(\widetilde{a})JP(\Phi - \Psi))\big)^T \in \ker W(V(a, b)). \tag{2.2}$$

Moreover, the operators

$$E_1 : \ker W(V(a, b)) \to \ker \operatorname{diag}\big(W(a) + H(b), W(a) - H(b)\big),$$
$$E_2 : \ker \operatorname{diag}\big(W(a) + H(b), W(a) - H(b)\big) \to \ker W(V(a, b)),$$

defined, respectively, by relations (2.1) *and* (2.2) *are mutually inverse.*

Proof. Consider the operators

$$A := \begin{pmatrix} I & 0 \\ W^0(\widetilde{b}) & W^0(\widetilde{a}) \end{pmatrix} \begin{pmatrix} I & I \\ J & -J \end{pmatrix}, \qquad B_1 := 2\begin{pmatrix} I & J \\ I & -J \end{pmatrix}, \tag{2.3}$$

$$B_2 := \operatorname{diag}(I, I) - \operatorname{diag}(P, Q)\begin{pmatrix} W^0(a) & W^0(b) \\ W^0(\widetilde{b}) & W^0(\widetilde{a}) \end{pmatrix} \operatorname{diag}(Q, P),$$

$$B_3 := \operatorname{diag}(I, I) + \operatorname{diag}(P, P)\begin{pmatrix} W^0(a - b\widetilde{b}\widetilde{a}^{-1}) & W^0(d) \\ -W^0(c) & W^0(\widetilde{a}^{-1}) \end{pmatrix} \operatorname{diag}(Q, Q).$$

Elementary but tedious computations show that the the operator

$$\operatorname{diag}\big(W(a)+H(b)+Q, W(a)-H(b)+Q\big)$$

can be represented as the product of three matrix operators, viz.

$$\begin{pmatrix} W(a)+H(b)+Q & 0 \\ 0 & W(a)-H(b)+Q \end{pmatrix} = B(W(V(a,b))) + \operatorname{diag}(Q,Q)A\,, \tag{2.4}$$

where $B := B_1B_2B_3$. The operator $A : L^p(\mathbb{R}) \times L^p(\mathbb{R}) \to L^p(\mathbb{R}) \times L^p(\mathbb{R})$ is invertible because a is invertible in G, and it is well known that all the operators B_1, B_2, B_3 are invertible as well. Therefore, relations (2.3)–(2.4) imply that for any $(\varphi,\psi)^T \in \ker W(V(a,b)))$, the element $A^{-1}((\varphi,\psi)^T)$ belongs to the set

$$\ker \operatorname{diag}\big(W(a)+H(b)+Q, W(a)-H(b)+Q\big) = \ker \operatorname{diag}\big(W(a)+H(b), W(a)-H(b)\big).$$

Hence

$$\operatorname{diag}(P,P)A^{-1}((\varphi,\psi)^T) = A^{-1}((\varphi,\psi)^T).$$

Computing the left-hand side of the last equation, one obtains relation (2.1). Analogously, if $(\Phi,\Psi)^T \in \ker \operatorname{diag}\big(W(a)+H(b), W(a)-H(b)\big)$, then $A((\Phi,\Psi)^T) \in \ker W(V(a,b))$ and $\operatorname{diag}(P,P)\,A((\Phi,\Psi)^T) = A((\Phi,\Psi)^T)$, so representation (2.2) follows.

Now let (φ,ψ) and (Φ,Ψ) be as above. Then

$$\operatorname{diag}(P,P)A \operatorname{diag}(P,P)A^{-1}((\varphi,\psi)^T) = AA^{-1}((\varphi,\psi)^T),$$
$$\operatorname{diag}(P,P)A^{-1} \operatorname{diag}(P,P)A((\Phi,\Psi)^T) = A^{-1}A((\Phi,\Psi)^T),$$

which completes the proof. □

From now on we will always assume that the generating functions a and b satisfy the condition

$$a\widetilde{a} = b\widetilde{b}. \tag{2.5}$$

Analogously to [**6**], relation (2.5) is called matching condition, and if a and b satisfy (2.5), then the duo (a,b) is called a matching pair. For each matching pair (a,b) one can assign another matching pair (c,d) with $c := \widetilde{b}\widetilde{a}^{-1}$ and $d := b\widetilde{a}^{-1}$. Such a pair (c,d) is called the subordinated pair for (a,b), and it is easily seen that the functions which constitutes a subordinated pair have a specific property, namely $c\tilde{c} = 1 = d\widetilde{d}$. Throughout this paper any function $g \in G$ satisfying the condition

$$g\widetilde{g} = 1,$$

is called matching function. In passing note that the matching functions c and d can also be expressed in the form

$$c = ab^{-1}, \quad d = \widetilde{b}^{-1}a.$$

Besides, if (c,d) is the subordinated pair for a matching pair (a,b), then $(\overline{d},\overline{c})$ is the subordinated pair for the matching pair $(\overline{a},\overline{\widetilde{b}})$ which defines the adjoint operator

$$(W(a)+H(b))^* = W(\overline{a}) + H(\overline{\widetilde{b}}) \tag{2.6}$$

for the operator $W(a)+H(b)$. Further, a matching pair (a,b) is called Fredholm, if the Wiener–Hopf operators $W(c)$ and $W(d)$ are Fredholm.

If (a,b) is a matching pair, then the corresponding matrix–function $V(a,b)$ takes the form

$$V(a,b)=\begin{pmatrix}0 & d\\ -c & \widetilde{a}^{-1}\end{pmatrix},$$

where (c,d) is the subordinated pair for the pair (a,b). Moreover, similarly to the corresponding representation of the matrix Toeplitz operator $T(V(a,b))$ from [**6**], the operator $W(V(a,b)))$ can be represented as the product of three matrix Wiener–Hopf operators

$$\begin{aligned}W(V(a,b)))&=\begin{pmatrix}0 & W(d)\\ -W(c) & W(\widetilde{a}^{-1})\end{pmatrix}\\ &=\begin{pmatrix}-W(d) & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & -I\\ I & W(\widetilde{a}^{-1})\end{pmatrix}\begin{pmatrix}-W(c) & 0\\ 0 & I\end{pmatrix},\end{aligned}\tag{2.7}$$

where the operator

$$D:=\begin{pmatrix}0 & -I\\ I & W(\widetilde{a}^{-1})\end{pmatrix}$$

in the right-hand side of (2.7) is invertible and

$$D^{-1}=\begin{pmatrix}W(\widetilde{a}^{-1}) & I\\ -I & 0\end{pmatrix}.$$

Note that a useful representation for the kernel of the block Toeplitz operator $T(V(a,b)))$ defined by a matching pair (a,b), has been derived recently. Following [**7**, Proposition 3.3], one can also obtain a similar result for the block Wiener-Hopf operator $W(V(a,b))$.

Proposition 2.1. *Let $(a,b)\in G\times G$ be a matching pair such that the operator $W(c)$, $c=ab^{-1}$, is invertible from the right. Then*

$$\ker W(V(a,b)))=\Omega(c)\dotplus\widehat{\Omega}(d)$$

where

$$\begin{aligned}\Omega(c)&:=\big\{(\varphi,0)^T:\varphi\in\ker W(c)\big\},\\ \widehat{\Omega}(d)&:=\big\{(W_r^{-1}(c)W(\widetilde{a}^{-1})s,s)^T:s\in\ker W(d)\big\},\end{aligned}$$

and $W_r^{-1}(c)$ is one of the right inverses for the operator $W(c)$.

Proof. It is clear that $\Omega(c)$ and $\widehat{\Omega}(d)$ are closed subspaces of $\ker W(V(a,b))$ and $\Omega(c)\cap\widehat{\Omega}(d)=\{0\}$.

If $(y_1,y_2)^T\in\ker W(V(a,b))$, then $W(d)y_2=0$, and $W(c)y_1=W(\widetilde{a}^{-1})y_2$. Since $W_r^{-1}(c)$ is left-invertible, the space $L^p(\mathbb{R}^+)$ is the direct sum of the closed subspaces $\ker W(c)$ and $\operatorname{im} W_r^{-1}(c)$, i.e., $L^p(\mathbb{R}^+)=\ker W(c)\dotplus\operatorname{im} W_r^{-1}(c)$. Consequently, the element y_1 can be represented in the form $y_1=y_{10}+y_{11}$, where $y_{10}\in\ker W(c)$ and $y_{11}\in\operatorname{im} W_r^{-1}(c)$. Moreover, there is a unique vector $y_3\in L^p(\mathbb{R}^+)$ such that $y_{11}=W_r^{-1}(c)y_3$, so we get

$$W(c)y_1=W(c)(W_r^{-1}(c)y_3+y_{10})=y_3=W(\widetilde{a}^{-1})y_2.$$

It implies that $y_1 = W_r^{-1}(c)W(\widetilde{a}^{-1})y_2 + y_{10}$, what leads to the representation

$$(y_1, y_2)^T = (W_r^{-1}(c)W(\widetilde{a}^{-1})y_2, y_2)^T + (y_{10}, 0)^T,$$

with $(W_r^{-1}(c)W(\widetilde{a}^{-1})y_2, y_2)^T \in \widehat{\Omega}(d)$ and $(y_{10}, 0)^T \in \Omega(c)$. □

Thus $\varphi \in \ker W(c)$ implies that $(\varphi, 0)^T \in \ker W(V(a, b)))$ and by Lemma 2.1

$$\begin{aligned} &\varphi - JQW^0(c)P\varphi \in \ker(W(a) + H(b)), \\ &\varphi + JQW^0(c)P\varphi \in \ker(W(a) - H(b)). \end{aligned} \tag{2.8}$$

It is even more remarkable that the functions $\varphi - JQW^0(c)P\varphi$ and $\varphi + JQW^0(c)P\varphi$ belong to the kernel of the operator $W(c)$ as well.

Proposition 2.2. *Let $g \in G$ be a matching function, i.e., $g\widetilde{g} = 1$. If $f \in \ker W(g)$, then $JQW^0(g)Pf \in \ker W(g)$ and $(JQW^0(g)P)^2 f = f$.*

Proof. If $g\widetilde{g} = 1$ and $f \in \ker W(g)$, then

$$\begin{aligned} W(g)(JQW^0(g)Pf) &= PW^0(g)PJQW^0(g)Pf = JQW^0(\widetilde{g})QW^0(g)Pf \\ &= JQW^0(\widetilde{g})W^0(g)Pf - JQW^0(\widetilde{g})PW^0(g)Pf = 0, \end{aligned}$$

and assertion (i) follows. On the other hand, for any $f \in \ker W(g)$ one has

$$\begin{aligned} (JQW^0(g)P)^2 f &= JQW^0(g)PJQW^0(g)Pf = PW^0(\widetilde{g})QW^0(g)Pf \\ &= PW^0(\widetilde{g})W^0(g)Pf - PW^0(\widetilde{g})PW^0(g)Pf = f, \end{aligned}$$

which completes the proof. □

Consider now the operator $\mathbf{P}(g) := JQW^0(g)P\big|_{\ker W(g)}$. Proposition 2.2 implies that $\mathbf{P}(g) : \ker W(g) \to \ker W(g)$ and $\mathbf{P}^2(g) = I$. Thus on the space $\ker W(g)$ the operators $\mathbf{P}^-(g) := (1/2)(I - \mathbf{P}(g))$ and $\mathbf{P}^+(g) := (1/2)(I + \mathbf{P}(g))$ are complementary projections generating a decomposition of $\ker W(g)$. Moreover, relations (2.8) lead to the following result.

Corollary 2.1. *Let (c, d) be the subordinated pair for a matching pair $(a, b) \in G \times G$. Then $\ker W(c) = \operatorname{im} \mathbf{P}^-(c) \dotplus \operatorname{im} \mathbf{P}^+(c)$, and the following relations hold*

$$\operatorname{im} \mathbf{P}^-(c) \subset \ker(W(a) + H(b)), \quad \operatorname{im} \mathbf{P}^+(c) \subset \ker(W(a) - H(b)). \tag{2.9}$$

Relations (2.9) show the influence of the operator $W(c)$ on the kernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$. Let us now clarify the role of another operator–viz. the operator $W(d)$, in the structure of the kernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$. Assume additionally that the operator $W(c)$ is invertible from the right. If $s \in \ker W(d)$, then the element $(W_r^{-1}(c)W(\widetilde{a}^{-1})s, s)^T \in \ker W(V(a, b)))$. By Lemma 2.1, the element

$$2\varphi^{\pm}(s) := W_r^{-1}(c)W(\widetilde{a}^{-1})s \mp JQW^0(c)PW_r^{-1}(c)W(\widetilde{a}^{-1})s \pm JQW^0(\widetilde{a}^{-1})s$$

belongs to the null space $\ker(W(a) \pm H(b))$ of the corresponding operator $W(a) \pm H(b)$.

LEMMA 2.2. *Let (c,d) be the subordinated pair for a matching pair $(a,b) \in G \times G$. If the operator $W(c)$ is right-invertible, then for every $s \in \ker W(d)$ the following relations*

$$(W(\widetilde{b}) + H(\widetilde{a}))\varphi_+(s) = \mathbf{P}^+(d)s, \quad (W(\widetilde{b}) - H(\widetilde{a}))\varphi_-(s) = \mathbf{P}^-(d)s,$$

hold. Thus the corresponding mappings $\varphi_+ : \operatorname{im}\mathbf{P}^+(d) \to \operatorname{im}\mathbf{P}^+(d)$ and $\varphi_- : \operatorname{im}\mathbf{P}^-(d) \to \operatorname{im}\mathbf{P}^-(d)$, are injective operators.

PROOF. Assuming that $s \in \ker W(d)$, one can show that the operator $W(\widetilde{b}) + H(\widetilde{a})$ sends $\varphi^+(s)$ into $\mathbf{P}^+(d)s$ and the operator $W(\widetilde{b}) - H(\widetilde{a})$ sends $\varphi^-(s)$ into $\mathbf{P}^-(d)s$. The proof of these facts is based on relations (1.3) and runs similarly to the proof of [**7**, Lemma 3.6]. $\square$

PROPOSITION 2.3. *Let (c,d) be the subordinated pair for a matching pair $(a,b) \in G \times G$. If the operator $W(c)$ is right-invertible, then*

$$\ker(W(a) + H(b)) = \varphi^+(\operatorname{im}\mathbf{P}^+(d)) \dotplus \operatorname{im}\mathbf{P}^-(c),$$
$$\ker(W(a) - H(b)) = \varphi^-(\operatorname{im}\mathbf{P}^-(d)) \dotplus \operatorname{im}\mathbf{P}^+(c).$$

PROOF. Using the invertibility of the operator E_1 and Proposition 2.1, one obtains

$$\ker \operatorname{diag}(W(a) + H(b), W(a) - H(b)) = E_1(\widehat{\Omega}(d)) \dotplus E_1(\Omega(c)).$$

Apparently, $\widehat{\Omega}(d) = \widehat{\Omega}_+(d) \dotplus \widehat{\Omega}_-(d)$, $\Omega(c) = \Omega_+(c) \dotplus \Omega_-(c)$, where

$$\begin{aligned}
\widehat{\Omega}_+(d) &= \big\{(W_r^{-1}(c)W(\widetilde{a}^{-1})s, s)^T : s \in \operatorname{im}\mathbf{P}^+(d)\big\},\\
\widehat{\Omega}_-(d) &= \big\{(W_r^{-1}(c)W(\widetilde{a}^{-1})s, s)^T : s \in \operatorname{im}\mathbf{P}^-(d)\big\},\\
\Omega_+(c) &= \big\{(s,0)^T : s \in \operatorname{im}\mathbf{P}^+(c)\big\},\\
\Omega_-(c) &= \big\{(s,0)^T : s \in \operatorname{im}\mathbf{P}^-(c)\big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\ker \operatorname{diag}(W(a) + H(b), W(a) - H(b))\\
&\quad = E_1(\widehat{\Omega}_+(d)) \dotplus E_1(\widehat{\Omega}_-(d)) \dotplus E_1(\Omega_+(c)) \dotplus E_1(\Omega_-(c)) = E_1(\ker W(V(a,b)).
\end{aligned}$$

It is clear that if $\phi \in \ker(W(a)+H(b))$, then $(\phi,0)^T \in \ker \operatorname{diag}(W(a)+H(b), W(a)-H(b))$. Now we want to find that uniquely defined element $(\alpha,\beta)^T$ from the kernel of the operator $W(V(a,b))$, which is sent into the element $(\phi,0)$ by the operator E_1. It can be uniquely represented in the form

$$(\alpha,\beta)^T = (W_r^{-1}(c)W(\widetilde{a}^{-1})s_+, s_+)^T + (W_r^{-1}(c)W(\widetilde{a}^{-1})s_-, s_-)^T + (v_+,0)^T + (v_-,0)^T,$$

where $s_\pm \in \operatorname{im}\mathbf{P}^\pm(d)$, $v_\pm \in \operatorname{im}\mathbf{P}^\pm(c)$. Then

$$\begin{aligned}
(\phi,0)^T &= E_1((\alpha,\beta)^T)\\
&= E_1((W_r^{-1}(c)W(\widetilde{a}^{-1})s_+, s_+)^T) + E_1((W_r^{-1}(c)W(\widetilde{a}^{-1})s_-, s_-)^T)\\
&\quad + E_1((v_+,0)^T) + E_1((v_-,0)^T)\\
&= (\varphi_+(s_+), \varphi_-(s_+))^T + (\varphi_+(s_-), \varphi_-(s_-))^T + (0,v_+)^T + (v_-,0)^T.
\end{aligned}$$

Thus

$$\phi = \varphi_+(s_+) + \varphi_+(s_-) + v_-, \quad 0 = \varphi_-(s_+) + \varphi_-(s_-) + v_+.$$

However, since $\varphi_+(s_-) \in \ker(W(a) + H(b))$ and $E_2((\varphi_+(s_-), 0)^T) = (\varphi_+(s_-), 0)^T$ according to Lemma 2.2, we get $\varphi_+(s_-) \in \operatorname{im} \mathbf{P}^-(c)$. Analogously, one can show that $\varphi_-(s_+) \in \mathbf{P}^+(c)$. It implies that $\varphi_-(s_-) = -(\varphi_-(s_+) + v_+) \in \operatorname{im} \mathbf{P}^+(c)$ and $E_2((0, \varphi_-(s_-))^T) = (\varphi_-(s_-), 0)^T$ because $\varphi_-(s_-) \in \operatorname{im} \mathbf{P}^+(c)$. On the other hand, due to Lemma 2.2, one has $E_2((0, \varphi_-(s_-))^T) = (\varphi_-(s_-), s_-)^T$. The comparison of the two expressions for the element $E_2((0, \varphi_-(s_-))^T)$ gives $s_- = 0$, and therefore $\varphi_-(s_-) = 0$. This implies $\varphi_-(s_+) = -v_+$. Consequently,

$$(\alpha, \beta)^T = (W_r^{-1}(c) W(\widetilde{a}^{-1}) s_+, s_+)^T - (\varphi_-(s_+), 0)^T + (v_-, 0)^T,$$

which leads to the relation

$$E_1((\alpha, \beta)^T) = (\varphi_+(s_+), \varphi_-(s_+))^T - (0, \varphi_-(s_+))^T + (v_-, 0)^T = (\varphi_+(s_+) + v_-, 0)^T.$$

Thus $\varphi_+(s_+) + v_- \in \ker(W(a) + H(b))$. This result shows that $\ker(W(a) + H(b))$ is the sum of its subspaces $\varphi_+(\operatorname{im} \mathbf{P}^+(d))$ and $\operatorname{im} \mathbf{P}^-(c)$. Recalling that $(W_r^{-1}(c) W(\widetilde{a}^{-1}) s_+, s_+)^T - (\varphi_-(s_+), 0)^T \in \widehat{\Omega}_+(d) \dotplus \Omega_+(c)$ and $(v_-, 0)^T \in \Omega_-(c)$, one finally obtains

$$\ker(W(a) + H(b)) = \varphi_+(\operatorname{im} \mathbf{P}^+(d)) \dotplus \operatorname{im} \mathbf{P}^-(c).$$

The relation

$$\ker(W(a) - H(b)) = \varphi_-(\operatorname{im} \mathbf{P}^-(d)) \dotplus \operatorname{im} \mathbf{P}^+(c)$$

can be verified analogously. □

Corollary 2.2. *Let (c, d) be the subordinated pair for a matching pair $(a, b) \in G \times G$ satisfying the conditions of Proposition 3.1. Then*

$$\dim \ker(W(a) + H(b)) = \dim \operatorname{im} \mathbf{P}^+(d) + \dim \operatorname{im} \mathbf{P}^-(c),$$

$$\dim \ker(W(a) - H(b)) = \dim \operatorname{im} \mathbf{P}^-(d) + \dim \operatorname{im} \mathbf{P}^+(c).$$

Remark 2.1. If $(a, b) \in G \times G$ is a Fredholm matching pair, i.e., if $W(c), W(d)$ are Fredholm operators, then $W(a) \pm H(b)$ are Fredholm operators and

$$\operatorname{ind}(W(a) + H(b)) + \operatorname{ind}(W(a) - H(b)) = \operatorname{ind} W(c) + \operatorname{ind} W(d). \tag{2.10}$$

We conjecture that if one of the operators $W(a)+H(b)$ or $W(a)-H(b)$ is Fredholm, then so is the other and relation (2.10) holds.

3. Kernels of Wiener–Hopf plus Hankel operators. Specification

In this section we study the kernels of Wiener–Hopf plus Hankel operators $W(a) + H(b)$ in the case where the generating functions $a, b \in G$ satisfy matching condition (2.5) and $W(c), W(d)$ are mainly Fredholm operators such that

$$0 \leqslant |\operatorname{ind} W(c)|, |\operatorname{ind} W(d)| \leqslant 1.$$

Recall that a is supposed to be invertible in G. In view of Theorem 1.1, on has $\nu(c) = \nu(d) = 0$ and $0 \leqslant |n(c)|, |n(d)| \leqslant 1$.

In order to formulate our first result we need the following lemma.

LEMMA 3.1. *Let* $\chi(t) := (t-i)/(t+i)$, $t \in \mathbb{R}$.

(i) *If the function* ψ *is defined by*

$$\psi(t) := \begin{cases} e^{-t} & \textit{if } t > 0, \\ 0 & \textit{if } t < 0, \end{cases}$$

then $W^0(\chi^{-1})\psi = -\widetilde{\psi}$.

(ii) *On each space* $L^p(\mathbb{R}^+)$, $1 \leqslant p \leqslant \infty$ *the operator* $W(\chi^{-1})$ *has a one-dimensional kernel generated by the function* $\psi_0(t) = e^{-t}$, $t > 0$.

PROOF. Assertion (i) can be obtained by using the relation

$$(W^0(\chi^{-1})g)(t) = g(t) - 2\int_t^\infty e^{t-s} g(s)\, ds, \quad -\infty < t < \infty,$$

which is valid for all $g \in L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, [**11**].

Assertion (ii) is well known. It can be proved by the differentiation of the identity

$$\varphi(t) = 2\int_t^\infty e^{t-s}\varphi(s)\, ds.$$

Moreover, one has

$$(W(\chi^{-1})g)(t) = g(t) - 2\int_t^\infty e^{t-s} g(s)\, ds, \quad 0 < t < \infty,$$

(see [**11**]). Note that assertion (ii) also follows from assertion (i). $\square$

Now we can derive the following version of the Coburn–Simonenko Theorem.

THEOREM 3.1. *Let* $a \in G$ *be invertible and let* A *denote any of the four operators* $W(a) - H(a\chi)$, $W(a) + H(a\chi^{-1})$, $W(a) \pm H(a)$. *Then at least one of the spaces* $\ker A$ *or* $\operatorname{coker} A$ *is trivial.*

PROOF. Part 1: Let us start with the operator $W(a) + H(a\chi)$. The function χ satisfies the relation $\widetilde{\chi} = \chi^{-1}$, so the duo $(a, a\chi)$ is a matching pair with the subordinated pair (c, d) with $c = \chi^{-1}$ and $d = a\widetilde{a}^{-1}\chi$. Moreover, the operator $W(\chi^{-1})$ is invertible from the right and one of its right inverses is the operator $W(\chi)$. Thus the theory of Section 2 applies. As it was pointed out earlier, the kernel of this operator is $\ker W(\chi^{-1}) = \{\mathbf{c}\psi_0 : \mathbf{c} \in \mathbb{C}\}$, where $\psi_0(t) = e^{-t}$, $t > 0$. In order to apply Proposition 2.3 we have to identify, in particular, the projections $\mathbf{P}^{\pm}(\chi^{-1})$ acting on the space $\ker W(\chi^{-1})$. But $\mathbf{P}^{+}(\chi^{-1})$ and $\mathbf{P}^{-}(\chi^{-1})$ are complimentary projections on the one-dimensional space $\ker W(\chi^{-1})$. Therefore, one of these projections is just the identity operator whereas the other one is the zero operator. Consider next the expression $JQW^0(\chi^{-1})P\psi_0$. By Lemma 3.1(i) one has

$$JQW^0(\chi^{-1})P\psi_0 = JQW^0(\chi^{-1})\psi = -JQ\widetilde{\psi},$$

so that $JQW^0(\chi^{-1})P\psi_0 = -\psi_0$ and $\mathbf{P}^{-}(\chi^{-1}) = I$ on $\ker W(\chi^{-1})$.

According to Proposition 2.3, the kernels of the operators $W(a)+H(a\chi)$ and $W(a)-H(a\chi)$ can be represented in the form

$$\begin{aligned}
&\ker(W(a)-H(a\chi)) = \varphi^-(\operatorname{im}\mathbf{P}^-(d)),\\
&\ker(W(a)+H(a\chi)) = \varphi^+(\operatorname{im}\mathbf{P}^+(d)) \dotplus \{\mathbf{c}\psi_0 : \mathbf{c}\in\mathbb{C}\}.
\end{aligned} \tag{3.1}$$

If $\dim\ker W(d) > 0$, then $\operatorname{coker}(W(a)\pm H(a\chi)) = \{0\}$. Indeed, relation (2.7) and the familiar Coburn–Simonenko Theorem for the operator $W(d)$ show that $\operatorname{coker} W(V(a,a\chi)) = \{0\}$. Taking into account representation (2.4), one obtains that the cokernel of each of the operators $W(a)+H(a\chi)$ and $W(a)-H(a\chi)$ contains the zero element only.

Let us now assume that $\ker W(d) = \{0\}$. Then the first relation (3.1) implies that $\ker(W(a)-H(a\chi)) = 0$. Hence, the operator $W(a)-H(a\chi)$ is subject to Coburn–Simonenko Theorem.

Part 2: Consider the operator $W(a)+H(a\chi^{-1})$ and note that $W(c)=W(\chi)$ is not right-invertible, so that Proposition 2.3 cannot be directly used in this situation. Nevertheless, the case at hand can be reduced to the operators studied. Thus the operators $W(a)\pm H(a\chi^{-1})$ can be represented in the form

$$W(a)\pm H(a\chi^{-1}) = (W(a\chi^{-1})\pm H(a\chi^{-1}\chi))W(\chi). \tag{3.2}$$

The proof of (3.2) follows from (1.4) and relation $H(\chi)W(\chi)=0$. Setting $\alpha := a\chi^{-1}$, we get

$$W(a)\pm H(a\chi^{-1}) = (W(\alpha)\pm H(\alpha\chi))W(\chi). \tag{3.3}$$

The operators of the form $W(\alpha)\pm H(\alpha\chi)$ in the right-hand side of (3.2) have been just studied, and we already know that the function ψ_0 belongs to the kernels of both operators $W(\alpha)+H(\alpha\chi)$ and $W(\chi^{-1})$. Since $W(\chi^{-1})W(\chi)=I$ it follows that $\psi_0\notin \operatorname{im} W(\chi)$. Consider now the projection $Q_0 := W(\chi)W(\chi^{-1})$ which projects the space $L^p(\mathbb{R}^+)$, $1\leqslant p\leqslant\infty$ onto $\operatorname{im} W(\chi)$ parallel to $\ker W(\chi^{-1})$.

Assume first that $\ker W(d)=\{0\}$ and note that for the matching pairs $(a, a\chi^{-1})$ and $(\alpha,\alpha\chi)$, the corresponding subordinated pairs (c,d) have the same element d, namely, $d = a\widetilde{a}^{-1}\chi$. Then (3.3) shows that $\ker(W(a)+H(a\chi^{-1}))=\{0\}$. Further, if $\dim\ker W(d)>0$, then the space $\ker(W(\alpha)+H(\alpha\chi))$ decomposes as follows

$$\ker(W(\alpha)+H(\alpha\chi)) = \ker W(\chi^{-1}) \oplus Q_0(\ker(W(\alpha)+H(\alpha\chi))).$$

However, as was already shown, the operator $W(\alpha)-H(\alpha\chi)$ is right-invertible and

$$\ker W(\chi^{-1}) \subset \ker W(\alpha)+H(\alpha\chi).$$

Therefore, relation (3.2) implies that the operator $W(a)+H(a\chi^{-1})$ maps $L^p(\mathbb{R}^+)$ onto $L^p(\mathbb{R}^+)$, so it is subject to the Coburn–Simonenko Theorem.

Part 3: It remains to consider the operators $W(a)\pm H(a)$. For these operators the element c in the corresponding subordinated pair is either 1 or -1, and our claim follows immediately from the Coburn–Simonenko Theorem for scalar Wiener–Hopf operators and from relations (2.7) and (2.4). □

Remark 3.1. The proof of Theorem 3.1 shows that this theorem remains true for more general generating functions, for instance, in the case where a and b belong to the algebras G_p, $1 \leqslant p \leqslant \infty$ studied in [**11**, Chapter VII].

The reader can also observe that, in fact, we have proved a bit more than Theorem 3.1 states. A more detailed result can be formulated as follows.

Corollary 3.1. *Let $a \in G$ be invertible. Then*

(i) *If* $\dim \ker W(d) = 0$, *then*

$$\ker(W(a) - H(a\chi)) = \{0\}, \quad \ker(W(a) + H(a\chi)) = \{\mathbf{c}\psi_0 : \mathbf{c} \in \mathbb{C}\},$$

and if $\dim \ker W(d) > 0$, *then* $\operatorname{coker}(W(a) \pm H(a\chi)) = \{0\}$.

(ii) *If* $\dim \ker W(d) = 0$, *then* $\ker(W(a) \pm H(a\chi^{-1})) = \{0\}$, *and if* $\dim \ker W(d) > 0$, *then* $\operatorname{coker}(W(a) + H(a\chi^{-1})) = \{0\}$.

Let us emphasize that the description of the projections $\mathbf{P}^{\pm}(\chi^{-1})$ did play an important role in our considerations. In the general case one has to study the projections $\mathbf{P}^{\pm}(g)$ for the functions g satisfying the relation $g\widetilde{g} = 1$. Because of the space restriction, we are not going to pursue this matter here. Nevertheless, let us consider the case where $\nu(g) = 0$ and $n(g) = -1$, which is one of the simplest generalization of the situation $g = \chi^{-1}$. In order to handle this case we need a result from [**11**, Chapter VII].

Proposition 3.1. *Each invertible function $g \in G$ admits the factorization of the form*

$$g(t) = g_-(t) e^{i\nu t} \Big(\frac{t-i}{t+i}\Big)^n g_+(t), \quad -\infty < t < \infty, \tag{3.4}$$

where $g_+^{\pm 1} \in G^+$, $g_-^{\pm 1} \in G^-$, $\nu = \nu(g)$ and $n = n(g)$. Moreover, under the agreement $g_-(0) = 1$, the factorization factors g_+ and g_- are uniquely defined.

Note that the proof of Theorem 1.1 is based on Proposition 3.1.

Definition 3.1. Suppose that $g \in G$ satisfies the condition $g\widetilde{g} = 1$ and set

$$\xi(g) = (-1)^n g(0), \quad n = n(g).$$

Theorem 3.2. *If $g \in G$ and $g\widetilde{g} = 1$, then $\xi(g) = \pm 1$ and the factorization* (3.4) *takes the form*

$$g(t) = \big(\xi(g)\, \widetilde{g}_+^{-1}(t)\big)\, e^{i\nu t} \Big(\frac{t-i}{t+i}\Big)^n g_+(t) \tag{3.5}$$

with $\widetilde{g}_+^{\pm 1}(t) \in G^-$ and $g_-(t) = \xi(g)\, \widetilde{g}_+^{-1}(t)$.

Proof. Using the condition $g^{-1} = \widetilde{g}$, we get from (3.4) that

$$g_+^{-1}(t) e^{-i\nu t} \Big(\frac{t-i}{t+i}\Big)^{-n} g_-^{-1}(t) = \widetilde{g}_-(t) e^{-i\nu t} \Big(\frac{t-i}{t+i}\Big)^{-n} \widetilde{g}_+(t),$$

where $\nu = \nu(g)$, $n = n(g)$.

Note that $\widetilde{g}_-^{\pm 1} \in G^+$, $\widetilde{g}_+^{\pm 1} \in G^-$, as easy computations show. Therefore,

$$g_+^{-1} \widetilde{g}_-^{-1} = g_- \widetilde{g}_+,$$

and $g_+^{-1}\widetilde{g}_-^{-1} \in G^+$, $g_-\widetilde{g}_+ \in G^-$. It follows that there is a constant $\xi \in \mathbb{C}$ such that $g_+^{-1}\widetilde{g}_-^{-1} = \xi = g_-\widetilde{g}_+$, and $g_- = \xi\,\widetilde{g}_+^{-1}$. For the function $g_0 = g_+g_-$ we have $g_0\widetilde{g}_0 = 1$. Therefore,

$$1 = g_0\widetilde{g}_0 = (\xi g_+\widetilde{g}_+^{-1})(\xi\widetilde{g}_+g_+^{-1}) = \xi^2.$$

For $t = 0$, which is one of the fixed points of the operator J, the equation $g_0 = \xi g_+\widetilde{g}_+^{-1}$ implies $g_0(0) = \xi$, and $g_0(0) = g(0)(-1)^n$ (see (3.5)). Thus we obtain that $\xi = g(0)(-1)^n$ which completes the proof. □

Now we again use the notation

$$\chi^{\pm1}(t) = \Big(\frac{t-i}{t+i}\Big)^{\pm1}, \quad t \in \mathbb{R}.$$

THEOREM 3.3. *Let $g \in G$, $g\widetilde{g} = 1$, $\nu(g) = 0$ and $n(g) = -1$. Then*

$$\operatorname{im}\mathbf{P}^{\pm}(g) = \Big\{\mathbf{c}\Big(\frac{1 \mp \xi(g)}{2}\Big)W(g_+^{-1})\psi_0 : \mathbf{c} \in \mathbb{C}\Big\}.$$

PROOF. It is easily seen that $\ker W(g) = \big\{\mathbf{c}W(g_+^{-1})\psi_0 : \mathbf{c} \in \mathbb{C}\big\}$, According to the definition of projections $\mathbf{P}^{\pm}(g)$ we have to compute the expression

$$JQW^0(g)PW(g_+^{-1})\psi_0.$$

We have

$$\begin{aligned} JQW^0(g)PW(g_+^{-1}) &= JQW^0(g_-)W^0(\chi^{-1})W^0(g_+)W^0(g_+^{-1})P \\ &= JQW^0(g_-)W^0(\chi^{-1})P. \end{aligned}$$

Recall that by Lemma 3.1, $W^0(\chi^{-1})P\psi_0 = W^0(\chi^{-1})\psi = -\widetilde{\psi}$, and using Theorem 3.2 we get

$$\begin{aligned} JQW^0(g)PW(g_+^{-1})\psi_0 &= -JQW^0(g_-)\widetilde{\psi} = -W^0(\widetilde{g}_-)\psi \\ &= -P\xi(g)W^0(g_+^{-1})P\psi_0 = -\xi(g)W^0(g_+^{-1})\psi_0, \end{aligned}$$

and we are done. □

The next result is a generalization of Theorem 3.1.

THEOREM 3.4. *Let $a, b \in G$ constitute a matching pair, a be invertible in G and let (c, d) be the subordinated pair for (a, b). If A denotes one of the following operators*

(i) *$W(a) \pm H(b)$ with $\nu(c) = 0$, $n(c) = 1$ and $\xi(c) = \pm1$;*
(ii) *$W(a) \mp H(b)$ with $\nu(c) = 0$, $n(c) = -1$ and $\xi(c) = \pm1$;*
(iii) *$W(a) \pm H(b)$ with $\nu(c) = 0$ and $n(c) = 0$*

considered on the space $L^p(\mathbb{R}^+)$, then at least one of the spaces $\ker A$ *or* $\operatorname{coker} A$ *is trivial.*

PROOF. The proof mimics that of Theorem 3.1 with minor modifications. First, we note that the case $\xi(c) = -1$ can be reduced to the case $\xi(c) = 1$ via rearrangements $W(a)+H(b) = W(a)-H(-b)$ and $W(a)-H(b) = W(a)+H(-b)$. Therefore, we only consider the situation $\xi(c) = 1$ in the cases (i) and (ii). Further, one has to use Theorem 3.3 instead of the description of the projections $\mathbf{P}^{\pm}(\chi^{\pm 1})$. Consider the operator $W(a)+H(b)$ in the case where $\nu(c)=0$ and $n(c)=1$. Representing the operator $W(a)\pm H(b)$ in the form

$$W(a)\pm H(b)=(W(a\chi^{-1})\pm H(b\chi))W(\chi),$$

we observe that $(a\chi^{-1}, b\chi)$ is a matching pair with the subordinated pair $(c\chi^{-2}, d)$ and $\operatorname{ind} W(c\chi^{-2})=-1$, $\operatorname{im}\mathbf{P}^{+}(c\chi^{-2})=\ker W(c\chi^{-2})=\{\mathbf{c}W(c_{+}^{-1})\psi_0 : c\in\mathbb{C}\}$. Let us also note that $\ker W(c\chi^{-2})=\ker W(c_{+}\chi^{-1})$ and $W(c_{+}\chi^{-1})\,W(c_{+}^{-1}\chi)=I$. Hence, $\ker W(c\chi^{-2})\cap\operatorname{im} W(c_{+}^{-1}\chi)=\{0\}$. Since obviously $\operatorname{im} W(c_{+}^{-1}\chi)=\operatorname{im} W(\chi)$, we obtain $\ker W(c\chi^{-1})\cap\operatorname{im} W(\chi)=\{0\}$.

Now one can proceed similarly to Part 2 in the proof of Theorem 3.1. □

Corollary 3.2. *Assume that $a,b\in G$ constitute a matching pair with the subordinated pair (c,d) such that $\xi(c)=1$. Then*

(i) *If* $\dim\ker W(d)=0$, *and* $\operatorname{ind} W(c)=1$, *then*

$$\ker(W(a)-H(b))=\{0\},\quad \ker(W(a)+H(b))=\{\mathbf{c}W(c_{+}^{-1})\psi_0:\mathbf{c}\in\mathbb{C}\},$$

and if $\dim\ker W(d)>0$, *then* $\operatorname{coker}(W(a)\pm H(b))=\{0\}$.

(ii) *If* $\dim\ker W(d)=0$, *and* $\operatorname{ind} W(c)=-1$, *then* $\ker(W(a)\pm H(b))=\{0\}$, *and if* $\dim\ker W(d)>0$, *then* $\operatorname{coker}(W(a)+H(b))=\{0\}$.

An interesting and important subclass of the operators considered in this paper comprises the identity plus Hankel operators. Let us specify the above results in this situation

Corollary 3.3. *If $b\in G$ is a matching function, then $(1,b)$ is a matching pair with the subordinated pair $(\widetilde{b},b)$, and if A denotes any of the operators*

(i) $I-H(b)$ *with* $\nu(\widetilde{b})=0$, $n(\widetilde{b})=-1$ *and* $\xi(\widetilde{b})=1$;
(ii) $I+H(b)$ *with* $\nu(\widetilde{b})=0$, $n(\widetilde{b})=1$ *and* $\xi(\widetilde{b})=1$;
(iii) $I\pm H(b)$ *with* $\nu(\widetilde{b})=0$ *and* $n(\widetilde{b})=0$,

considered on the space $L^p(\mathbb{R}^+)$, then $\ker A$ *or* $\operatorname{coker} A$ *is trivial.*

Now we revisit Theorem 3.1 and consider the operators $W(a)\pm H(a\chi)$ and $W(a)\pm H(a\chi^{-1})$ under additional assumptions.

$\mathbf{1^0}$. Suppose that $\nu(a)=n(a)=0$ and $b=a\chi$. The subordinated pair (c,d) is given by the elements $c=\chi^{-1}$ and $d=a\widetilde{a}^{-1}\chi$. Thus

$$\operatorname{ind} W(c)=1,\quad \operatorname{ind} W(d)=-1,\quad \xi(c)=\xi(d)=1.$$

According to (2.10) we have

$$\operatorname{ind}(W(a)+H(a\chi))+\operatorname{ind}(W(a)-H(a\chi))=0. \tag{3.6}$$

Further, by Corollary 3.1(i) we also have

$$\ker(W(a)-H(a\chi))=0,\quad \ker(W(a)+H(a\chi))=\{\mathbf{c}\psi_0:\mathbf{c}\in\mathbb{C}\}.$$

In order to describe the cokernels of the above operators we make use of the adjoint operators. If $p\in[1,\infty)$, then according to (2.6) the adjoint operators have the form $W(\overline{a})\pm H(\overline{\widetilde{a}}\chi)$, and the duo $(\overline{a},\overline{\widetilde{a}}\chi)$ is a matching pair with the subordinated pair

$(\overline{d},\overline{c})$, so that $\operatorname{ind} W(\overline{d}) = 1, \operatorname{ind} W(\overline{c}) = -1$ and $\xi(\overline{d}) = 1$. By Corollary 3.2(ii), $\ker(W(\overline{a}) - H(\overline{\widetilde{a}}\chi)) = \{0\}$, which finally proves that the operator $W(a) - H(a\chi)$ is invertible. Note that this result is also true for the space $L^\infty(\mathbb{R}^+)$. Indeed, the operator $W(\overline{a}) - H(\overline{\widetilde{a}}\chi)$ acts on the space $L^1(\mathbb{R}^+)$ and the above considerations show that $\dim\ker(W(\overline{a}) - H(\overline{\widetilde{a}}\chi)) = 0$. The adjoint of this operator acts on the space $L^\infty(\mathbb{R}^+)$ and is equal to the operator $W(a) - H(a\chi)$, the kernel of which is trivial. Therefore, the operator $W(\overline{a}) - H(\overline{\widetilde{a}}\chi)$ is invertible on the space $L^1(\mathbb{R}^+)$. Consequently, its adjoint $W(a) - H(a\chi)$ is invertible on $L^\infty(\mathbb{R}^+)$. Then relation (3.6) immediately implies that $\operatorname{ind}(W(a) + H(a\chi)) = 0$. Note that the operator $W(a) + H(a\chi)$ provides an example of operators where both spaces $\ker(W(a) + H(a\chi))$ and $\operatorname{coker}(W(a) + H(a\chi))$ are nontrivial.

$\mathbf{2^0}$. Suppose that $\nu(a) = 0$, $n(a) = -1$ and $b = a\chi$. For the subordinated pair (c, d) we have $c = \chi^{-1}$ and $d = a\widetilde{a}^{-1}\chi$ so that $\operatorname{ind} W(c) = 1$, $\operatorname{ind} W(d) = 1$, $\xi(d) = 1$. Since $\operatorname{ind} W(d) = 1$, Corollary 3.1(i) indicates that $\operatorname{coker}(W(a) \pm H(a\chi)) = \{0\}$. besides, $\dim\ker(W(a) \pm H(a\chi)) = 1$ by Proposition 2.3.

$\mathbf{3^0}$. Suppose that $\nu(a) = n(a) = 0$ and $b = a\chi^{-1}$. Since $c = \chi$, the operator $W(c)$ is not invertible from the right. Write

$$W(a) \pm H(a\chi^{-1}) = (W(a\chi^{-1}) \pm H(a\chi^{-1}\chi))W(\chi), \tag{3.7}$$

and set $\alpha := a\chi^{-1}$. The operators $W(\alpha) \pm H(\alpha\chi)$ are considered in $\mathbf{2^0}$, so we have

$$\dim\ker(W(\alpha) \pm H(\alpha\chi)) = 1$$
$$\dim\operatorname{coker}(W(\alpha) \pm H(\alpha\chi)) = 0.$$

According to the Part 2 in the proof of Theorem 3.1, one has $\ker(W(a)+H(a\chi^{-1})) = \{0\}$. This and the relation $\dim\operatorname{coker}(W(\alpha) + H(\alpha\chi)) = 0$ show the invertibility of the operator $W(a) + H(a\chi^{-1})$. Due to Proposition 2.3 (see also (3.1)) we know that the kernel of the operator $W(\alpha) - H(\alpha\chi)$ is spanned on the element

$$\begin{aligned}\kappa = W(\chi)W(\widetilde{\alpha}^{-1})W(d_+^{-1})\psi_0 + JQW^0(\chi^{-1})PW^0(\chi)PW(\widetilde{\alpha}^{-1})W(d_+^{-1})\psi_0 \\ - JQW^0(\widetilde{\alpha}^{-1})PW(d_+^{-1})\psi_0,\end{aligned} \tag{3.8}$$

where we used the fact that $W(\chi)$ is a right inverse for the operator $W(\chi^{-1})$ and where d_+^{-1} arises from the factorization (3.5) of the function $d = a\widetilde{a}^{-1}\chi^{-1}$. Note that the first term in (3.8) belongs to the set $\operatorname{im} W(\chi)$, whereas the second one is equal to zero. Thus the operator $W(a) - H(a\chi^{-1})$ is invertible if and only if $H(\alpha^{-1})W(d_+^{-1})\psi_0 \notin \operatorname{im} W(\chi)$. On the other hand, if this condition is not satisfied, the operator $W(a) - H(a\chi^{-1})$ presents an example of a Wiener–Hopf plus Hankel operator with one-dimensional kernel and cokernel.

$\mathbf{4^0}$. Suppose that $\nu(a) = 0$, $n(a) = 1$ and $b = a\chi^{-1}$. Let us use representation (3.7) and set $\alpha = a\chi^{-1}$. It follows from Part $\mathbf{1^0}$ that $W(\alpha) - H(\alpha\chi)$ is invertible whereas the operator $W(\alpha)+H(\alpha\chi)$ has one-dimensional kernel and cokernel. Since

$$\ker(W(\alpha) + H(\alpha\chi)) = \{\mathbf{c}\psi_0 : \mathbf{c} \in \mathbb{C}\} \cap \operatorname{im} W(\chi) = \{0\},$$

we conclude that the operator $W(a)+H(a\chi^{-1})$ has trivial kernel and a cokernel of dimension 1. Of course, the same conclusion is valid for the operator $W(a)-H(a\chi)$.

It is worth noting that a similar consideration with natural amendments can be used in the contest of Theorem 3.4. Let us restrict ourselves to the operators $I+H(b)$ with the generating function b satisfying the condition $b\widetilde{b}=1$. Then $(1,b)$ is a matching pair with the subordinated pair $(\widetilde{b},b)$.

$\mathbf{5^0}$. Suppose that $\nu(b)=n(b)=0$. Then the operators $W(b)$ and $W(\widetilde{b})$ are invertible and relations (2.4), (2.7) already show that $I+H(b)$ and $I-H(b)$ are invertible operators.

Assume next that $\nu(b)=0$ but $n(b)=1$ and $\xi(\widetilde{b})=1$. Then $\operatorname{ind} W(\widetilde{b})=1$ and $\operatorname{ind} W(b)=-1$. By Corollary 3.2(i), one has

$$\ker(I-H(b))=\{0\}, \quad \ker(I+H(b))=\{\mathbf{c}W(b_+)\psi_0 : \mathbf{c}\in\mathbb{C}\}.$$

Similarly to Part $\mathbf{1^0}$ one shows that the operator $I-H(b)$ is invertible and $\operatorname{ind}(I+H(b))=0$.

Finally, let us assume that $\nu(b)=0$, $n(b)=-1$ and $\xi(\widetilde{b})=1$. Since $\operatorname{ind} W(\widetilde{b})=-1$, we will use the relation $I\pm H(b)=(W(\chi^{-1})\pm H(b\chi))W(\chi)$. It is clear that $(\chi^{-1},b\chi)$ is a matching pair with the subordinated pair $(\widetilde{b}\chi^{-2},b)$ and $\operatorname{ind} W(\widetilde{b}\chi^{-2})=\operatorname{ind} W(b)=1$. Analogously to Part $\mathbf{2^0}$ we obtain that

$$\operatorname{coker}(W(\chi^{-1})\pm H(b\chi))=\{0\}.$$

Moreover, by Proposition 2.3, $\dim\ker(W(\chi^{-1})\pm H(b\chi))=1$ and since

$$\ker(W(\chi^{-1})+H(b\chi))=\{\mathbf{c}W(b_+)\psi_0 : \mathbf{c}\in\mathbb{C}\}\cap \operatorname{im} W(\chi)=\{0\},$$

the operator $I+H(b)$ is invertible. If $\ker W(\chi^{-1})-H(b\chi)\cap \operatorname{im} W(\chi)=\{0\}$, then $I-H(b)$ is invertible. Otherwise, $\operatorname{ind}(I-H(b))=0$, but this operator is not invertible.

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